# Frobenius Algebras and Classical Proof Nets

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- Categorical logic is an appropriate mathematical language for providing semantics of proofs

| | | |
|---|---|---|
| (*-Autonomous categories | / | Multiplicative linear logic |
| CCC | / | Intuitionistic logic) |

- Classical Logic – a notoriously difficult problem

| | | |
|---|---|---|
| Heyting Algebras | : | CCC |
| Boolean Algebras | : | ??? |

Before mid 2000's:

- Joyal's paradox
- Parigot, Selinger, Ong – $\lambda\mu-$calculus, Control categories
- Girard – LC, Coherence spaces

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Double negation not isomorphic to an object, non-symmetric, connectives are not bifunctors, semantics is not a category

Last 6-7 years:

- Došen, Petrić
- Robinson, Führman, Pym
- Belin, Hyland, Robinson, Urban
- Lamarche, Strassburger

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Different axiomatiozations of "the Boolean category"

Concrete denotational semantics [Novaković, Lamarche - SD09, CT10] – Posets and Bimodules / Comparisons

- Objects: Posets
- Maps: $(M, \leq) \xrightarrow{f} (N, \leq)$ is a relation $f \subseteq M \times N$ s.t.:
  $m\ f\ n, \quad m' \leq m \quad$ implies $\quad m'\ f\ n \quad$ (down-closed to the left)
  $m\ f\ n, \quad m \leq n' \quad$ implies $\quad m\ f\ n' \quad$ (and up-closed to the right).
- Composition: Ordinary relational
- Identity: $\mathrm{Id}_M = \{\, (m, m') \mid m \leq m' \,\}$

MLL:

- $\mathbf{1}$ and $\perp \rightsquigarrow \{*\}$
- $\mathrm{a} \rightsquigarrow$ poset $a$;
- $\mathrm{A} \otimes \mathrm{B} \rightsquigarrow A \times B$, (bi)functorial,
- $\mathrm{A}^{\perp} \rightsquigarrow A^{op}$, contravariant functor,
- $A \parr B = (\mathrm{A}^{\perp} \otimes \mathrm{B}^{\perp})^{\perp} \rightsquigarrow (A^{op} \times B^{op})^{op} = A \times B = A \otimes B$.
- Natural bijeciton:

$$\frac{A \otimes B \to C}{A \to B^{\perp} \parr C.}$$

$$\frac{}{\vdash \mathrm{a}^{\perp}, \mathrm{a}} \qquad \rightsquigarrow \quad \mathrm{Id}_a = \{(x,y) \in a \times a \mid x \leq y\}$$

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{B}}{\vdash \Gamma, \mathrm{A} \parr \mathrm{B}} \parr \qquad \rightsquigarrow \quad \text{do nothing}$$

$$\frac{\vdash \Gamma, \mathrm{A} \quad \vdash \mathrm{B}, \Sigma}{\vdash \Gamma, \mathrm{A} \otimes \mathrm{B}, \Sigma} \otimes \qquad \rightsquigarrow \quad \begin{array}{l}\text{given } f \text{ for } \Gamma \times A \text{ and } g \text{ for } B \times \Sigma, \text{ take } f \times g \\ \text{for } \Gamma \times A \times B \times \Sigma\end{array}$$

$$\frac{\vdash \Gamma, \mathrm{A} \quad \vdash \mathrm{A}^{\perp}, \Sigma}{\vdash \Gamma, \Sigma} \mathit{Cut} \qquad \rightsquigarrow \quad \begin{array}{l}\text{given } f \text{ for } \Gamma \times A \text{ and } g \text{ for } A^{\perp} \times \Sigma, \text{ take} \\ \{(\gamma, \delta) \mid \exists x \in A : (\gamma, x) \in f, (x, \delta) \in g\} \text{ for } \Gamma \times \Sigma\end{array}$$

$$\frac{\vdash \Gamma \quad \vdash \Sigma}{\vdash \Gamma, \Sigma} \mathit{Mix} \qquad \rightsquigarrow \quad \begin{array}{l}\text{given } f \text{ for } \Gamma \text{ and } g \text{ for } \Sigma, \text{ take } f \times g \\ \text{for } \Gamma \times \Sigma.\end{array}$$

Going classical:

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Equip each object $A$ with a commutative monoid $\nabla, \amalg$ and a cocomutative comonoid $\Delta, \Pi$.

