On varieties generated by standard BL-algebras

Zuzana Haniková

Institute of Computer Science, AS CR
182 07 Prague, Czech Republic
zuzana@cs.cas.cz

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Mission

Theorem

If $\mathbb{V}$ is a subvariety of $\mathbb{BL}$ generated by a set of standard BL-algebras, then $\mathbb{V}$ is also generated by a finite set of standard BL-algebras.

As a consequence:

- each such variety $\mathbb{V}$ is finitely axiomatizable (because of the finite-class case [Esteva, Godo, Montagna 04, Galatos 04])
- the equational theory of $\mathbb{V}$ is coNP-complete (because of the finite-class case [Baaz, Hájek, Montagna, Veith 02, Hanikova 02])

BL-algebras form the equivalent algebraic semantics of the Basic Logic; both introduced in [Hájek 98]

Definition

A BL-algebra is an algebra $\mathbf{A} = \langle A, *, \to, \wedge, \vee, 0, 1\rangle$ such that:

1. $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice
2. $\langle A, *, 1\rangle$ is a commutative monoid
3. for all $x, y, z \in A$, $z \leq (x \to y)$ iff $x * z \leq y$
4. for all $x, y \in A$, $x \wedge y = x * (x \to y)$
5. for all $x, y \in A$, $(x \to y) \vee (y \to x) = 1$

BL-algebras form a variety $\mathbb{BL}$.
Each BL-algebra is a subdirect product of BL-chains, so the variety $\mathbb{BL}$ is generated by BL-chains.

Standard BL-algebras

A BL-algebra is standard iff its domain is the real unit interval $[0,1]$, and its lattice order is the usual order of reals.

Let **A** be a standard BL-algebra. Then its monoidal operation $*$ is continuous w. r. t. the order topology, hence a continuous t-norm. Moreover, we have

$$x \rightarrow^{\mathbf{A}} y = \max\{z \mid x *^{\mathbf{A}} z \leq y\}.$$

Thus **A** is uniquely determined by $*^{\mathbf{A}}$; often, the notation is $[0,1]_*$.

Standard BL-algebras generate the variety $\mathbb{BL}$
[Hájek 98; Cignoli, Esteva, Godo and Torrens 00]

Examples of standard BL-algebras

t-norm	st. BL-alg.	$x * y$	$x \rightarrow y$ for $x > y$
Łukasiewicz	$[0,1]_{Ł}$	$\max(0, x+y-1)$	$1 - x + y$
Gödel	$[0,1]_{G}$	$\min(x,y)$	y
product	$[0,1]_{\Pi}$	$x \cdot y$	y/x

$x \in [0,1]$ is *idempotent* w. r. t. $*$ iff $x * x = x$.
For each standard BL-algebra $[0,1]_*$, its idempotent elements form a closed subset of $[0,1]$.

The complement of this set is a union of countably many disjoint open intervals; on the closure of each of these, $*$ is isomorphic to

- the Łukasiewicz t-norm $*_{Ł}$ on $[0,1]$, or
- the product t-norm $*_{\Pi}$ on $[0,1]$.

[Mostert, Shields 57]

Ordinal sum of BL-chains

Definition

Let I be a linearly ordered set with minimum i_0 and let $\{\mathbf{A}_i\}_{i \in I}$ be a family of BL-chains s. t. $\mathbf{A}_i \cap \mathbf{A}_j = 1^{\mathbf{A}_i} = 1^{\mathbf{A}_j}$ for $i \neq j \in I$.
The ordinal sum $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ of $\{\mathbf{A}_i\}_{i \in I}$ is as follows:

