Free algebras via a functor on partial algebras

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Radboud Universiteit Nijmegen

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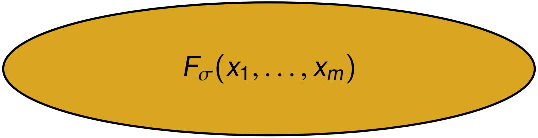

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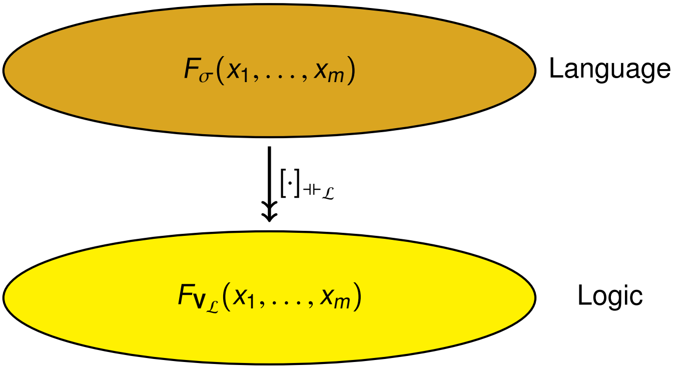

Free algebra as colimit of a chain

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- Regard $F_{\mathbf{V}_{\mathcal{L}}}(x_1, \ldots, x_n)$ as colimit of a chain of finite algebras in the reduced signature, and add the additional operation(s) step-by-step:

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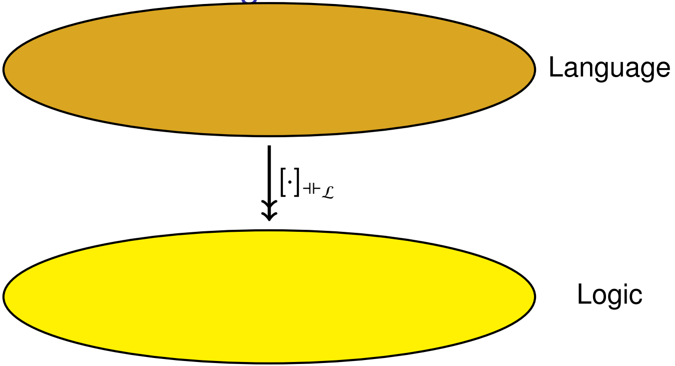

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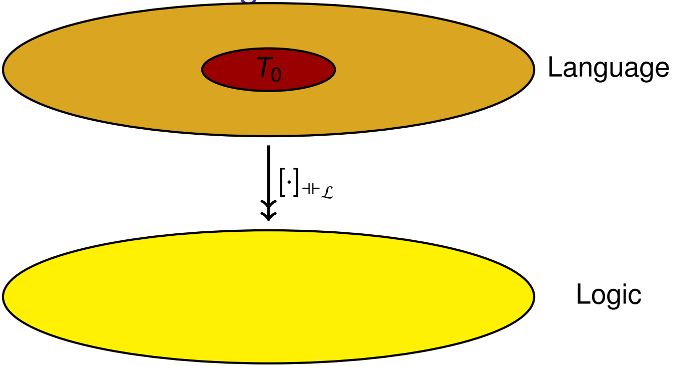

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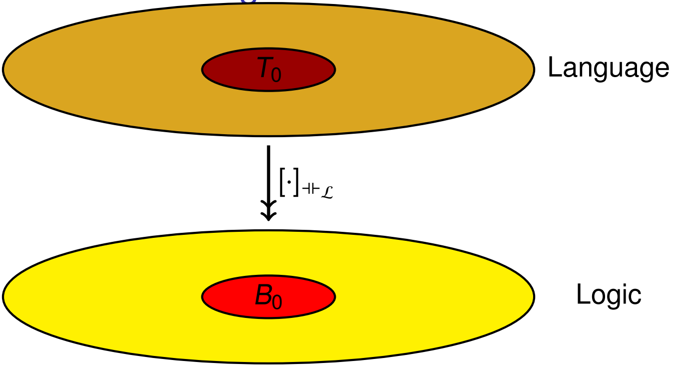