i) $\nabla_A : A \otimes A \longrightarrow A$

ii) $\amalg_A : \mathbf{1} \longrightarrow A$

iii) $\Delta_A : A \longrightarrow A \otimes A$

iv) $\Pi_A : A \longrightarrow \mathbf{1}$.

$$\frac{\vdash \Gamma}{\vdash \Gamma, A}\ \mathit{Weak}$$

$\leadsto$ given $f : 1 \to \Gamma$, take

$f \otimes \amalg$

for $1 \xrightarrow{\sim} 1 \otimes 1 \to \Gamma, A$;

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ \mathit{Contr}$$

$\leadsto$ given $f : 1 \to \Gamma, A, A$ take

$\Gamma \otimes \nabla \circ f$

for $1 \to \Gamma, A, A \to \Gamma, A$;

$\mathbb{Z}$ :

$$\begin{array}{llllll} (j,k)^{\nabla_a} i & \text{iff} & j+k \leq i + C; & *^{\amalg_a} i & \text{iff} & C \leq i. \\ i^{\Delta_a}(j,k) & \text{iff} & i \leq j+k; & i^{\amalg_a} * & \text{iff} & i \leq 0; \end{array} \tag{1}$$

- 'Weird' Church numerals
- Curry-Howard correspondence does not hold
- ...
- The assigned bialgebra structure on an object is a Frobenius algebra!

## Definition (Frobenius algebra)

Let $(\mathbf{C}, \otimes, \mathbf{1})$ be a SMC, and $A$ an object of it.
A *Frobenius algebra* is a sextuple

$$(A, \Delta, \Pi, \nabla, \amalg)$$

where $(A, \nabla, \amalg)$ is a commutative monoid, $(A, \Delta, \Pi)$ a co-commutative comonoid, where the following diagram commutes:

$$\begin{tikzcd}
A \otimes A \arrow[r, equal] \arrow[d, "\Delta\otimes\mathrm{Id}"'] & A \otimes A \arrow[r, equal] \arrow[d, "\nabla"'] & A \otimes A \arrow[d, "\mathrm{Id}\otimes\Delta"] \\
A \otimes A \otimes A \arrow[d, "\mathrm{Id}\otimes\nabla"'] & A \arrow[d, "\Delta"'] & A \otimes A \otimes A \arrow[d, "\nabla\otimes\mathrm{Id}"] \\
A \otimes A \arrow[r, equal] & A \otimes A \arrow[r, equal] & A \otimes A
\end{tikzcd}$$

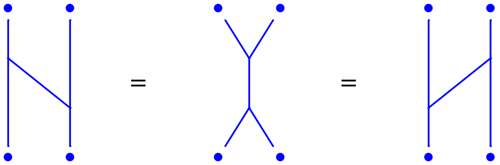


Figure: A diagram version of Frobenius equations

A Frobenius algebra is *thin* if for every $k \geq 0$, the $\mathbf{1} \longrightarrow \mathbf{1}$ map

$$\amalg \circ \underbrace{\nabla \circ \Delta \circ \cdots \circ \nabla \circ \Delta}_{k} \circ \amalg$$

is the identity.

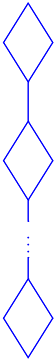


Figure: A diagram version of the Thinness axiom equations

A Frobenius algebra is *thin* if for every $k \geq 0$, the $\mathbf{1} \longrightarrow \mathbf{1}$ map

$$\Pi \circ \underbrace{\nabla \circ \Delta \circ \cdots \circ \nabla \circ \Delta}_{k} \circ \amalg$$

is the identity.

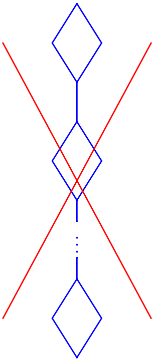


Figure: A diagram version of the Thinness axiom equations

The following is well-known.

## Proposition

*The tensor of two Frobenius algebras is also a Frobenius algebra, where the monoid and comonoid operations are defined as usual in an SMC. It is thin if both factors are.*

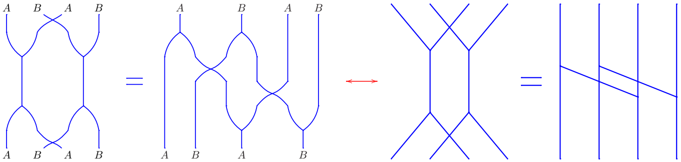



Figure: Diagrams of (one of) Frobenius equations for a composite type

### Definition

A Frobenius category **C**:

- a symmetrical monoidal category
- every object $A$ is equipped with a *thin* Frobenius algebra structure $(A, \nabla_A, \amalg_A, \Delta_A, \amalg)$
- the algebra on the tensor of two objects is the usual tensor algebra.

