# Modal and Intuitionistic Natural Dualities via the Concept of Structure Dualizability

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TACL, Marseille, July 26-30, 2011

# Outline

# Nishida philosophy and Stone duality

Kitaro Nishida (1870-1945) was a philosopher of Kyoto school.

- Nishida considered experience as having a person, rather than a person having experience.
  - The notion of a person is derived from pure experiences and is just a way to organize them. Obvious analogy with phenomenology.
- A person is a bundle of experiences.

Now, Stone duality comes into the picture. An implication of it:

- An object is a bundle of properties.
  - Point-free top.: spaces are bundles of predicates on them.
  - Abramsky: programs are bundles of observable properties.

Stone duality could be placed in a broader context of duality b/w subjects and objects (Piet Hut) or epistemology and ontology.

# Finitary and infinitary Stone dualities

Finitary Stone dualitlies:

- involve finitary operations and compact specs.
  - where universal algebra seems useful.
- includes Stone duality for Boolean algebras.
- often needs a form of AC (ontologically demanding).

Infinitary Stone dualities:

- involve infinitary operations and non-compact specs.
  - where categorical algebra seems useful.
- includes Isbell-Papert duality b/w frames and topo. spaces.
- (sometimes) avoids AC (epistemologically more certain).

In this talk we focus on the former, esp. the theory of natural dualities as a general theory of finitary Stone dualities.

# The theory of natural dualities (TND)

TND is a univ.-alg. theory of dualities and discusses:

- when a duality holds for $\mathbb{ISP}(M)$ for a finite algebra $M$.
    - $M$ with the discrete top. works as a schizophrenic object. We later consider other topologies in rel. to Heyt. alg.
- Reference: “Natural dualities for the working algebraist" (Davey and Clark, CUP).

TND encompasses:

- Stone duality for the class of Boolean algebras, which is $\mathbb{ISP}(\mathbf{2})$ where **2** is the two-element Boolean algebra.
- Priestley duality for the class of distributive lattices, which is $\mathbb{ISP}(\mathbf{2})$ where **2** is the two-element distributive lattice.
- Cignoli duality for the class of $\mathrm{MV}_n$ algebras, which is $\mathbb{ISP}(\mathbf{n})$ where **n** is $\{0, 1/(n-1), ..., 1\}$ (as an MV-algebra).

# Category theory vs. universal algebra

Some categorical general theories of dualities are useful for dual adjunctions, but not for dual equivalences:

- “Concrete dualities" (Porst and Tholen, 1991).
  - “Enriched logical connections" (Kurz and Velebil, preprint).

Porst and Tholen says: (given a dual adjunction)

- the main task for establishing a duality in a concrete situation is now to identify Fix($\epsilon$) and Fix($\eta$). This can be a very hard problem, and this is where categorical guidance comes to an end (p.102).

TND gives us dual equivalences, though assumptions are stronger (such ass. can be described well in univ.-alg. terms).

# TND does not encompass JT, KKV, or Esakia dualities

TND does not encompass:

- Jónsson-Tarski duality for the class of modal algebras, which is not $\mathbb{ISP}(M)$ for any finite algebra $M$.
  - This states: ModalAlg $\simeq$ RelBoolSp (or DesGenFrm).
  - Teheux duality for modal $\mathrm{MV}_n$ algs. generalize this.
- Kupke-Kurz-Venema's coalgebraic duality for modal algs.
  - ModalAlg $\simeq$ Coalg($V$). Some contributions by Abramsky.
- Esakia duality for the class of Heyting algebras, which is not $\mathbb{ISP}(M)$ for any finite algebra $M$.

We address the problem with the help of new notions of $\mathbb{ISP}_{\mathbb{M}}$ and $\mathbb{IS}_{\mathbb{R}}\mathbb{P}$. We have $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2}) =$ ModalAlg. $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(\mathbf{2}) =$ HeytAlg.

# $\mathbb{ISP}$ Modalized

$L :=$ a finite algebra with a lattice reduct.