1. the domain is $A = \bigcup_{i \in I} A_i$
2. $0^{\mathbf{A}} = 0^{\mathbf{A}_{i_0}}$ and $1^{\mathbf{A}} = 1^{\mathbf{A}_{i_0}}$
3. the ordering is $x \leq^{\mathbf{A}} y$ iff $\begin{cases} x, y \in A_i \text{ and } x \leq^{\mathbf{A}_i} y \\ x \in A_i \setminus \{1^{\mathbf{A}_i}\} \text{ and } y \in A_j \text{ and } i < j \end{cases}$
4. $x *^{\mathbf{A}} y = \begin{cases} x *^{\mathbf{A}_i} y \text{ if } x, y \in A_i \\ \min^{\mathbf{A}}(x, y) \text{ otherwise} \end{cases}$
5. $x \rightarrow^{\mathbf{A}} y = \begin{cases} 1^{\mathbf{A}} & \text{if } x \leq^{\mathbf{A}} y \\ x \rightarrow_i y & \text{if } x, y \in A_i \\ y & \text{otherwise} \end{cases}$

Standard BL-algebras as ordinal sums

Theorem

Each standard BL-algebra is an ordinal sum of a family of BL-algebras, each of whom is an isomorphic copy of either $[0,1]_{\text{Ł}}$ or $[0,1]_{\text{G}}$ or $[0,1]_{\Pi}$ or 2 (the two-element Boolean algebra).

The elements of the sum are called components; we have Ł-components (isomorphic to $[0,1]_{\text{Ł}}$), G-components (isomorphic to $[0,1]_{\text{G}}$), Π-components (isomorphic to $[0,1]_{\Pi}$), and 2-components (isomorphic to $\{0,1\}_{\text{Boole}}$).

Gödel components are those maximal w. r. t. inclusion.

For a standard BL-algebra one can write $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$, where the ordered set I, as well as the isomorphism type of each of the $\mathbf{A}_i$'s, are uniquely determined by $\mathbf{A}$.

Each class of isomorphism of standard BL-algebras is given by a corresponding ordinal sum of symbols out of Ł, G, Π and 2.

Remarks on partial embeddability

For each $c \in (0,1)$, the BL-algebra $[0,1]_{Ł}$ is isomorphic to the *cut product algebra* $([c,1], *_c, \to_c, c, 1)$ where

$$\begin{aligned} x *_c y &= \max(c, x *_\Pi y) \\ x \to_c y &= x \to_\Pi y \end{aligned}$$

The element c is called the *cut*.

As a consequence, $[0,1]_\Pi$ is partially embeddable into $[0,1]_{Ł} \oplus [0,1]_{Ł}$.

Moreover, any standard BL-algebra without Ł-components is partially embeddable into any infinite sum of Π-components.

Standard BL-algebras generating $\mathbb{BL}$

The class of all standard BL-algebras generates the variety $\mathbb{BL}$.

The same is true about particular examples of standard BL-algebras.

Theorem

A standard BL-algebra $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ generates the variety $\mathbb{BL}$ iff $\mathbf{A}_{i_0}$ is an Ł-component and for infinitely many $i \in I$, $\mathbf{A}_i$ is an Ł-component.

This is a consequence of a theorem of [Aglianò, Montagna 03], which gives a characterization of BL-generic chains.

Standard BL-algebras generating $\mathbb{SBL}$

The variety $\mathbb{SBL}$ is a subvariety of $\mathbb{BL}$ given by the identity

$$(x \wedge (x \to 0)) \to 0 = 1$$

A standard BL-algebra is an SBL-algebra iff the first component in its ordinal sum is *not* an Ł-component.

Theorem

A standard SBL-algebra $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ *generates the variety* $\mathbb{SBL}$ *iff* $\mathbf{A}_{i_0}$ *is not an Ł-component and for infinitely many* $i \in I$, $i \neq i_0$, $\mathbf{A}_i$ *is an Ł-component.*

Canonical BL-algebras

[Esteva, Godo, Montagna 04]

Definition

A standard BL-algebra is *canonical* iff its sum is either $\omega Ł$ or $\Pi \oplus \omega Ł$, or a finite sum of expressions from among Ł, G, Π and $\omega\Pi$, where no G is preceded or followed by another G, and no $\omega\Pi$ is preceded or followed by a G, a Π or another $\omega\Pi$.