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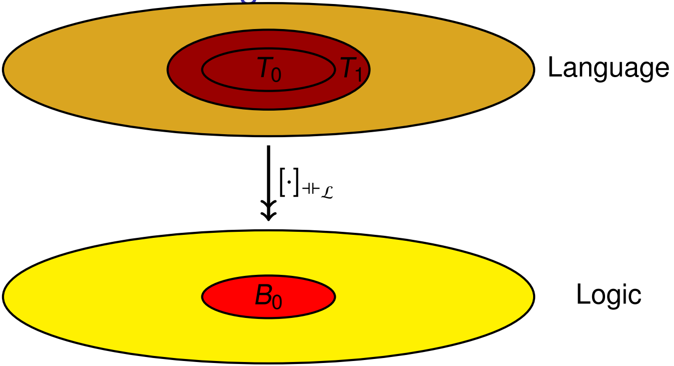

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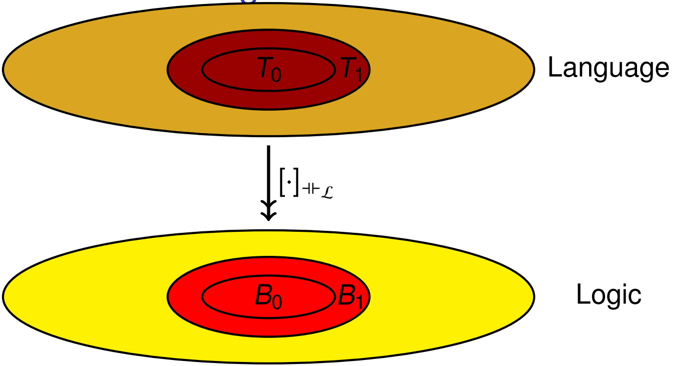

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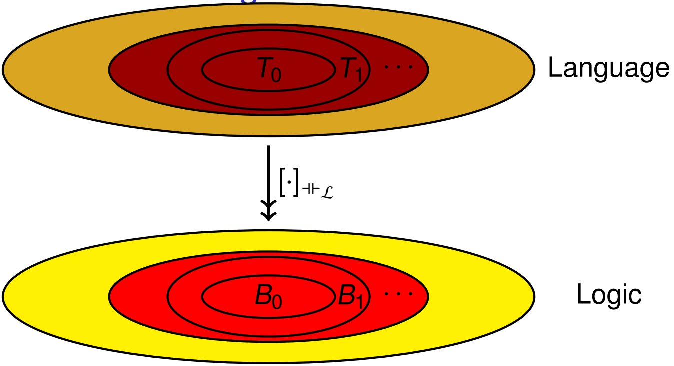

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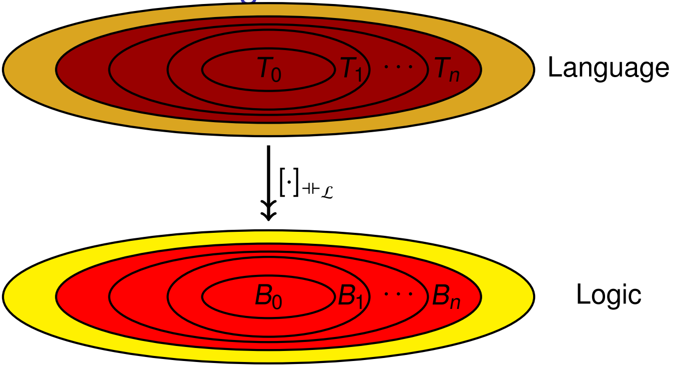

