Continuous Metrics

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In analogy to the situation for continuous lattices which were introduced by Dana Scott as precisely the injective T_0 spaces via the (nowadays called) Scott topology, we study those metric spaces which correspond to injective T_0 approach spaces and characterise them as precisely the continuous lattices equipped with an unitary and associative $[0, \infty]$-action. This result is achieved by a detailed analysis of the notion of cocompleteness for approach spaces.

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$x \leq y \leq z \Rightarrow x \leq z$	

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$f : (X, \delta_X) \to (Y, \delta_Y)$ is in App if $\delta_X(x, A) \geq \delta_Y(f(x), f(A))$.

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Proposition. For a metric space X, TFAE.

(i) X is injective (with respect to isometries).

(ii) $y_X : X \to [0,\infty]^{X^{\mathrm{op}}}$ has a left inverse.

(iii) y_X has a left adjoint.

(iv) X is cocomplete.

Tensored metric spaces

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Theorem. Let $X = (X, d)$ be a metric space. Then the TFAE.

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Theorem. The category of cocomplete metric spaces is equivalent to the category of $\mathsf{Sup}^{[0,\infty]}$ of $[0, \infty]$-actions preserving supremum in both variables.

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We recover the metric by taking

$$d(x, y) = \inf\{u \in [0, \infty] \mid x \leq y + u\}$$

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Every absolutely cocomplete approach space is exponentiable in App and the *full* subcategory of App defined by these spaces is Cartesian closed.