Theorem

For each standard BL-algebra, there is a canonical BL-algebra generating the same variety.

In particular, there are only countably many subvarieties of $\mathbb{BL}$ that are generated a single standard BL-algebra.

Canonical BL-algebras and subvarieties of $\mathbb{BL}$

Two canonical BL-algebras are isomorphic iff they are given by the same finite ordinal sum of symbols.

Non-isomorphic canonical BL-algebras generate distinct subvarieties of $\mathbb{BL}$.

Hence, there is a 1-1 correspondence between

subvarieties of $\mathbb{BL}$ given by a single standard BL-algebra
and
ωŁ, $\Pi \oplus \omega$Ł, and finite sums out of the symbols Ł, G, Π, $\omega\Pi$.

The above words are called *canonical BL-expressions*.

The problem

Given a class $\mathbb{C}$ of standard BL-algebras, find
a finite class $\mathbb{C}'$ of standard BL-algebras s. t. $\mathbf{Var}(\mathbb{C}) = \mathbf{Var}(\mathbb{C}')$.

Without loss of generality, we may assume:

1. $\mathbb{C}$ is a class of canonical BL-algebras
2. the isomorphism classes in $\mathbb{C}$ are represented by canonical BL-expressions

Therefore, we may assume $\mathbb{C}$ (and $\mathbb{C}'$) is a class of canonical BL-expressions.

We use the notation $\mathbf{Var}(\mathbb{C})$, ... in the obvious sense.

A plan for the proof

Definition

For canonical BL-expressions $\mathbf{A}$, $\mathbf{B}$, let $\mathbf{A} \preceq \mathbf{B}$ iff $\mathbf{Var}(\mathbf{A}) \subseteq \mathbf{Var}(\mathbf{B})$.

$\preceq$ is a partial order on canonical BL-expressions.

For any two canonical BL-expressions, we have

$$\mathbf{A} \preceq \mathbf{B} \text{ iff } \mathbf{Var}(\mathbf{A}) \subseteq \mathbf{Var}(\mathbf{B}) \text{ iff } \mathbf{Var}(\{\mathbf{A}, \mathbf{B}\}) = \mathbf{Var}(\mathbf{B}).$$

Theorem

Let $\mathbb{K}$, $\mathbb{L}$ be two non-empty classes standard BL-algebras. Then the following are equivalent:

- $\mathbf{Var}(\mathbb{K}) \subseteq \mathbf{Var}(\mathbb{L})$;
- $\mathbb{K}$ *is partially embeddable to* $\mathbb{L}$.

[Esteva, Godo, Montagna 04]

A partition on canonical BL-expressions

Let $\mathbb{L}$ denote the class of canonical BL-expressions,
$\mathbb{L}_{\text{Ł}}$ the elements of $\mathbb{L}$ starting with an Ł-component and
$\mathbb{L}_{\overline{\text{Ł}}}$ the elements of $\mathbb{L}$ not starting with an Ł-component.

For each $i \in (\mathbb{N} \cup \{\omega\}) \setminus \{0\}$, denote
$\mathbb{L}^{i}_{\text{Ł}}$ the class of canonical BL-expressions starting with an Ł-component and with exactly i Ł-components altogether.

For each $i \in \mathbb{N} \cup \{\omega\}$, denote
$\mathbb{L}^{i}_{\overline{\text{Ł}}}$ the class of canonical BL-expressions *not* starting with an Ł-component and with exactly i Ł-components

A partition on $\mathbb{C}$

We decompose the given class $\mathbb{C}$ of canonical BL-expressions along these lines:

$\mathbb{C}^i_{Ł} = \mathbb{C} \cap \mathbb{L}^i_{Ł}$ and $\mathbb{C}_{Ł} = \bigcup_{i \in (\mathbb{N} \cup \{\omega\}) \setminus \{0\}} \mathbb{C}^i_{Ł}$
(all algebras in $\mathbb{C}$ starting with an Ł-component).
Analogously for $\mathbb{C}_{\overline{Ł}}$.