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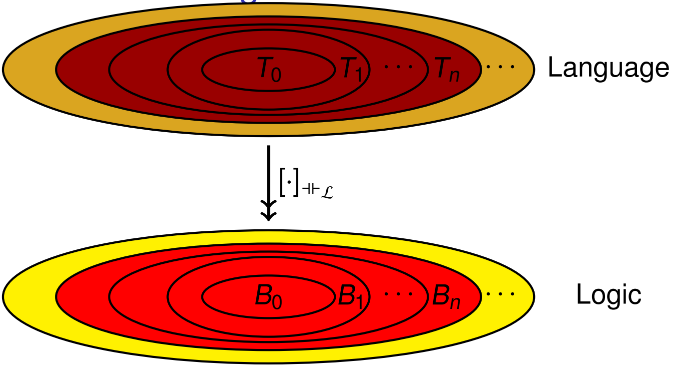

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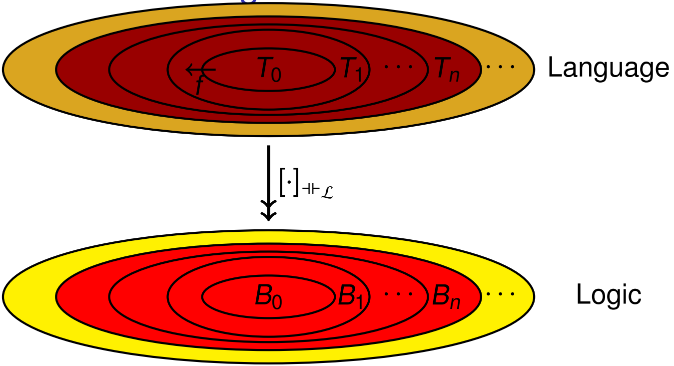

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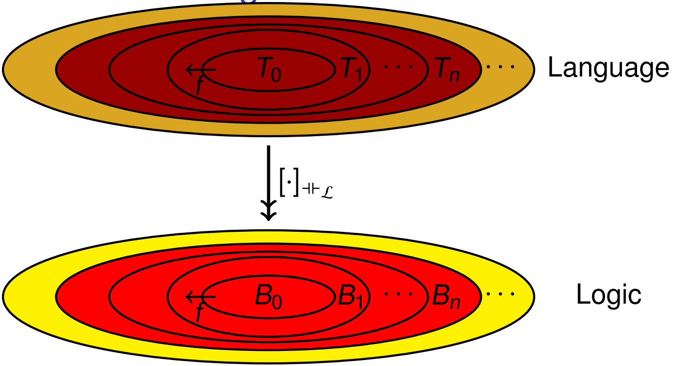

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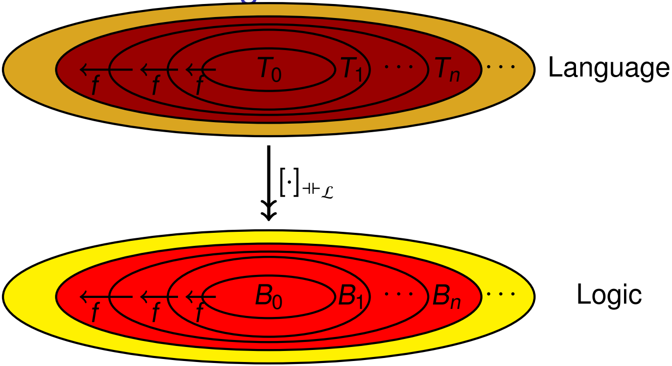

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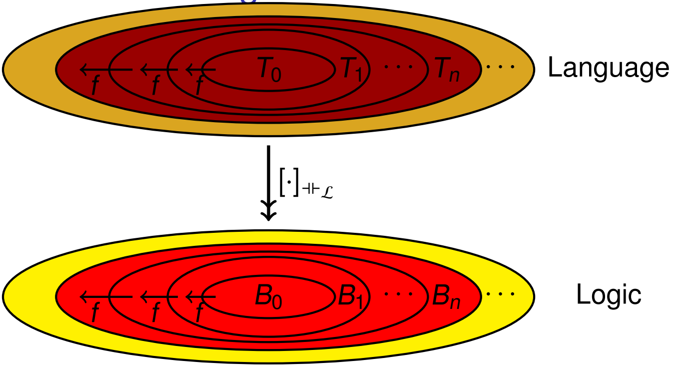