Frobenius algebras have gained a lot of attention

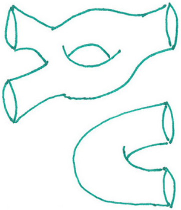
- closely related to 2-dimensional Topologica Quantum Field Theories (TQFTs) [Dij89, Koc04], and can be stated as follows.

## Theorem

*The free Frobenius category* **F** *on one object generator is equivalent to the two following categories.*

1. *The category of bounded Riemann surfaces up to a homeomorphism*

| | |
|---|---|
| Objects: | *finite disjoint unions of m circles* |
| Maps: | *A map $m \to n$ is a Riemann surface (with boundary) whose boundary is the disjoint sum $m + n$,* *Two surfaces are identified modulo homeomorphism.* |
| Composition: | *gluing, forgetting the boundaries in the middle* |
| Thin: | *every connected component has a nonempty boundary* |



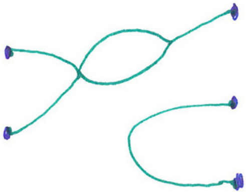
2. The category of finitary graphs (the node set is finite), up to a homology

**Objects:** finite sets $[m] = \{0, 1, \dots, m-1\}$, seen as discrete topological spaces

**Maps:** $[m] \to [n]$ is a topological graph $G$ (i.e. a CW-complex of dimension one), with an injective function $[m+n] \to G$
Two graphs are identified if they are equivalent modulo homology

**Composition:** also gluing.

**Thin:** every connected components of $G$ is in the image of the injective function $[m+n] \to G$



- A free Frobenius category is defined only up to equivalence of categories, with the standard universal property associated to that situation
- The two characterizations in Theorem 3 happen to be skeletal categories and are isomorphic
- Our nonstandard notion of Frobenius category requires thinness; maps in the standard, non-thin free Frobenius category can contain several "floating" components that do not touch the border.

Since homology is much more technical than homotopy, we prefer to replace the second result above with:

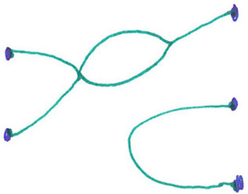
2'. The category of finitary graphs, up to a *homotopy*

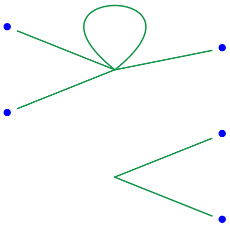
**Objects:** finite sets $[m] = \{0, 1, \ldots, m-1\}$, seen as discrete topological spaces

**Maps:** $[m] \to [n]$ is a topological graph $G$ (i.e. a CW-complex of dimension one), with an injective function $[m+n] \to G$
Two graphs are identified if they are equivalent modulo *homotopy* in $(m+n)/\mathit{Top}$, where homotopies are defined to be constant on $[m+n]$.

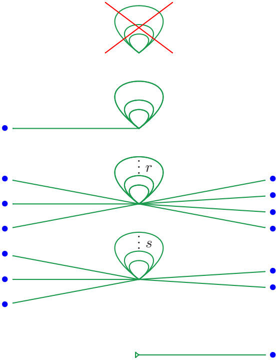
**Composition:** gluing.

**Thin:** every connected components of $G$ is in the image of the injective function $[m+n] \to G$





Every map in **F** can be represented by a graph $G$ of the following form, where every connected component is a "star" whose central node has $n$ loops attached to it, with $n \geqslant 0$.



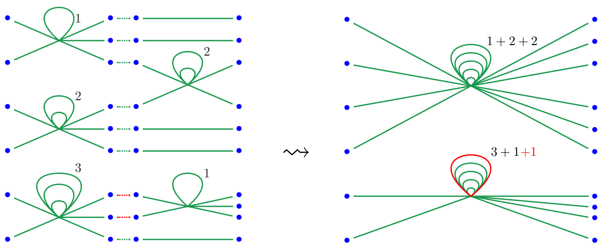



**Fig. 2.** Composition.

# Proposition

*The category* **F** *is compact-closed, the dual of an object being the object itself.*

More generally, *any* Frobenius category is compact-closed.