The classes $\mathbb{C}_{Ł}$ and $\mathbb{C}_{\overline{Ł}}$ will be addressed separately.
Clearly, $\mathbb{C}_{Ł}$ generates $\mathbb{BL}$ or its subvariety and
$\mathbb{C}_{\overline{Ł}}$ generates $\mathbb{SBL}$ or its subvariety.

Substituting the generators of a variety

Lemma

Let $\mathbb{K} = \bigcup_{i \in I} \mathbb{K}_i$, $\mathbb{L} = \bigcup_{i \in I} \mathbb{L}_i$ *be classes of algebras in the same language. Assume* $\mathbf{Var}(\mathbb{K}_i) = \mathbf{Var}(\mathbb{L}_i)$ *for each* $i \in I$. *Then* $\mathbf{Var}(\mathbb{K}) = \mathbf{Var}(\mathbb{L})$.

Proof:
$\mathbf{HSP}(\mathbb{K}) = \mathbf{HSP}(\bigcup_{i \in I} \mathbb{K}_i) = \mathbf{HSP}(\bigcup_{i \in I} \mathbf{HSP}(\mathbb{K}_i)) ==$
$\mathbf{HSP}(\bigcup_{i \in I} \mathbf{HSP}(\mathbb{L}_i)) = \mathbf{HSP}(\bigcup_{i \in I} \mathbb{L}_i) = \mathbf{HSP}(\mathbb{L})$.

Classes generating $\mathbb{BL}$

Lemma

Whenever $\{k \in \mathbb{N} \mid \mathbb{C}_{Ł}^{k}$ *is nonempty*$\}$ *is infinite or* $\mathbb{C}_{Ł}^{\omega}$ *is nonempty, we have* $\mathbf{Var}(\mathbb{C}_{Ł}) = \mathbb{BL}$.

Then we have $\mathbf{Var}(\mathbb{C}_{Ł}) = \mathbf{Var}(\omega Ł) = \mathbb{BL}$.

If the above conditions are not satisfied, then there is a $k_0 \in \mathbb{N}$ such that each expression in $\mathbb{C}_{Ł}$ has at most k_0 Ł-components.
Then $\mathbb{C}_{Ł}$ generates a proper subvariety of $\mathbb{BL}$.

Lemma

Whenever $\{k \in \mathbb{N} \mid \mathbb{C}^{k}_{\overline{Ł}}$ *is nonempty*$\}$ *is infinite or* $\mathbb{C}^{\omega}_{\overline{Ł}}$ *is nonempty, we have* $\mathbf{Var}(\mathbb{C}_{\overline{Ł}}) = \mathbb{SBL}$.

Then we have $\mathbf{Var}(\mathbb{C}_{\overline{Ł}}) = \mathbf{Var}(\Pi \oplus \omega Ł) = \mathbb{SBL}$.

If the above conditions are not satisfied, then there is a $k_0 \in \mathbb{N}$ such that each expression in $\mathbb{C}_{\overline{Ł}}$ has at most k_0 Ł-components.
Then $\mathbb{C}_{Ł}$ generates a proper subvariety of $\mathbb{SBL}$.

Bounded number of Ł-components

Consider the classes $\mathbb{C}_{\text{Ł}}$ and $\mathbb{C}_{\overline{\text{Ł}}}$ separately. The case when the number of Ł-components in elements of each of the classes is unbounded has been addressed.

It remains to find a method of solution for the case when there is an upper bound $k_0 \in \mathbb{N}$ on the number of Ł-components of each element of $\mathbb{C}_{\text{Ł}}$ ($\mathbb{C}_{\overline{\text{Ł}}}$).