**Definition (modal power w.r.t. Kripke frame)**

For a Kripke frame $(S, R)$, the modal power of $L$ w.r.t. $(S, R)$ is $L^S \in \mathbb{ISP}(L)$ equipped with an operation $\Box_R$ on $L^S$ defined by

$$(\Box_R f)(w) = \bigwedge\{f(w')\ ;\ wRw'\}$$

where $f \in L^S$ and $w \in S$.

Without a lattice reduct: replace $\bigwedge$ with a binary operation on $L$.

**Definition (modal power)**

A modal power of $L$ is defined as
the modal power of $L$ w.r.t. $(S, R)$ for a Kripke frame $(S, R)$.

# Generations of modal algebras

**Definition ($\mathbb{ISP}_{\mathbb{M}}$)**

$\mathbb{ISP}_{\mathbb{M}}(L)$ denotes the class of all isomorphic copies of subalgebras of modal powers of $L$.

The class of modal algs. $\neq \mathbb{ISP}(L)$ for any single alg. $L$.

**Proposition**

- $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2}) =$ *the class of modal algebras.*
- $\mathbb{ISP}_{\mathbb{M}}(\mathbf{n}) =$ *the class of modal* $\mathrm{MV}_n$ *algebras, which were introduced by Hansoul and Teheux in 2006.*

We have a similar fact for algebras of Fitting's mv. modal logic.

# Generations of Heyting algebras

Assume that

- $L$ has a binary operation $*$ ;
- $L$ is $*$-residuated in the following sense:

An ordered algebra $A$ with $*$ is $*$-residuated iff
for any $a, b \in A$, there is $c \in A$ s.t., $\forall x \in A$ $a * x \leq b$ iff $x \leq c$.

**Definition ($\mathbb{IS}_{\mathbb{R}}\mathbb{P}$)**

$\mathbb{IS}_{\mathbb{R}}\mathbb{P}(L)$ denotes the class of all isomorphic copies of $*$-residuated subalgebras of direct powers of $L$.

**Proposition**

*If* $* = \wedge$, *then* $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(\mathbf{2})$ = *the class of all Heyting algebras.*

# Structure dualizability

Our task is to develop duality theory for $\mathbb{ISP}_{\mathbb{M}}(L)$ and $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(L)$.

- We do this via the notion of structure dualizability.

Given an alg. $\Omega$ and a collection $\mathcal{X}$ of top. and rel. on $\Omega$,

- $\Omega$ is dualizab. w.r.t. $\mathcal{X}$ iff $\mathrm{Hom}_{\mathrm{Alg}}(\Omega^n, \Omega) = \mathrm{Hom}_{\mathcal{X}}(\Omega^n, \Omega)$.

The dualizability conditions we assume are:

- For $\mathbb{ISP}_{\mathbb{M}}(L)$, we let $\mathcal{X} = \{M \mid M \text{ is a subalg. of } L\}$.
- For $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(L)$, we let $\mathcal{X} =$ the Alexandrov topology on $L$.
    - Then $L$ may be called an “intuitionistic primal" alg.
      Note: a primal alg. is an alg. dualiz. w.r.t. the discrete top.

Classical $=$ discrete top. Intuitionistic $=$ Alexandrov top.

# Keimel-Werner's semi-primal duality

Our duality for $\mathbb{ISP}_{\mathbb{M}}(L)$ is developed based on Keimel-Werner's semi-primal duality. $\mathrm{BoolSp}$ = the class of all Bool. spaces.

**Definition (Category $\mathbf{BS}_L$)**

An object in $\mathbf{BS}_L$ is $\alpha : \mathrm{SubAlg}(L) \to \mathrm{BoolSp}$ s.t.

- $L_3 = L_1 \cap L_2$ implies $\alpha(L_3) = \alpha(L_1) \cap \alpha(L_2)$.

An arrow $f : \alpha \to \beta$ in $\mathbf{BS}_L$ is a conti. map $f : \alpha(L) \to \beta(L)$ s.t. $\forall M \in \mathrm{SubAlg}(L)\ (x \in \alpha(M) \Rightarrow f(x) \in \beta(M))$.

**Theorem (Keimel and Werner 1974)**

$\mathbf{ISP}(L) \simeq \mathbf{BS}_L^{\mathrm{op}}$. *Stone duality for BA is the case* $L = \mathbf{2}$.