## Definition (Linking)

We define a *linking* to be a triple

$$P = (P, \mathcal{C}omp_P, \mathcal{G}en_P)$$

where

- $P$ is a finite set
- $\mathcal{C}omp_P$ is the set of classes of a *partition* of the set $P$. Its elements are called *components*.
- the function $\mathcal{G}en_P : \mathcal{C}omp_P \to \mathbb{N}$ (called *genus*) assigns a natural number to each component in $\mathcal{C}omp_P$

A map $m \to n$ in $\mathbf{F}$ can be described as a linking on the set $m + n$.

The relevance of the "Frobenius equations" for proof theory is due to the fact that they address the contraction-against-contraction case in cut elimination

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \bar{\mathrm{a}}, \mathrm{a}}\,\mathit{Ax} \quad \dfrac{}{\vdash \bar{\mathrm{a}}, \mathrm{a}}\,\mathit{Ax}}{\vdash \bar{\mathrm{a}}, \mathrm{a}, \bar{\mathrm{a}}, \mathrm{a}}\,\mathit{Mix}}{\vdash \bar{\mathrm{a}}, \bar{\mathrm{a}}, \mathrm{a}}\,\mathit{Contr} \qquad \dfrac{\dfrac{\dfrac{}{\vdash \bar{\mathrm{a}}, \mathrm{a}}\,\mathit{Ax} \quad \dfrac{}{\vdash \bar{\mathrm{a}}, \mathrm{a}}\,\mathit{Ax}}{\vdash \bar{\mathrm{a}}, \mathrm{a}, \bar{\mathrm{a}}, \mathrm{a}}\,\mathit{Mix}}{\vdash \bar{\mathrm{a}}, \mathrm{a}, \mathrm{a}}\,\mathit{Contr}}{\vdash \bar{\mathrm{a}}, \bar{\mathrm{a}}, \mathrm{a}, \mathrm{a}}\,\mathit{Cut}
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$$

**Fig. 3.** Two proofs identified by Frobenius equations

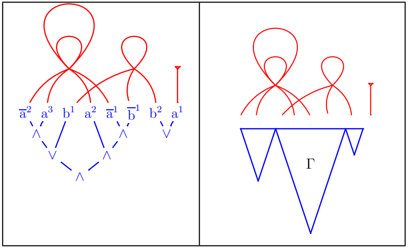
## Definition (F-prenet)

We define an *F-prenet* to be a pair

$$P \triangleright \Gamma$$

where

- $\Gamma$ is a sequent
- $P = (P, \mathcal{C}omp_P, \mathcal{G}en_P)$ is a linking
- there is a bijection between the underlying set $P$ and the set of literals of $\Gamma$ (for which there is no need to make it explicit)
- every class in $\mathcal{C}omp_P$ contains only atoms of the same type and their negations.



Fix a calculus: the calculus CL [LS05]

$$\frac{}{\vdash \bar{\mathrm{a}}, \mathrm{a}}\ Ax$$

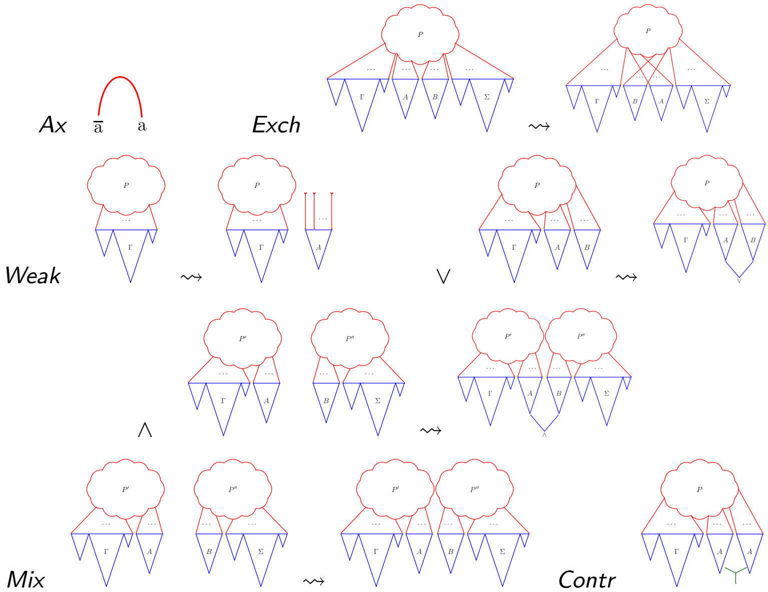
$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{B}}{\vdash \Gamma, \mathrm{A} \vee \mathrm{B}}\ \vee \qquad \frac{\vdash \Gamma, \mathrm{A} \quad \vdash \mathrm{B}, \Sigma}{\vdash \Gamma, \mathrm{A} \wedge \mathrm{B}, \Sigma}\ \wedge$$

$$\frac{\vdash \Gamma, \mathrm{A} \quad \vdash \overline{\mathrm{A}}, \Sigma}{\vdash \Gamma, \Sigma}\ Cut$$