Recall the partition: for $1 \leq k \leq k_0$, we have

$$\mathbb{C}^k_{\text{Ł}} = \{\mathbf{A} \in \mathbb{C}_{\text{Ł}} \mid \mathbf{A} \text{ has exactly } k \text{ Ł-components}\}$$

and analogously for $\mathbb{C}_{\overline{\text{Ł}}}$ and the partition $\mathbb{C}^k_{\overline{\text{Ł}}}$, $k \leq k_0$.

Ordinal sums in $\mathbb{L}^k$

The class $\mathbb{L}^k$ consist of all canonical BL-expressions with exactly k Ł-components. The class $\mathbb{L}^0$ has no Ł-components.

For a canonical BL-expression $\mathbf{A} \in \mathbb{L}^k$, we may write

$$\mathbf{A} = \mathbf{A}_0 \oplus \text{Ł} \oplus \mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_{k-1} \oplus \text{Ł} \oplus \mathbf{A}_k$$

where each $\mathbf{A}_j$, $j \leq k$ is either the empty sum $\emptyset$, or a finite ordinal sum of G's and Π's, or $\infty\Pi$.
In particular, each expression $\mathbf{A}_j$ is an element of $\mathbb{L}^0$.
(We consider $\emptyset$ as an element of $\mathbb{L}^0$.)

$\preceq$ on $\mathbb{L}^k_{\text{Ł}}$ and $\mathbb{L}^k_{\overline{\text{Ł}}}$

Fix a $k \in \mathbb{N}$.

Theorem

Let $\mathbf{A}$, $\mathbf{B}$ *be two canonical BL-expressions in* $\mathbb{L}^k_{\overline{\text{Ł}}}$, *where*
$\mathbf{A} = \mathbf{A}_0 \oplus \text{Ł} \oplus \mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_{k-1} \oplus \text{Ł} \oplus \mathbf{A}_k$ *and*
$\mathbf{B} = \mathbf{B}_0 \oplus \text{Ł} \oplus \mathbf{B}_1 \oplus \cdots \oplus \mathbf{B}_{k-1} \oplus \text{Ł} \oplus \mathbf{B}_k$.
Then $\mathbf{A} \preceq \mathbf{B}$ *iff for each* $j \leq k$, $\mathbf{A}_j \preceq \mathbf{B}_j$.

In other words, $\preceq$ on $\mathbb{L}^k_{\overline{\text{Ł}}}$ is the product order of $k+1$ factors (each being $\mathbb{L}^0$ ordered by $\preceq$).

Analogously for $\mathbb{L}^k_{\text{Ł}}$.

$\preceq$ on $\mathbb{L}^0$

The elements of $\mathbb{L}^0$ are the following expressions: the empty sum $\emptyset$, finite ordinal sums of G- and Π-components, and the expression $\omega\Pi$.
We define $\emptyset \preceq \mathbf{A}$ for any BL-expression $\mathbf{A}$, and $\emptyset \preceq \emptyset$.

Properties of $\preceq$ on $\mathbb{L}^0$:

- $\infty\Pi$ is the top element of $\mathbb{L}^0$ and $\emptyset$ is the bottom element;
- if $\mathbf{A}, \mathbf{B} \in \mathbb{L}^0$ are finite sums of G's and Π's, then $\mathbf{A} \preceq \mathbf{B}$ iff $\mathbf{A}$ is a subsum of $\mathbf{B}$.

Theorem

$\preceq$ on $\mathbb{L}^0$ is a w. q. o.

$\preceq$ on $\mathbb{L}_{Ł}^{k}$ and $\mathbb{L}_{\overline{Ł}}^{k}$ revisited

It is well known that if $(L_1, \leq_1)$, $(L_2, \leq_2)$ are w.q.o.'s, then so is their product $(L_1, \leq_1) \times (L_2, \leq_2)$.

Theorem

$\preceq$ is a w. q. o. on $\mathbb{L}_{Ł}^{k}$ and on $\mathbb{L}_{\overline{Ł}}^{k}$.

In particular, there are no infinite $\preceq$-antichains.

