# A final Vietoris coalgebra beyond compact spaces and a generalized Jónsson-Tarski duality

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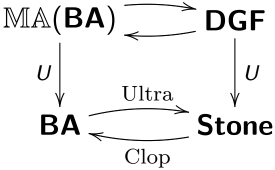
# Jónsson-Tarski duality is . . .

1. in old days . . .



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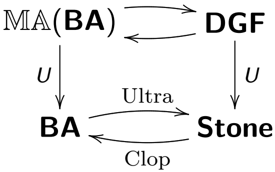
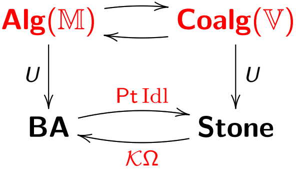
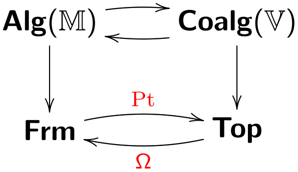



2. nowadays . . .



# Is is true in general?

The *main purpose* of this talk:



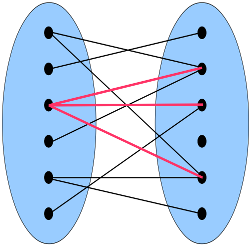

and an application, the final $\mathbb{V}$-coalgebra.
Let's see how far we can go.

# Kripke frames are $\mathcal{P}$-coalgebras

## Definition

A *Kripke frame* $\langle X, R\rangle$ consists of

1. a set $X$ and
2. a relation $R$ of $X$



$$\begin{aligned}\xi_R : X &\to \mathcal{P}X \\ x &\mapsto \{y : xRy\}\end{aligned}$$

# Bounded morphisms are $\mathcal{P}$-coalgebra morphisms

**Definition**

A *bounded morphism* $f : \langle X, R\rangle \to \langle Y, S\rangle$ is a functional bisimulation.

$$\begin{array}{ccc} X & \xrightarrow{\ \xi\ } & \mathcal{P}X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{P}f} \\ Y & \xrightarrow[\ \eta\ ]{} & \mathcal{P}Y \end{array}$$

That is, $f$ is a $\mathcal{P}$-coalgebra morphism.

# Descriptive general frames are $\mathbb{V}$-coalgebras

## Definition

A *descriptive general frame* is

1. a Kripke frame $\langle X, \xi : X \to \mathcal{P}X \rangle$
2. on a Stone space $\langle X, \mathcal{B} \rangle$

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2. on a Stone space $\langle X, \mathcal{B} \rangle$
3. $\xi(x) \in \mathcal{K}X$
4. $\mathcal{B}$ is closed under
   1. $\Box V = \{x : \xi(x) \subseteq V\}$
   2. $\Diamond V = \{x : \xi(x) \cap V \neq \emptyset\}$.

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$$\xi_i : \langle X, \mathcal{B}_i \rangle \to \langle \mathcal{K}X, ? \rangle$$

What is the finest topology making every $\xi_i$ continuous?

**Definition**

A Vietoris topology $\mathbb{V}X = \langle \mathcal{K}X, \tau \rangle$ of a Stone space $X$ is a Stone space with $\tau$ generated by

1. $\Box V = \{K \in \mathcal{K}X : K \subseteq V\}$
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$$\mathcal{K}\Omega X \xrightarrow{\Box} \Omega\mathbb{V}X \xrightarrow{\xi^{-1}} \Omega X$$

$$\begin{aligned} \xi^{-1}(\Box V) &= \{x : \xi(x) \in \Box V\} \\ &= \{x : \xi(x) \subseteq V\} \end{aligned}$$

similarly for $\xi^{-1}(\Diamond V)$.

# Bounded morphisms are $\mathbb{V}$-coalgebra morphisms

## Definition

A morphism $f : \langle X, R, \mathcal{B}\rangle \to \langle Y, S, \mathcal{C}\rangle$ is a bounded morphism and also $f^{-1}(C) \in \mathcal{B}$ for any $C \in \mathcal{C}$ (continuity).