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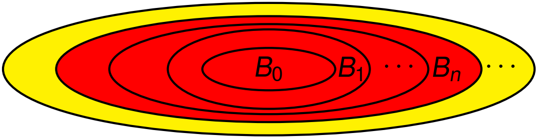
$$F_{\mathbf{V}_{\mathcal{L}}}(x_1, \ldots, x_m) = \text{colim}_{n \geq 0}\, B_n$$

Research Question



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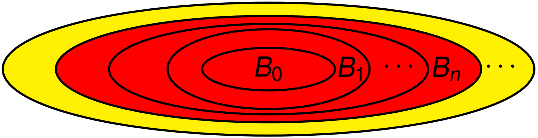

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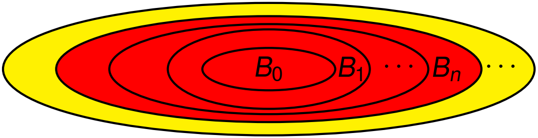

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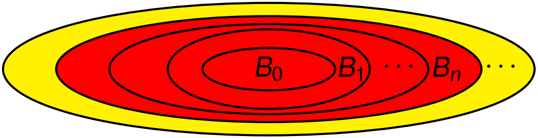

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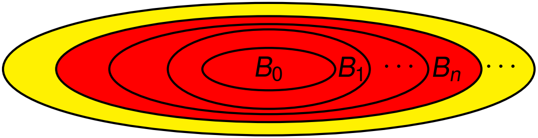

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- Not always, since logics can be undecidable.
- We give general sufficient conditions under which this is possible (known cases follow as particular instances).

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- A homomorphism $h : A \to B$ of partial algebras is a function which preserves all total operations, and preserves the partial operation f whenever defined.
- A homomorphism $h : A \to B$ is image-total if the image of h is contained in the domain of f^B.

Free image-total functor

Definition

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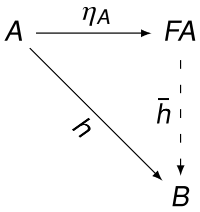
A functor $F : \mathbf{pV} \to \mathbf{pV}$ is free image-total if there is a component-wise image-total natural transformation $\eta : 1_{\mathbf{pV}} \to F$ such that, for all image-total $h : A \to B$, there exists a unique $\bar{h} : FA \to B$ making the following diagram commute:



Free image-total functor

Main theorem

Theorem

Let $\eta : 1 \to F$ be a free image-total functor and $A_0 \in \mathbf{pV}$. Let A_ω be the partial algebra-colimit of the image-total chain $\{\eta_{F^n(A_0)} : F^n(A_0) \to F^{n+1}(A_0)\}_{n \geq 0}$.
If A_ω is in $\mathbf{V}$, then A_ω is the free total $\mathbf{V}$-algebra over A_0.

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Category-theoretic arguments. □

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Now, to apply this theorem:

- We construct a free image-total functor for any set of quasi-equations,
- We give sufficient conditions under which $A_\omega \in \mathbf{V}$.

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Construction

- Let $\mathcal{E}$ be a set of quasi-equations (of rank at most 1) axiomatizing the variety $\mathbf{V}$.

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- η_A is the composite

$$A \rightarrowtail A + F_{\mathbf{V}^-}(\mathbf{f}A) \twoheadrightarrow F_{\mathcal{E}}(A).$$

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$F_{\mathcal{E}}$ is a free image-total functor with universal arrow η. Furthermore, if $A_0 \in \mathbf{pV}$ is such that each component $\eta_{F^n_{\mathcal{E}}(A_0)} : F^n_{\mathcal{E}}(A_0) \to F^{n+1}_{\mathcal{E}}(A_0)$ is an embedding, then $A_\omega \in \mathbf{V}$.

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Corollary

If $A_0 \in \mathbf{pV}$ is such that each component $\eta_{F^n_\mathcal{E}(A_0)} : F^n_\mathcal{E}(A_0) \to F^{n+1}_\mathcal{E}(A_0)$ is an embedding, then A_ω is the free total $\mathbf{V}$-algebra over A_0.