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{A}}{\vdash \Gamma, \mathrm{A}}\ Contr \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \mathrm{A}}\ Weak$$

$$\frac{\vdash \Gamma \quad \vdash \Sigma}{\vdash \Gamma, \Sigma}\ Mix$$

Figure 3: System CL



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Every $n$-ary introduction rule of CL

$$\frac{\vdash \Gamma_1 \quad \vdash \Gamma_2 \quad \cdots \quad \vdash \Gamma_n}{\vdash \Gamma}$$

can be transformed into a family of $n$ morphisms $P_i \triangleright \Gamma_i \to Q \triangleright \Gamma$ in the following *syntactic category*.

## Definition (Syntactic Category)

Let $\mathcal{F}Synt$ have F-prenets for objects, where a map

$$f : P \triangleright \Gamma \to Q \triangleright \Delta$$

is given by an ordinary function on the underlying set of literals

$$f : P \to Q \qquad\qquad (= \mathcal{L}it(\Gamma) \to \mathcal{L}it(\Delta))$$

such that

1. for every formula $\mathrm{A}$, $f$ maps $\mathcal{L}it(A)$ to a subset of $\mathcal{L}it(\Delta)$ which defines a subformula of a formula in $\Delta$, while preserving the syntactic left-right order on literals.
2. for every $C \in \mathcal{C}omp_P$, one has that $f(C) \subseteq \mathcal{L}it(\Delta)$ is contained in a component $C' \in \mathcal{C}omp_Q$, with $\mathcal{G}en_P(C) \leq \mathcal{G}en_Q(C')$.

## Definition

In the category $\mathcal{F}Synt$, we define the families of cospans *Mix* and $\wedge$ to be

$$\begin{tikzcd}[column sep=small] P_l \rhd \Gamma \arrow[dr, "Mix : l"] & & P_r \rhd \Gamma \arrow[dl, "Mix : r"'] \\ & P_l \uplus P_r(\Gamma \curlyvee \Gamma) \rhd \Gamma & \end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small] P_l \rhd \Gamma, \mathrm{A} \wedge \mathrm{B}, \mathrm{A} \arrow[dr, "\wedge : l"] & & P_r \rhd \mathrm{B}, \mathrm{A} \wedge \mathrm{B}, \Gamma \arrow[dl, "\wedge : r"'] \\ & Q \rhd \Gamma, \mathrm{A} \wedge \mathrm{B} & \end{tikzcd}$$

where $Q$ is $P_l \uplus P_r\big(\Gamma \curlyvee \Gamma, \mathrm{A} \curlyvee \mathrm{A} \curlyvee \mathrm{A}, \mathrm{B} \curlyvee \mathrm{B} \curlyvee \mathrm{B}\big)$.

## Definition

An *anodyne map* $P \triangleright \Gamma \xrightarrow{\ominus} Q \triangleright \Delta$ is a syntactic map that can be decomposed

$$P \triangleright \Gamma \xrightarrow{\sim} Q \triangleright \Delta_1 \xrightarrow{\vee} \cdots \xrightarrow{\vee} Q \triangleright \Delta_n = \Delta.$$

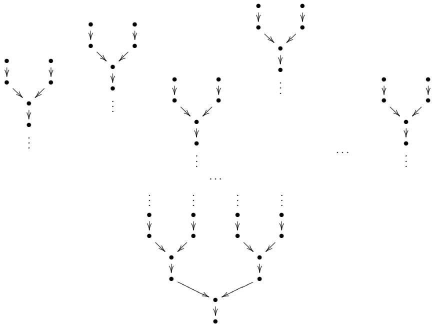
We write

$$[P \triangleright \Gamma] \xrightarrow{\boxminus} P \triangleright \Gamma$$

to denote the anodyne map whose domain is the sequent where all outer disjunctions have been removed.

# Definition (Correctness diagram)

A *correctness diagram* $\mathcal{T} : \mathsf{T} \to \mathcal{F}Synt$ is a diagram (functor) of the type



(1)

for which:

1. the branchings are $\wedge$- and *Mix*-cospans ;
2. vertical maps are anodyne;
3. every leaf $Q \triangleright \Delta$ is s.t. $\mathcal{C}omp_Q = \{\{\mathrm{a}, \bar{\mathrm{a}}\}, \{x_1\}, \dots, \{x_m\}\}$ and $\mathcal{G}en_Q$ is 0 everywhere.