## Fact

$\mathbb{V}f(K) = f[K]$ *is compact*
$\mathbb{V}f$ *is continuous.*

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X & \xrightarrow{\xi} & \mathcal{P}X \\
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# Modal algebras are $\mathbb{M}$-algebras

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Modal algebra = Boolean algebra + unary operators $\Box, \Diamond$ subject to normal modal logic laws

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## Definition

A modal algebra construction $\mathbb{M}A$ of a Boolean algebra is

$$
\begin{aligned}
\mathbb{BA}\langle \Diamond A \cup \Box A \mid \; & \Box(a \wedge b) = \Box a \wedge \Box b \\
& \Diamond(a \vee b) = \Diamond a \vee \Diamond b \\
& \Diamond(a \wedge b) \geq \Diamond a \wedge \Box b \\
& \Box(a \vee b) \leq \Diamond a \vee \Box b \; \rangle
\end{aligned}
$$

A $\mathbb{M}$-algebra is an interpretation:

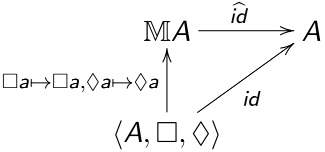
$$\mathbb{M}A \xrightarrow[\alpha]{} A \mapsto \langle A, \Box, \Diamond \rangle$$

$\Box a = \alpha(\Box a), \Diamond a = \alpha(\Diamond a).$

A $\mathbb{M}$-algebra is an interpretation:

$$\mathbb{M}A \xrightarrow[\alpha]{} A \;\mapsto \langle A, \Box, \Diamond \rangle$$

$\Box a = \alpha(\Box a), \Diamond a = \alpha(\Diamond a)$. Conversely, we can obtain an interpretation by freeness:



# Modal algebra morphisms are $\mathbb{M}$-algebra morphisms

**Definition**

A *modal algebra morphism* $f$ is a Boolean algebra morphism and $f(\Box_A a) = \Box_B f(a)$, $f(\Diamond_A a) = \Diamond_B f(a)$ and relations.

$$\begin{array}{ccc} A & \xrightarrow{\Box_A} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[\Box_B]{} & B \end{array}$$

Modal algebra morphisms are $\mathbb{M}$-algebras morphisms.

## Definition

Given $f : A \to B$, define

1. $\mathbb{M}f(\Box a) = \Box f(a)$ and $\mathbb{M}f(\Diamond a) = \Diamond f(a)$
2. $\mathbb{M}f$ is a Boolean algebra homomorphism by freeness of $\mathbb{M}A$.

$$\begin{array}{ccc} \mathbb{M}A & \xrightarrow{\alpha} & A \\ {\scriptstyle \mathbb{M}f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{M}B & \xrightarrow[\beta]{} & B \end{array}$$

$$\begin{aligned} f(\Box_A a) &= (f \circ \alpha)(\Box a) \\ &= \beta(\Box f(a)) = \Box_B f(a) \end{aligned}$$

# Short summary

**Fact**

1. $\mathbf{DGF} \cong \mathbf{Coalg}(\mathbb{V})$ *and* $\mathbb{MA}(\mathbf{BA}) \cong \mathbf{Alg}(\mathbb{M})$
2. *The classical Jónsson-Tarski duality:* $\mathbf{DGF} \cong \mathbb{MA}(\mathbf{BA})^{\mathrm{op}}$
3. *The (co)-algebra viewpoint:* $\mathbf{Coalg}(\mathbb{V}) \cong \mathbf{Alg}(\mathbb{M})^{\mathrm{op}}$

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## Question

1. What is the relationship between $\mathbb{M}$ and $\mathbb{V}$?
2. An extension of $\mathbb{M}$ and $\mathbb{V}$?

$$\begin{array}{ccc} \mathbf{BA} & \xrightarrow{\mathbb{M}} & \mathbf{BA} \\ {\scriptstyle \mathrm{Idl}}\downarrow & & \downarrow{\scriptstyle \mathrm{Idl}} \\ \mathbf{Frm} & \xrightarrow[\mathbb{M}_F]{} & \mathbf{Frm} \end{array} \qquad \begin{array}{ccc} \mathbf{Stone} & \xrightarrow{\mathbb{V}} & \mathbf{Stone} \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ \mathbf{Top} & \xrightarrow[\mathbb{V}']{} & \mathbf{Top} \end{array}$$

# Duality between algebras and coalgebras

**Fact**

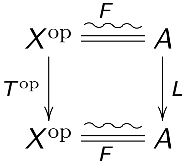
$$\mathbf{Coalg}(T)^{\mathrm{op}} \equiv \mathbf{Alg}(T^{\mathrm{op}})$$

where

$$T^{\mathrm{op}} : X^{\mathrm{op}} \to X^{\mathrm{op}}$$
$$x \mapsto x$$
$$f^{\mathrm{op}} \mapsto (Tf)^{\mathrm{op}}$$

# Dual functors

$T$ is *dual* to $L$ if




i.e. $LF \cong FT$.

## Fact

$$\mathbf{Coalg}(T)^{\mathrm{op}} \cong \mathbf{Alg}(T^{\mathrm{op}}) \cong \mathbf{Alg}(L)$$

*if* $X^{\mathrm{op}} \cong A$ *and* $LF \cong FT$.

