Divisible pseudo-BCK-algebras

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Divisible porims/residuated lattices

A porim (partially ordered residuated integral monoid) is a structure $(A; \cdot, \to, \rightsquigarrow, 1, \leq)$ such that $(A; \cdot, 1, \leq)$ is an integral pomonoid and, for all $x, y, z \in A$,

$$z \leq x \to y \ \Leftrightarrow \ z \cdot x \leq y, \quad \text{and} \quad z \leq x \rightsquigarrow y \ \Leftrightarrow \ x \cdot z \leq y.$$

A porim A is called divisible if

$$x \leq y \quad \Leftrightarrow \quad (\exists a, b \in A) \quad x = y \cdot a = b \cdot y$$

or, equivalently, A satisfies the identity

$$x \cdot (x \rightsquigarrow y) = (y \to x) \cdot y.$$

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$$z \leq \underbrace{x \rightarrow y}_{y/x} \Leftrightarrow z \cdot x \leq y, \quad \text{and} \quad z \leq \underbrace{x \rightsquigarrow y}_{x \backslash y} \Leftrightarrow x \cdot z \leq y.$$

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Pseudo-BCK-algebras

Pseudo-BCK-algebras are subalgebras of the $\{\to, \rightsquigarrow, 1\}$-reducts of porims.

A pseudo-BCK-algebra is an algebra $(A; \to, \rightsquigarrow, 1)$ of type $\langle 2, 2, 0 \rangle$ together with a partial order $\leq$ satisfying the following conditions (for all $x, y, z \in A$):

- $x \leq 1$,
- $x \leq y \iff x \to y = 1 \iff x \rightsquigarrow y = 1$,
- $1 \to x = x$, $1 \rightsquigarrow x = x$,
- $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$, $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$.

(Georgescu & Iorgulescu: non-commutative generalization of BCK-algebras)

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$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z),$$
$$(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) = (y \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow z).$$

We call a pseudo-BCK-algebra divisible if it satisfies these identities.

Vetterlein:

$$(x \to y) \to (x \to z) = x \to ((x \rightsquigarrow y) \to z),$$
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n-potent algebras

A porim is n-potent ($n \in \mathbb{N}$) if $x^n = x^{n+1}$.

Notation:

$$x^n \to y = x \to (\ldots \to (x \to y) \ldots)$$
$$x^n \rightsquigarrow y = x \rightsquigarrow (\ldots \rightsquigarrow (x \rightsquigarrow y) \ldots)$$

We call a pseudo-BCK-algebra n-potent if for all x, y,

$$x^n \to y = 1 \quad \text{iff} \quad x^{n+1} \to y = 1.$$

Equivalently:

$$x^n \to y = x^{n+1} \to y \quad \text{or} \quad x^n \rightsquigarrow y = x^{n+1} \rightsquigarrow y$$

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Normal pseudo-BCK-algebras

By a relative congruence of a pseudo-BCK-algebra A we mean a congruence θ such that A/θ is a pseudo-BCK-algebra.

Pseudo-BCK-algebras are relatively 1-regular.

We say that a pseudo-BCK-algebra is normal iff the 1-classes of relative congruences can be characterized as follows:

- $1 \in K$;
- if $a \in K$ and $a \to b \in K$ (or $a \rightsquigarrow b \in K$), then $b \in K$.

Theorem

Every divisible n-potent pseudo-BCK-algebra is normal.

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Subdirectly irreducible normal divisible pseudo-BCK-algebras

Ordinal sums

Let A and B be pseudo-BCK-algebras such that $A \cap B = \{1\}$. Their ordinal sum is the pseudo-BCK-algebra $A \oplus B = (A \cup B; \to, \rightsquigarrow, 1)$ where

$$x \to y = \begin{cases} x \to^A y & \text{if } x, y \in A, \\ x \to^B y & \text{if } x, y \in B, \\ 1 & \text{if } x \in A \setminus \{1\} \text{ and } y \in B, \\ y & \text{if } x \in B \text{ and } y \in A \setminus \{1\}, \end{cases}$$

and $\rightsquigarrow$ is defined in the same way.

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Subdirectly irreducible normal divisible pseudo-BCK-algebras

Cone algebras

A cone algebra is a divisible pseudo-BCK-algebra satisfying the identity

$$(x \to y) \rightsquigarrow y = (y \rightsquigarrow x) \to x.$$

An algebra $(A; \to, \rightsquigarrow, 1)$ is a cone algebra iff there exists a lattice-ordered group $(G; \cdot, ^{-1}, 1, \leq)$ such that $(A; \to, \rightsquigarrow, 1)$ is isomorphic to a subalgebra of the algebra $(G^-; \to, \rightsquigarrow, 1)$ where $G^- = \{x \in G \mid x \leq 1\}$ and

$$x \to y = yx^{-1} \wedge 1 \quad \text{and} \quad x \rightsquigarrow y = x^{-1}y \wedge 1.$$

An MV-algebra is a bounded cone algebra where $x \to y = x \rightsquigarrow y$.

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Subdirectly irreducible normal divisible pseudo-BCK-algebras

Theorem

A non-trivial normal divisible pseudo-BCK-algebra A is subdirectly irreducible iff it is of the form $A = B \oplus C$ where C is a non-trivial subdirectly irreducible linearly ordered cone algebra.

Theorem

Every n-potent divisible pseudo-BCK-algebra is a BCK-algebra.
Every finite divisible pseudo-BCK-algebra is a BCK-algebra.

(Jipsen & Montagna: divisible integral residuated lattices = integral GBL-algebras)

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Poset products

(Jipsen & Montagna: integral GBL-algebras)

Let $(I;\leq)$ be a poset and let $\{A_i \mid i \in I\}$ be a family of MV-chains (with the same 0 and 1). Let $A = \bigotimes_{i\in I} A_i$ be the subset of $\prod_{i\in I} A_i$ defined as follows:

$$a \in A \quad \text{iff} \quad \text{whenever } a(i) \neq 1, \text{then } a(j) = 0 \text{ for all } j < i.$$

If we put

$$(a \to b)(i) = \begin{cases} a(i) \to b(i) & \text{if } a(j) \leq b(j) \text{ for all } j < i, \\ 0 & \text{otherwise,} \end{cases}$$

then A becomes a bounded BCK-algebra.

Theorem

Every finite divisible BCK-algebra embeds into a poset product of linearly ordered MV-algebras.

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Let Γ be the set of all completely meet-irreducible relative congruences of A, ordered by inclusion. Then for every $\gamma \in \Gamma$, A/γ is subdirectly irreducible, so $A/\gamma = B_\gamma \oplus C_\gamma$ where C_γ is a finite MV-chain. Then A embeds into the poset product $\bigotimes_{\gamma \in \Gamma} C_\gamma$.

THANK YOU!