## Definition (Correct F-nets)

An F-prenet $P \triangleright \Gamma$ is a CL-*correct F-net*, (or simply an *F-net*) if it is at the root of a *correctness diagram*

This can be strengthened by forcing the anodyne maps in a correctness diagram always to be $\square$-maps.

## Theorem (Sequentialization)

*Correct F-nets are precisely those F-prenets that come from* CL *without Cut.*

Given a linking $P$ let

- $|P|$ stand for the size of its underlying set,
- $|\mathcal{C}omp_P|$ be the number of components,
- $|\mathcal{G}en_P|$ be the sum of all genera in $P$, i.e. $|\mathcal{G}en_P| = \sum_{C \in \mathcal{C}omp_P} \mathcal{G}en_P(C)$.

The following observation is crucial to the proof:

## Lemma (Counting axiom links in an F-prenet)

*If an F-prenet $P \triangleright \Gamma$ corresponds to a* CL *proof, then*

$$|Ax| = |P| - |\mathcal{C}omp_P| + |\mathcal{G}en_P|,$$

*where $|Ax|$ is the number of axioms in the proof.*
*(Corollary: any correctness diagram for this proof will have the same number of leaves).*

### Theorem

*Given an F-prenet, its* CL*-correctness (*CL*-sequentializability) can be checked in finite time, i.e. the* CL*-correctness criterion yields a decision procedure for* CL*-correct F-nets.*

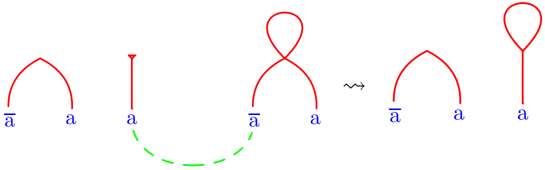
Strong evidence that the procedure is NP-complete, actually.

Cut:

- We define a *cut formula* to be $\mathrm{A} \diamondvert \overline{\mathrm{A}}$, where $- \diamondvert -$ is a new binary connective that is only allowed to appear as a root in a sequent
- Our original goal is to normalize these prenets with cuts by means of composition in **F**

[This is quite different to Hyland's [Hyl04]. It more resembles [LS05] with an "interaction category" construction [Hyl04, Section 3] on sets and relations, where composition is defined by the means of a trace operator]

Immediate problems:




For the resulting F-prenet to come from a proof we need the singleton component to come from a weakening, but this cannot happen according to our interpretation since its genus is $> 0$.

These issues can be dealt with by changing the deductive system and we define a new sound and complete calculus for classical logic, FL.

$$\frac{}{\vdash \overline{\mathrm{a}}, \mathrm{a}\ ;}\ Ax \qquad \frac{\vdash \Gamma\ ;\ \Delta}{\vdash \Gamma\ ;\ \Delta, \mathrm{a}, \mathrm{a}, \ldots, \mathrm{a}, \overline{\mathrm{a}}, \overline{\mathrm{a}}, \ldots, \overline{\mathrm{a}}}\ MulWeak$$

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{B}\ ;\ \Delta}{\vdash \Gamma, \mathrm{A} \vee \mathrm{B}\ ;\ \Delta}\ \vee_l \qquad \frac{\vdash \Gamma, \mathrm{A}\ ;\ \Delta, \mathrm{B}}{\vdash \Gamma, \mathrm{A} \vee \mathrm{B}\ ;\ \Delta}\ \vee_c \qquad \frac{\vdash \Gamma\ ;\ \Delta, \mathrm{A}, \mathrm{B}}{\vdash \Gamma\ ;\ \Delta, \mathrm{A} \vee \mathrm{B}}\ \vee_r$$

$$\frac{\vdash \Gamma_1, \mathrm{A}\ ;\ \Delta_1 \quad \vdash \mathrm{B}, \Gamma_2\ ;\ \Delta_2}{\vdash \Gamma_1, \mathrm{A} \wedge \mathrm{B}, \Gamma_2\ ;\ \Delta_1, \Delta_2}\ \wedge_l \qquad \frac{\vdash \Gamma\ ;\ \Delta, \mathrm{A}, \mathrm{B}}{\vdash \Gamma\ ;\ \Delta, \mathrm{A} \wedge \mathrm{B}}\ \wedge_r$$

$$\frac{\vdash \Gamma_1, \mathrm{A}\ ;\ \Delta_1 \quad \vdash \Gamma_2\ ;\ \mathrm{B}, \Delta_2}{\vdash \Gamma_2\ ;\ \mathrm{A} \wedge \mathrm{B}, \Gamma_1, \Delta_1, \Delta_2}\ \wedge_c$$