# Topological spaces and Frames

A dual adjunction ...

$$\mathbf{Frm} \underset{\Omega}{\overset{\mathrm{Pt}}{\rightleftarrows}} \mathbf{Top}$$

$\Omega X$ = the complete lattice of open sets with

$$\bigvee S \wedge a = \bigvee_{s \in S} (s \wedge a)$$

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$\mathrm{Pt}\, A$ = the space of frame homomorphisms $f : A \to 2$ with open sets

$$U_a = \{\varphi \in \mathrm{Pt}\, A : \varphi(a) = \top\}$$

or, neighbourhoods systems.

# Sober spaces and spatial frames

**Definition**

A frame $A$ is spatial if the unit is an iso.

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A frame $A$ is spatial if the unit is an iso.
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A *cheat* dual equivalence . . .

$$\mathbf{SFrm} \underset{\Omega}{\overset{\mathsf{Pt}}{\rightleftarrows}} \mathbf{Sob}$$

## Definition

A modal algebra construction $\mathbb{M}_F A$ of a frame $A$ is

$$\mathbf{Frm}\langle \Box A \cup \Diamond A \mid \text{ normal modal logic laws with the following } \rangle$$

for any *directed* $S \subseteq A$

1. $\Box(\bigvee S) = \bigvee_{s \in S} \Box s$
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$$\begin{array}{ccc} \mathbf{BA} & \xrightarrow{\mathbb{M}} & \mathbf{BA} \\ {\scriptstyle \mathrm{Idl}}\downarrow & & \downarrow{\scriptstyle \mathrm{Idl}} \\ \mathbf{Frm} & \xrightarrow[\mathbb{M}_F]{} & \mathbf{Frm} \end{array}$$

$\mathbb{M}_F$ is an extension of $\mathbb{M}$ along $\mathrm{Idl}$.

# Stably locally compact frames

Unfortunately, **SFrm** is not closed under $\mathbb{M}_F$.
Stably locally compact frames (locales) are closed under $\mathbb{M}_F$, i.e. $\mathbb{M}_F$ : **SLCFrm** $\to$ **SLCFrm**.

**Definition**

A stably locally compact frame is

1. a continuous domain
2. $x \ll y_1$ and $x \ll y_2 \Rightarrow x \ll y_1 \wedge y_2$

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## Example

1. the ideal completion of Boolean algebras
2. the ideal completion of distributive lattices
3. compact regular frames
4. ...

# Stably locally compact spaces

By dual equivalence, $\mathbf{SLCFrm} \cong \mathbf{SLCSp}^{\mathrm{op}}$:

**Definition**

A space $X$ is stably locally compact if $X \in \mathbf{Sob}$ and $\Omega X \in \mathbf{SLCFrm}$.

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**Example**

1. Stone spaces
2. coherent spaces, i.e. Priestly spaces
3. compact Hausdorff spaces
4. ...

# Is $\mathbb{M}_F$ dual to any functor?

**Definition**

$\mathbb{V}' : \mathbf{Top} \to \mathbf{Top}$ $\mathbb{V}'X = \langle \mathcal{KL}X, \tau \rangle$ where

1. $\mathcal{L}X$ = the set of intersections of open and closed sets
2. $\tau = \Box U \vee \Diamond U$

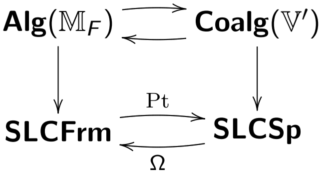
$$\Box U = \{L : L \subseteq U\}, \Diamond U = \{L : L \cap U \neq \emptyset\}$$

where $U \in \Omega X$.

$$\begin{array}{ccc} \mathbf{SLCFrm} & \xrightarrow{\mathrm{Pt}} & \mathbf{SLCSp} \\ \downarrow{\scriptstyle \mathbb{M}_F} & & \downarrow{\scriptstyle \mathbb{V}'} \\ \mathbf{SLCFrm} & \xrightarrow[\mathrm{Pt}]{} & \mathbf{SLCSp} \end{array}$$

# A generalized modal duality

Finally . . .



# A canonical Kripke structure

**Fact**

*A descriptive general Kripke frame $\langle X, R, \mathcal{B} \rangle$ is canonical if and only if $\xi_R : X \to \mathbb{V}X$ is a final $\mathbb{V}$-coalgebra.*

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Now, turn to our setting:
How to find a $\mathbb{V}'$-coalgebra?