Theorem

Let $\{\mathbf{A}_i\}_{i\in I}$ *be a* $\preceq$*-chain in* $\mathbb{L}^k_{\text{Ł}}$*. Then there is a* $\sup(\{\mathbf{A}_i\}_{i\in I})$ *in* $\mathbb{L}^k_{\text{Ł}}$*, and* $\mathbf{Var}(\{\mathbf{A}_i\}_{i\in I}) = \mathbf{Var}(\sup(\{\mathbf{A}_i\}_{i\in I}))$*. Analogously for* $\mathbb{L}^k_{\text{Ł}}$*.*

Let $\{\mathbf{A}_i\}_{i\in I}$ be a $\preceq$-chain in $\mathbb{C}$. We say that $\{\mathbf{A}_i\}_{i\in I}$ is *maximal* in $\mathbb{C}$ iff no element of $\mathbb{C}$ can be added on top. Clearly, each $\mathbf{A} \in \mathbb{C}$ belongs to some maximal chain.

$\preceq$-chains

Lemma

Let $\mathbb{C} \subseteq \mathbb{L}^k_{\text{Ł}}$. Let $\{\mathbf{A}_i\}_{i\in I}$, $\{\mathbf{B}_{i'}\}_{i'\in I'}$ be two maximal $\preceq$-chains in $\mathbb{K}$. If $\{\mathbf{B}_{i'}\}_{i'\in I'}$ has a top element in $\mathbb{K}$, then $\sup(\{\mathbf{A}_i\}_{i\in I}) \not\prec \sup(\{\mathbf{B}_{i'}\}_{i'\in I'})$.

Corollary

Let $\mathbb{C} \subseteq \mathbb{L}^k_{\text{Ł}}$. Let $\{\mathbf{A}_i\}_{i\in I}$, $\{\mathbf{B}_{i'}\}_{i'\in I'}$ be two maximal $\preceq$-chains in $\mathbb{K}$. If $\sup(\{\mathbf{A}_i\}_{i\in I}) \prec \sup(\{\mathbf{B}_{i'}\}_{i'\in I'})$, then $\{\mathbf{B}_{i'}\}_{i'\in I'}$ has no top element in $\mathbb{K}$, and there is a $j \in \{1, \ldots, k\}$ such that for each $i' \in I'$, $(\mathbf{B}_{i'})_j$ is a finite sum, whereas $(\sup(\{\mathbf{B}_{i'}\}_{i'\in I'}))_j = \omega\Pi$.

Analogously for $\mathbb{C} \subseteq \mathbb{L}^k_{\text{Ł}}$).

Iterating the suprema construction

Assume $\mathbb{C}^k_{\text{Ł}}$ is a given class of canonical BL-expressions in $\mathbb{L}^k_{\text{Ł}}$. Let us denote $\mathbb{C}_0 = \mathbb{C}^k_{\text{Ł}}$.
For $n \in \mathbb{N}$, define

$$\mathbb{C}_{n+1} = \{\mathbf{A} | \mathbf{A} = \sup(\{\mathbf{A}_i\}_{i \in I}) \text{ for some maximal chain } \{\mathbf{A}_i\}_{i \in I} \text{ in } \mathbb{C}_n\}$$

Theorem

- $\mathbf{Var}(\mathbb{C}_n) = \mathbf{Var}(\mathbb{C}_{n+1})$ *for each* $n \in \mathbb{N}$
- *There is an* $n \leq k + 2$ *such that*
  1. $\mathbb{C}_n = \mathbb{C}_{n+1}$
  2. $\mathbb{C}_n$ *is finite*

Conclusion

Given a class $\mathbb{C}$ of canonical BL-expressions, we can find a finite class $\mathbb{C}'$ of canonical BL-expressions such that $\mathbf{Var}(\mathbb{C}) = \mathbf{Var}(\mathbb{C}')$.

Therefore, the logic of any class of standard BL-algebras is

1. axiomatic extension of BL
2. finitely axiomatizable
3. coNP-complete