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- (Finite) Duality theory:
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  - Partial **KB** algebras $\leftrightarrow$ Sets with an equivalence relation $\sim$ and a quasi-symmetric relation R satisfying $R \circ \sim \subseteq R$

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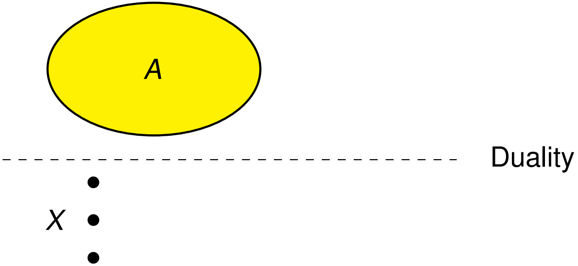

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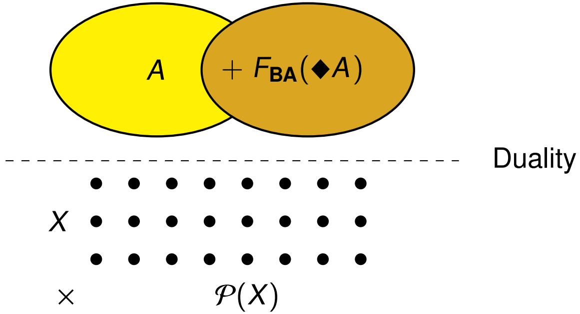

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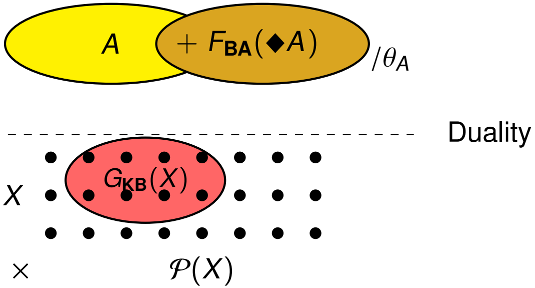

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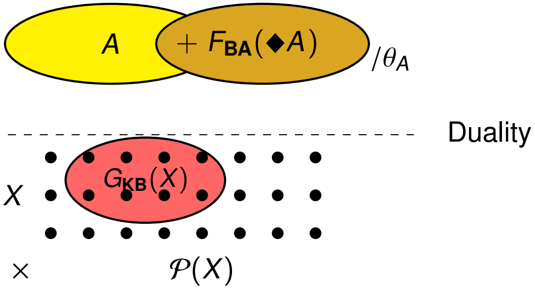

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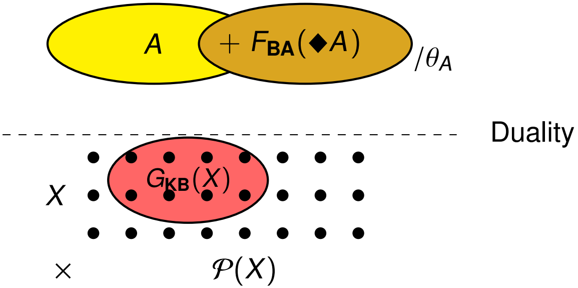
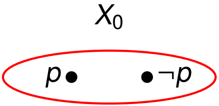


- Using correspondence theory, one can explicitly calculate a first-order definition of the points in $G_{\mathbf{KB}}(X, R, \sim)$
- These points are normal forms for **KB**.