# $\mathbf{RBS}_L \simeq \mathrm{Coalg}(\mathrm{V}_L)$

**Definition (Category $\mathbf{RBS}_L$)**

An object in $\mathbf{RBS}_L$ is $(\alpha, R)$ such that $\alpha$ is in $\mathbf{BS}_L$ and a relation $R$ on $\alpha(L)$ satisfies:

- $R[w]$ is closed and $R^{-1}[X]$ is clopen;
- $\forall M \in \mathrm{SubAlg}(L)\ (w \in \alpha(M)$ implies $R[w] \subset \alpha(M))$.

An arrow $f : (\alpha_1, R_1) \to (\alpha_2, R_2)$ in $\mathbf{RBS}_L$ is an arrow $f : \alpha_1 \to \alpha_2$ in $\mathbf{BS}_L$ with the usual conditions of p-morphisms.

**Definition ($L$-Vietoris functor $\mathrm{V}_L : \mathbf{BS}_L \to \mathbf{BS}_L$)**

Object: $\mathrm{V}_L(\alpha) := (\mathrm{V}(\alpha(L)), \mathrm{V} \circ \alpha)$.
Arrow: $\mathrm{V}_L(f)$ is defined by $\mathrm{V}_L(f)(F) = f(F)$ for $F \in \mathrm{V}(S)$.

# Duality for $\mathbb{ISP}_{\mathbb{M}}(L)$

**Theorem (Duality for $\mathbb{ISP}_{\mathbb{M}}(L)$)**

$$\mathbf{Coalg}(\mathrm{V}_L)^{\mathrm{op}} \simeq \mathbf{ISP_M}(L) \simeq \mathbf{RBS}_L^{\mathrm{op}}.$$

- This is a modal extension of KW semi-primal duality.
- JT and KKV dualities for modal algs. are the cases $L = \mathbf{2}$.

Moreover, this gives new coalgebraic dualities for many-valued modal logics by Teheux-Hansoul and by Fitting.

**Proposition**

*If* $\mathrm{SubAlg}(L)$ *and* $\mathrm{SubAlg}(L')$ *are order isomorphic, then categories* $\mathbf{ISP_M}(L)$ *and* $\mathbf{ISP_M}(L')$ *are equivalent.*

$\mathbf{ISP_M}(L)$ can be described as $\mathbf{Alg}(F)$ for a functor $F$ on $\mathbf{ISP}(L)$.

# Duality for $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(L)$

Assume $L$ is dual. w.r.t. $\mathcal{X}$ := the Alex. top. on $L$.

**Theorem (Duality for $\mathbb{ISP}(L)$)**

$\mathbf{ISP}(L) \simeq \mathbf{CohSp}^{\mathrm{op}}$.

$\mathbf{HeytSp}_L$ := the cat. of coh. sp. $S$ s.t. $\mathrm{Hom}_{\mathbf{CohSp}}(S, L)$ is $*$-resi.

**Corollary (Duality for $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(L)$)**

$\mathbf{IS_RP}(L) \simeq \mathbf{HeytSp}_L^{\mathrm{op}}$

This implies duality for intuitionistic Ł$_n$-valued logic with truth constants (in the sense of Kripke-style semantics).

- $\mathbf{HeytSp}_{Ł_n}$ coincides with the cat. of coh. sp. $S$ s.t. any Bool. combi. of comp. op. is comp. op. (or Esakia sp.).

# Conclusions

We extended natural duality theory so that it subsumes:

- Kupke-Kurz-Venema duality and Jönsson-Tarski duality for modal algebras, and Esakia duality for Heyting algebras.

This was done via the new concepts of $\mathbb{ISP}_{\mathbb{M}}$ and $\mathbb{IS}_{\mathbb{R}}\mathbb{P}$.

- $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2}) = \text{ModalAlg}$. $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(\mathbf{2}) = \text{HeytAlg}$.
  - Recall: $\mathbb{ISP}(M) \neq$ ModalAlg or HeytAlg for any $M$.
- Structure dualizability seems useful to search for dualities.

Our general results imply new coalgebraic and topological dualities for some modal and intuitionistic many-valued logics.

- Maru., Natural duality, modality, and coalgebra, to appear in *J. Pure Appl. Algebra*.