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{A}\ ;\ \Delta}{\vdash \Gamma, \mathrm{A}\ ;\ \Delta}\ Contr_l \qquad \frac{\vdash \Gamma\ ;\ \Delta, \mathrm{A}, \mathrm{A}}{\vdash \Gamma\ ;\ \Delta, \mathrm{A}}\ Contr_r$$

$$\frac{\vdash \Gamma, \mathrm{A}\ ;\ \Delta, \mathrm{A}}{\vdash \Gamma, \mathrm{A}\ ;\ \Delta}\ Contr_c$$

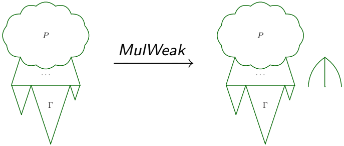
$$\frac{\vdash \Gamma\ ;\ \Delta_1 \quad \vdash \Delta\ ;\ \Delta_2}{\vdash \Gamma, \Delta\ ;\ \Delta_1, \Delta_2}\ Mix$$

$$\frac{\vdash \Gamma, \mathrm{A}\ ;\ \Delta_1 \quad \vdash \overline{\mathrm{A}}, \Delta\ ;\ \Delta_2}{\vdash \Gamma, \Delta\ ;\ \Delta_1, \Delta_2}\ Cut_l \qquad \frac{\vdash \Gamma\ ;\ \Delta, \mathrm{A}\overline{\mathrm{A}}}{\vdash \Gamma\ ;\ \Delta}\ Cut_r$$

$$\frac{\vdash \Gamma, \mathrm{A}\ ;\ \Delta_1 \quad \vdash \Delta\ ;\ \overline{\mathrm{A}}, \Delta_2}{\vdash \Delta\ ;\ \Gamma, \Delta_1, \Delta_2}\ Cut_c$$

**Fig. 4.** System FL.

- The stoup keeps track of the part that is sure to come from weakening and to allow the introduction of arbitrary linking configurations
- The intended interpretation of *MulWeak* is adding to the linking a *single* component of genus zero



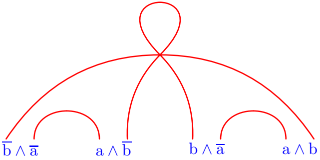

- Correctness for FL needs to accommodate the new connective for cut,
- We introduce another cospan in the syntactic category of F-prenets $\mathcal{F}Synt$
- We relax the definition of anodyne map to allow for

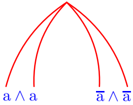
$$P \triangleright \Gamma \hookrightarrow\!\!\circ\!\!\twoheadrightarrow P \uplus Q \triangleright \Delta$$

- The sequentializability theorem and the correctness procedure are restated
- This time, for FL-correct net we have $|Ax| \leq |P| - |\mathcal{C}omp_P| + |\mathcal{G}en_P|$.

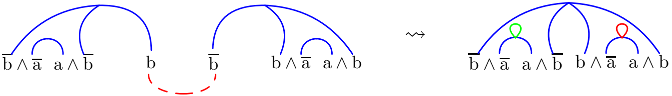
- F-prenets do form a category which is equivalent to the free Frobenius category generated by the set of literal types (an atom and its negation have the same "type")
- We can consider FL-correct (and CL-correct) nets to be a class of maps in that category, which is not closed under composition.

Some examples:

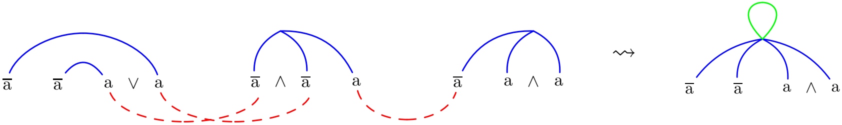



Correct F-prenets are calculus-dependent

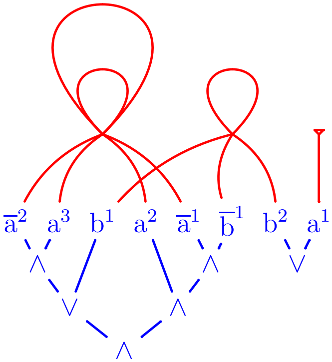



Minimal amount of loops that need to be added is not uniquely determined...