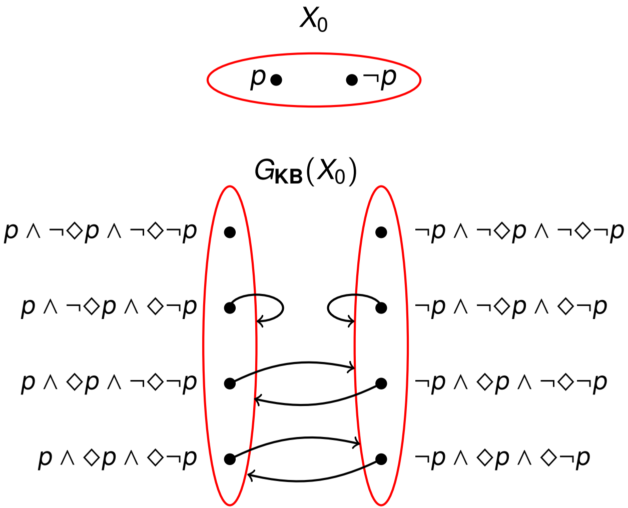
The chain for **KB**

First steps



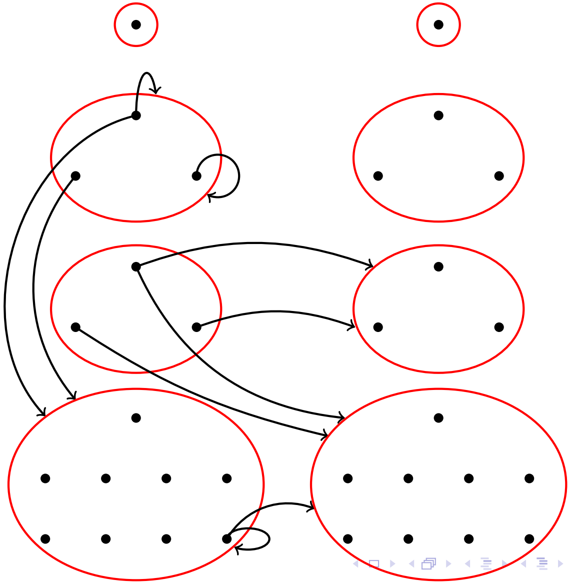
The chain for **KB**

First steps



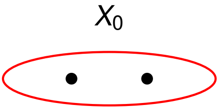
The chain for **KB**

(part of) $G^2_{\mathbf{KB}}(X_0)$



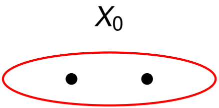
The chain for **S4**

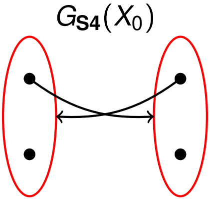
First steps



The chain for **S4**

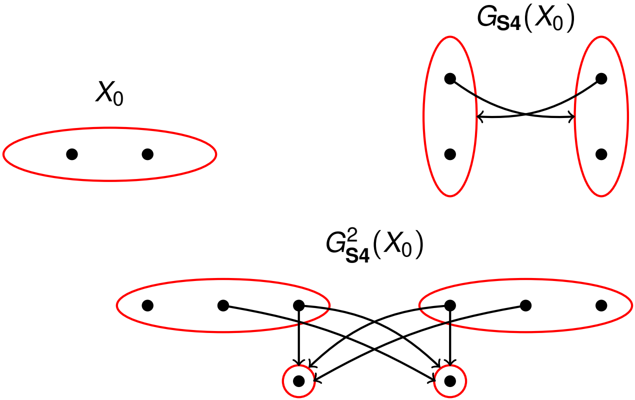
First steps





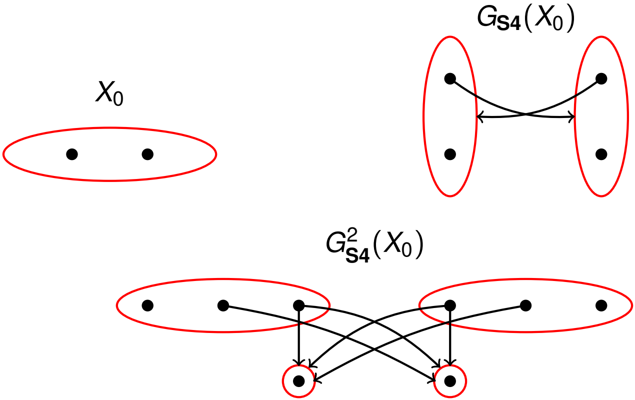
The chain for **S4**

First steps



The chain for **S4**

First steps



(...)

Free algebras via a functor on partial algebras

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