Algebraic Semantics and Model Completeness for Intuitionistic Public Announcement Logic

Minghui Ma, Alessandra Palmigiano, Mehrnoosh Sadrzadeh

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PAL

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1. $\langle\alpha\rangle p \leftrightarrow (\alpha \wedge p)$
2. $\langle\alpha\rangle\neg\varphi \leftrightarrow (\alpha \wedge \neg\langle\alpha\rangle\varphi)$
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Not amenable to a standard algebraic treatment.

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- Generalize epistemic update on algebras to much wider classes of algebras.
- Dualize back to relational models for non classically based logics.

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Let $[b]_a$ be the equivalence class of $b \in \mathbb{A}$. Let

$$\mathbb{A}^a := \mathbb{A}/{\equiv_a}$$

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Let $\pi_a : \mathbb{A} \to \mathbb{A}^a$ be the canonical projection.

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Crucial remark

Each $\equiv_a$-equivalence class has a canonical representant. Hence, the map $i' : \mathbb{A}^a \to \mathbb{A}$ given by $[b] \mapsto b \wedge a$ is injective.

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- If $\mathbb{A} = \mathcal{F}^+$ for some Kripke frame $\mathcal{F}$, then $\mathbb{A}^a \cong_{BAO} \mathcal{F}^{a+}$.

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from which we get

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$M^\alpha := (\mathbb{A}^\alpha, V^\alpha)$ s.t. $\mathbb{A}^\alpha = \mathbb{A}^{[\![\alpha]\!]_M}$ and $V^\alpha : \mathrm{AtProp} \to \mathbb{A}^\alpha$ is $\pi \circ V$, i.e. $[\![p]\!]_{M^\alpha} = V^\alpha(p) = \pi(V(p)) = \pi([\![p]\!]_M)$ for every p.

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Interaction with logical constants

$\langle\alpha\rangle\bot = \bot$

$[\alpha]\top = \top$

Preservation of facts

$\langle\alpha\rangle p = \alpha \wedge p$

$[\alpha]p = \alpha \to p$

Interaction with disjunction

$\langle\alpha\rangle(\varphi \vee \psi) = \langle\alpha\rangle\varphi \vee \langle\alpha\rangle\psi$

$[\alpha](\varphi \vee \psi) = \alpha \to (\langle\alpha\rangle\varphi \vee \langle\alpha\rangle\psi)$

Interaction with conjunction

$\langle\alpha\rangle(\varphi \wedge \psi) = \langle\alpha\rangle\varphi \wedge \langle\alpha\rangle\psi$

$[\alpha](\varphi \wedge \psi) = [\alpha]\varphi \wedge [\alpha]\psi$

Interaction with implication

$\langle\alpha\rangle(\varphi \to \psi) = \alpha \wedge (\langle\alpha\rangle\varphi \to \langle\alpha\rangle\psi)$

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Interaction with $\Diamond$

$\langle\alpha\rangle\Diamond\varphi = \alpha \wedge \Diamond\langle\alpha\rangle\varphi$

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Interaction with $\Box$

$\langle\alpha\rangle\Box\varphi = \alpha \wedge \Box[\alpha]\varphi$

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  - R is an (equivalence) relation on W s.t.

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Work in progress:
Intuitionistic account of Muddy Children Puzzle.