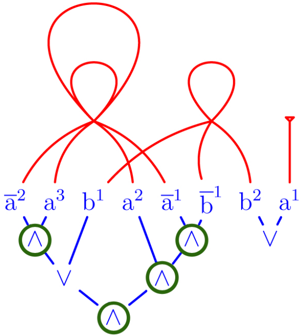



... and it depends on the order in which normalization is done.

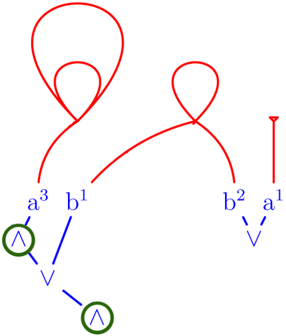
Conjunctive switching:



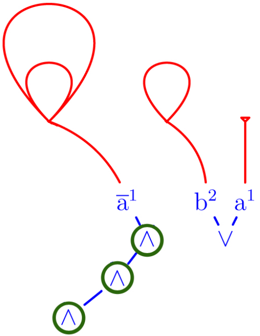
Conjunctive switching:



Conjunctive switching:



Conjunctive switching:



## Definition

For an F-prenet $P \triangleright \Gamma$ for which every switching yields a component with atoms of opposite polarity, we say that is a *sound* net.

Appears as the Lamarche-Strassburger condition on $\mathbb{B}$-nets in [LS05].

## Proposition

*A* (CL- / FL-) *correct F-net is sound.*

## Theorem

*Sound F-prenets define a category.*

The large category of F-prenets (as usual objects are formulas and a map $\mathrm{A} \to \mathrm{B}$ is a $P \triangleright \overline{\mathrm{A}}, \mathrm{B})$ has an order enrichment.

## Definition

Let $P \triangleright \Gamma, Q \triangleright \Gamma$ be two linkings over the same sequent. We write

$$P \leqslant Q$$

if

- $\mathcal{C}omp_P = \mathcal{C}omp_Q$ and
- $\mathcal{G}en_P \leqslant \mathcal{G}en_Q$, i.e, the genus functions are ordered pointwise.

**Theorem**

*The set of* FL*-correct nets is up-closed under the* $\leq$ *order.*

**Theorem**

*Let* $P \triangleright \Gamma$ *be a sound net. Then there exists an* FL*-correct linking* $Q \geqslant P$.

So we can obtain a category by cheating on our original goal and define a composition that "fattens" the one given by ordinary Frobenius categories.

## Definition

- F-prenet $P \triangleright \Gamma$
- $|\wedge|$ - the number of conjunctions
- $|\Gamma|$- the number of literals in $\Gamma$.
  We define a *bonus* to be the value

$$B(P \triangleright \Gamma) = |\wedge| \cdot |\Gamma| \cdot \frac{3^{|\wedge|+1} - 1}{2}$$

- $\lceil P \triangleright \Gamma \rceil^B$ is obtained by adding $B(P \triangleright \Gamma)$ many loops to every component of $P \triangleright \Gamma$,
- $\lfloor P \triangleright \Gamma \rfloor_B$ be the F-prenet obtained by subtracting $B(P \triangleright \Gamma)$ many loops from every component of $P \triangleright \Gamma$, if possible, $P \triangleright \Gamma$ otherwise.

## Theorem

*For every sound net $P \triangleright \Gamma$, $\lceil P \triangleright \Gamma \rceil^B$ is* FL-*correct.*

## Definition

Define:

$$(P \triangleright \Gamma, A) \diamond (Q \triangleright \overline{A}, \Sigma) = \begin{cases} (P \triangleright \Gamma, A) \circ (Q \triangleright \overline{A}, \Sigma), & \text{if } (P \triangleright \Gamma, A) \circ (Q \triangleright \overline{A}, \Sigma) \\ & \text{is either } P \triangleright \Gamma, A \text{ or} \\ & Q \triangleright \overline{A}, \Sigma \\ \left\lceil \lfloor P \triangleright \Gamma, A \rfloor_B \circ \lfloor Q \triangleright \overline{A}, \Sigma \rfloor_B \right\rceil^B, & \text{otherwise} \end{cases}$$

where the $\circ$ is the standard "Frobenius" composition.

## Theorem

*The $\diamond$ composition of two correct F-nets yields a correct net, it is associative and has a unit for each F-prenet.*

### Theorem

*Every F-prenet in the category of sound F-prenets is obtained by cut elimination/Frobenius composition applied on correct F-nets.*

Thank you!

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