On Reiterman Conversion

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We study several kinds of categories of algebras over a general category $\mathcal{C}$. Reiterman's (by himself unpublished) result, which enables to connect different approaches, is extended to see more connections between various descriptions of algebras.

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Functor algebras

Definition (Algebra for a functor)

Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathcal{C}$ be a functor.

F-algebra: a pair (A, α), where $\alpha : FA \to A$ is a morphism in $\mathcal{C}$

a *morphism* of F-algebras $\phi : (A, \alpha) \to (B, \beta)$ is $\phi : A \to B$ in $\mathcal{C}$ such that the diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{F\phi} & FB \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\phi} & B \end{array}$$

Alg F: the category of F-algebras and F-algebra morphisms.

f-algebraic category: a category concretely isomorphic to **Alg** F for some F.

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Monad algebras

Definition (Algebra for a monad)

Let $\mathcal{C}$ be a category and $\mathbf{M} = (M, \eta, \mu)$ be a monad on $\mathcal{C}$.

M-algebra: an M-algebra (A, α) satisfying Eilenberg-Moore identities:

$$\alpha \circ \mu_A = \alpha \circ M\alpha,$$

$$\alpha \circ \eta_A = \mathrm{id}_A$$

M−alg: the category of **M**-algebras and M-algebra morphisms (Eilenberg-Moore category for **M**).

monadic category: category isomorphic to **M−alg** for some monad **M**

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Polymeric categories [J.P. 2009]

Polymer of an algebra

A *n-polymer* of an F-algebra (A, α) is a morphism $\alpha^{(n)} : F^n(A) \to A$ in $\mathcal{C}$ defined recursively:

$$\alpha^{(0)} = \mathrm{id}_A, \quad \alpha^{(n+1)} = \alpha \circ F\alpha^{(n)}.$$

$\alpha^{(1)}$: $FA \xrightarrow{\alpha} A$,

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polymeric term - natural transformation $\phi : G \to F^m$ for some *domain functor* $G : \mathcal{C} \to \mathcal{C}$ and *arity* $m \in \mathcal{O}rd$

polymeric identity - a pair $(\phi, \psi)_p$ of polymeric terms with the same domain

satisfaction of polymeric identity $(\phi, \psi)_p$ by an F-algebra (A, α) for $\phi : G \to F^m$, $\psi : G \to F^n$:

$$(A, \alpha) \models (\phi, \psi)_p \text{ iff } \alpha^{(m)} \circ \phi_A = \alpha^{(n)} \circ \psi_A$$

i.e.

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L-algebraic categories [J.P. 2010 thesis]

Limits of concrete categories

The metacategory **Con** $\mathcal{C}$ of concrete categories and concrete functors over some base category $\mathcal{C}$ is complete (even some large limits exist).

Definition

A limit of a concrete diagram whose objects are f-algebraic categories is called an *l-algebraic category*

Observation

Every polymeric category is l-algebraic.

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Every polymeric category is l-algebraic.

Example

- A polymeric term in **Alg** τ for a signature τ corresponds to a τ-tree having all branches, which have the variables in leaves, of the same length. Hence commutative groupoids form a polymeric variety while semigroups do not.
- An Eilenberg-Moore category $\mathbf{M}-\mathbf{alg}$ for a monad $\mathbf{M} = (M, \eta, \mu)$ is a polymeric variety of M-algebras induced by $(\eta, \mathrm{id}_{\mathrm{Id}})_p, (\mu, \mathrm{id}_{M^2})_p$. Hence, every monadic category is polymeric, thus I-algebraic.
- Expression of the monoids as a polymeric category. Associativity is not polymeric identity so we need a different functor. We can use the monadicity of category of monoids, namely the *word monad* $\mathbf{M} = (M, \eta, mu)$ gained from generating a free monoid: $\eta_A : A \to A^*$ is inclusion of generators and $\mu_A : (A^*)^* \to A^*$ is the concatenation of components of a chain. Then *Monoids* $= \mathbf{M}-\mathbf{alg}$, which is monadic $\implies$ polymeric.

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Expression of the monoids as an I-algebraic category.

The assignment $(A, \alpha) \mapsto (A, \alpha^{(n)})$ induces a functor $P_n : \textbf{Alg } M \to \textbf{Alg } M^n$ for every finite n. The assignment $\textbf{Alg} -$ is a contravariant functor $\textbf{End}\mathcal{S}et \to \textbf{Con }\mathcal{S}et$ hence there are functors $\textbf{Alg } \eta : \textbf{Alg } M \to \textbf{Alg } \mathrm{Id}$ and $\textbf{Alg } \mu : \textbf{Alg } M \to \textbf{Alg } M^2$. Then category of monoids can be expressed as the concrete limit of

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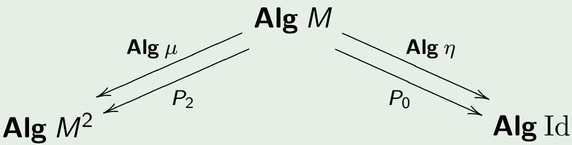

Algebras for a type [Rosický 1977], [Linton 1969]

Let $\mathcal{C}$ be a general category. Let Ω be a class of *operation symbols*.