**Fact**

*A final $\mathbb{V}'$-coalgebra corresponds to an initial $\mathbb{M}_F$-algebra.*

# $\mathbb{M}_F$-initial sequence

Base case (the initial object in **Frm**):

$$2 \dashrightarrow \cdots$$

Inductive case:

$$\to \mathbb{M}_F^n 2 \to \mathbb{M}_F^{n+1} 2 \to$$

and transfinitely by colimits.

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**Fact**

*If the sequence converges, i.e.* $\alpha : \mathbb{M}_F{}^\kappa 2 \cong \mathbb{M}_F \mathbb{M}_F^\kappa 2$ *for some* $\kappa$*, then*

$$\alpha^{-1} : \mathbb{M}_F \mathbb{M}_F^\kappa 2 \to \mathbb{M}_F^\kappa 2$$

*is an initial* $\mathbb{M}_F$*-algebra.*

$\bigvee$ is infinitary, so it is not clear whether $\mathbb{M}_F$ converges.

## Fact

1. $2 = \mathrm{Idl}2$
2. $\mathbb{M}_F$ *preserves coherences, i.e.*

$$\begin{array}{ccc} \mathbf{DLat} & \xrightarrow{\mathbb{M}_D} & \mathbf{DLat} \\ {\scriptstyle \mathrm{Idl}}\downarrow & & \downarrow{\scriptstyle \mathrm{Idl}} \\ \mathbf{Frm} & \xrightarrow[\mathbb{M}_F]{} & \mathbf{Frm} \end{array}$$

3. $\mathbb{M}_D$ *is the same as* $\mathbb{M}_F$ *but it is a free distributive construction.*

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3. $\mathbb{M}_D$ *is the same as* $\mathbb{M}_F$ *but it is a free distributive construction.*
4. **DLat** *is finitary, so* $\mathbb{M}_D$ *converges at* $\omega$*.*

$$2 \dashrightarrow \mathbb{M}_D 2 \longrightarrow \mathbb{M}_D^2 2 \longrightarrow \cdots \longrightarrow \mathbb{M}_D^\omega 2 \cong \mathbb{M}_D \mathbb{M}_D^\omega 2$$

$\mathbb{M}_D^\omega$ is a colimit.

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$\mathbb{M}_D^\omega$ is a colimit. $\mathbb{M}_F$-initial sequence converges at $\omega$:

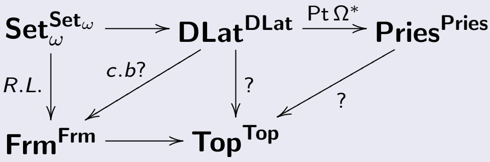
$$\begin{aligned}
\mathbb{M}_F \mathbb{M}_F^\omega 2 &\cong \mathbb{M}_F \mathrm{Idl} \mathbb{M}_D^\omega 2 && \{\mathrm{Idl} \text{ preserves colimits}\} \\
&\cong \mathrm{Idl} \mathbb{M}_D \mathbb{M}_D^\omega 2 && \{\mathbb{M}_F \text{ preserves coherence}\} \\
&\cong \mathrm{Idl} \mathbb{M}_D^\omega 2 && \{\mathbb{M}_D \text{ converges at } \omega\} \\
&\cong \mathbb{M}_F^\omega 2 && \{\mathrm{Idl} \text{ preserves colimits}\}
\end{aligned}$$

## Future work

1. $\mathbb{V}$ : **Stone** $\to$ **Stone** can be obtained by $\mathcal{P}_\omega$ via Pro-completion.
2. $\mathbb{M}_F$ : **Frm** $\to$ **Frm** can be obtained by $\mathcal{P}_\omega$ via relation lifting. See Y. Venema's *Generalized Powerlocales via Relation Lifting*.
3. $$\mathbf{Set}_\omega^{\mathbf{Set}_\omega} \xrightarrow{R.L.} \mathbf{Poset}_\omega^{\mathbf{Poset}_\omega} \longrightarrow \mathbf{DLat}^{\mathbf{DLat}}$$

Put things together . . .

1. A big picture:




where $c.b.$ is the change of base from **Poset** to **DCPO**.

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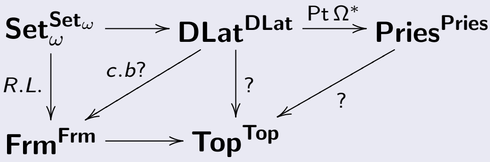


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2. A description of $\mathrm{Pt} \circ \mathbb{M}_T \circ \Omega$ where $T : \mathbf{Set}_\omega \to \mathbf{Set}_\omega$?

Thank you for your attention!
For details, please see my extended abstract, Vietoris locales by P. Johnstone, and Stone coalgebras by C. Kupke.