Stably supported quantales with a given support

David Kruml
Masaryk University, Brno
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INVESTMENTS IN EDUCATION DEVELOPMENT

Quantales

Sup-lattice — complete lattice, homomorphisms preserve arbitrary joins.
Quantale — sup-lattice with associative multiplication which distrubutes over joins.
Involutive quantale — quantale + involution, provided that

$$a^{**} = a,$$
$$(ab)^* = b^* a^*,$$
$$\left(\bigvee a_i\right)^* = \bigvee a_i^*.$$

Quantales are residuated (adjoints of the right/left action exist).
0 — bottom element, 1 — top element, e — unit (need not exists), $r \cdot 1 \leq r$ — right-sided element, $1 \cdot l \leq l$ — left-sided element, both rules — two-sided element.

Examples of involutive quantales

(1) Every frame is an involutive quantale with multiplication $\wedge$ and trivial involution.
(2) Binary relations Rel X on X set with $\bigvee, \circ, ^*$.

$$\rho_A = X \times A, \qquad \lambda_A = A \times X$$

(3) (J. Wick Pelletier, J. Rosický 97) Quantale of endomorphisms $\mathcal{Q}(S)$ on a sup-lattice S.

$$\rho_a(b) = \begin{cases} a & b \neq 0, \\ 0 & b = 0, \end{cases} \qquad \lambda_a(b) = \begin{cases} 1 & b \not\leq a, \\ 0 & b \leq 0. \end{cases}$$

If S is self-dual with a duality $'$, then $\mathcal{Q}(S)$ is involutive:

$$\alpha^*(x) = \left(\bigvee\{y \mid \alpha(y) \leq x'\}\right)'$$

Stably supported quantales

(Resende 2003) *Support* — sup-lattice endomorphism $\varsigma : Q \to Q$, s.t.

$$\varsigma a \leq e,$$
$$\varsigma a \leq a^* a,$$
$$\varsigma a \leq \varsigma a a$$

for any $a \in Q$.
The support is called *stable* if

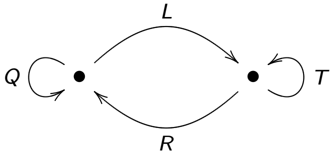
$$\varsigma a = e \wedge a$$

for every a.
Examples: (1) Rel X.
(2) Quantales on étale groupoids.
Remark: $\downarrow e$ is a frame. SSQ is an involutive quantale "with enough projections".

Problems

(1) For a given self-dual sup-lattice S, is there a Girard quantale where S is the lattice of right- (left-) sided elements? [Yes, J. Egger & D. Kruml 2009.]
(2) (A. Palmigiano) F, what are the SSQ where F appears as $\downarrow e$?

Triads categorically



$$T \otimes T \to T \qquad Q \otimes Q \to Q$$
$$R \otimes T \to R \qquad Q \otimes R \to R$$
$$T \otimes L \to L \qquad L \otimes Q \to L$$
$$L \otimes R \to T \qquad R \otimes L \to Q$$

16 pentagonal coherence axioms
+ some of the 6 triangular axioms for unital objects.

Solutions

$$R \otimes T \otimes L \rightrightarrows R \otimes L \longrightarrow Q_0$$

$$\begin{array}{ccccc}
Q_1 & \longrightarrow & L \multimap L & \rightrightarrows & (T \otimes L) \multimap L \\
\downarrow & & \downarrow & & \\
R \multimap R & \longrightarrow & (L \otimes R) \multimap T & & \\
\downdownarrows & & & & \\
(R \otimes T) \multimap R & & & &
\end{array}$$

$Q_0 = R \otimes_T L$, $Q_1 = \{(\phi, \psi) \mid \phi(l)r = l\psi(r)$ for any $r \in R, l \in L$, and ϕ, ψ are T-module endomorphisms of L, R, resp.$\}$.

Involutive triads

Triad (L, T, R) is *involutive* if T is commutative, and there is a T-module isomorphism $L \cong R$ making the inner product $L \times R \to T$ symmetric.
If (L, T, R) is involutive, then Q_0, Q_1 are involutive quantales: $(r \otimes l)^* = l \otimes r, (\alpha, \beta)^* = (\beta^*, \alpha^*)$.
Practical assumptions: $L = R$ and $T \subseteq L$ is an open frame homomorphism, i.e. it has both adjoints and satisfies Frobenius reciprocity condition

$$|a \wedge t| = |a| \wedge t$$

for $a \in F, t \in T$ and left adjoint $| - | : L \to T$.
It induces an involutive triad (L, T, L) with inner product $(l, r) \mapsto |l \wedge r|$.

Main result

Let $T \subseteq L$ be an open subframe. Then solution Q_1 of involutive triad (L, T, L) is a SSQ, s.t. L, T appears as lattices of left/two-sided elements respectively (and thus L as the support as well).

Examples: (1) If $L = T, |-| = id$, then $Q_1 \cong L$.

(2) $T = 2, |0| = 0$ and $|x| = 1$ otherwise.

Remark: The construction works also for OML L and its centre T with the same assumption (the solution is no more a SSQ).

Thank you!