Definition

type: on $\mathcal{C}$ with the domain Ω - a mapping $t : \Omega \to (\mathrm{Ob}\mathcal{C})^2$, and we write $t(\sigma) = (t_0(\sigma), t_1(\sigma))$ for $\sigma \in \Omega$.

t-algebra: a pair (A, S) made up of a $\mathcal{C}$-object A and a mapping $S : \Omega \to \mathrm{Mor}\mathcal{S}et$ such that $S(\sigma) : \hom(t_0(\sigma), A) \to \hom(t_1(\sigma), A)\ \forall \sigma \in \Omega$

t-algebra morphism $f : (A, S) \to (B, T)$ - a morphism $f : A \to B$ such that $\forall \sigma \in \Omega$ the diagram commutes

$$\begin{array}{ccc} \hom(t_0(\sigma), A) & \xrightarrow[S(\sigma)]{} & \hom(t_1(\sigma), A) \\ \Big\downarrow{\scriptstyle \hom(t_0(\sigma), f)} & & \Big\downarrow{\scriptstyle \hom(t_1(\sigma), f)} \\ \hom(t_0(\sigma), B) & \xrightarrow{T(\sigma)} & \hom(t_1(\sigma), B) \end{array}$$

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$t-\mathbf{alg}$: the (meta)category of t-algebras and their morphisms

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Equational theories [Rosický 1977]

Terms of type t

- σ is the term of arity-pair $t(\sigma)$ for every $\sigma \in \Omega$.
- there is a term $\overline{f}$ of (Y, X) (*morphism-constant*) for every $f : X \to Y$
- composition $p \cdot q$ is a term of arity-pair (Z, X) if q, p are terms of arity-pairs (Z, Y) and (Y, X), respectively.
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$\mathcal{T}(t)$ - the class of all terms of type t

t-equation - a pair of t-terms of the same arity-pair

Evaluation of terms on an algebra

For every t-algebra (A, S) there is an *evaluation of terms* on (A, S) given by term-extension $\overline{S} : \mathcal{T}(t) \to \mathrm{Mor}\mathcal{S}et$ of the mapping S.

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Observation

Every algebraic category over cocomplete base category is l-algebraic.

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Reiterman conversion

We want a connection between functor presentation and type presentation of algebras.

Reiterman theory

Given a functor $F : \mathcal{C} \to \mathcal{C}$, let Ω contain symbols σ_X of arity-pair (X, FX) for every object X in $\mathcal{C}$ and $\mathcal{I}$ be the closure of a class of equations

$$(\overline{Ff} \cdot \sigma_X, \sigma_Y \cdot \overline{f})$$

labeled by all morphisms $f : Y \to X$ in $\mathcal{C}$. We have obtained the *Reiterman theory* $(t, \mathcal{I})$. Given $X \in \mathrm{Ob}\mathcal{C}$, F-algebra (A, α) and a morphism $h : X \to A$ we set

$$R_\alpha(\sigma_X)(h) = \alpha \circ Fh.$$

The assignment $R : \mathbf{Alg}\, F \to (t, \mathcal{I})\mathbf{-alg}$ given by $(A, \alpha) \mapsto (A, R_\alpha)$ is an isomorphism.

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Theorem (Reiterman)

Every f-algebraic category is algebraic.

Reiterman isomorphism can be restricted so that for any polymeric variety $\mathbf{Alg}\,(F,\mathcal{P})$ there is an extension of the Reiterman theory $\mathcal{I}'$ such that

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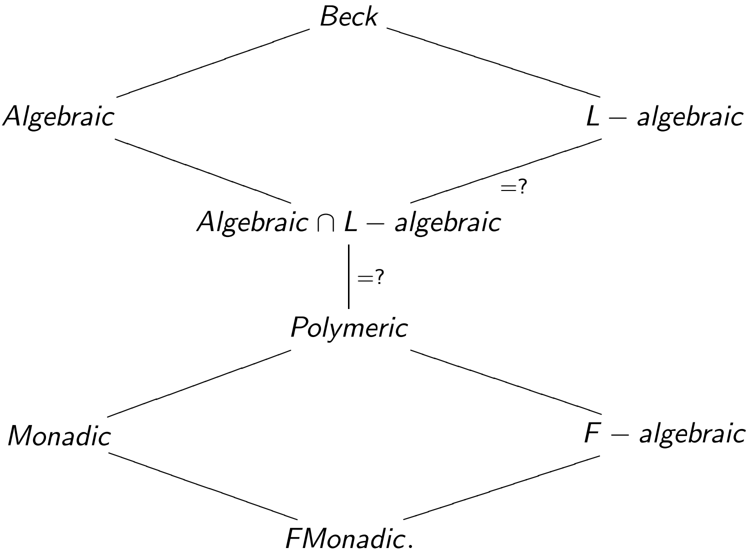

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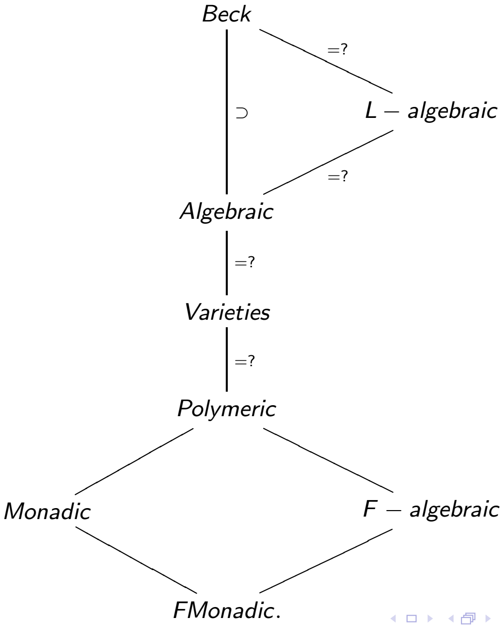

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Thank you for your attention.