Topological duality for arbitrary lattices via the canonical extension

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- Goal: Reconsider topological duality for arbitrary lattices in the light of the developments of canonical extension.

Canonical extensions and duality

Historical overview

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- Gehrke, Harding (2001): Canonical extension of arbitrary lattice – Hartung duality in algebraic form

Canonical extensions and duality

Distributive lattices

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- The canonical extension can be captured by purely lattice-theoretic properties (without referring to the duality):

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Theorem

Any lattice L can be embedded in a complete lattice L^δ in a dense and compact way:

Moreover, the completion L^δ is the unique dense and compact completion of L up to isomorphism.

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- *(dense) The lattice L both $\bigvee\bigwedge$-generates and $\bigwedge\bigvee$-generates L^δ,*
- *(compact) If $S, T \subseteq L$ and $\bigwedge S \leq \bigvee T$ in L^δ, then there exist finite $S' \subseteq S$, $T' \subseteq T$ such that $\bigwedge S' \leq \bigvee T'$ in L.*

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Topological duality for lattices

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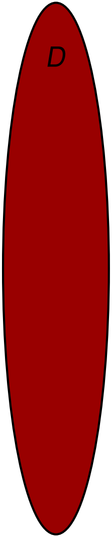

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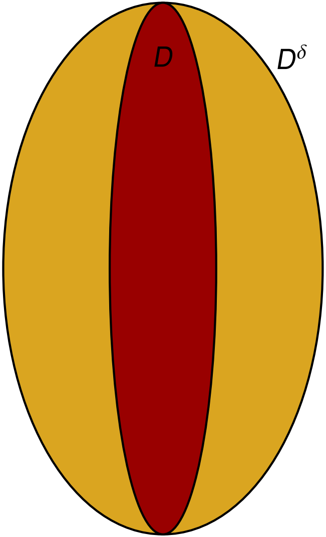

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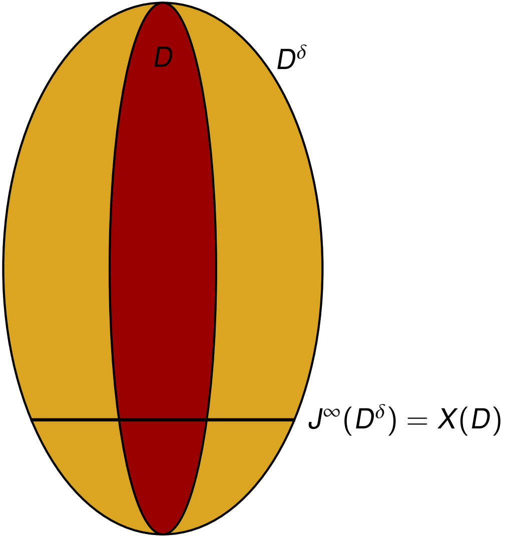

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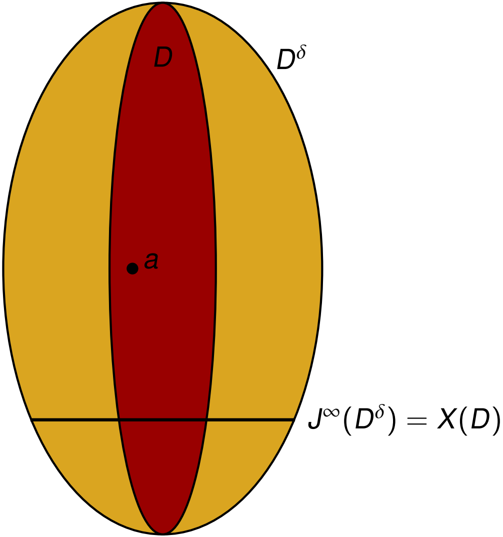

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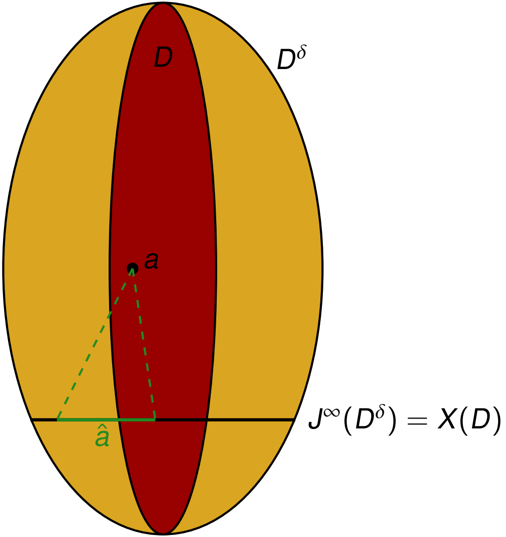

Duality for lattices

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BA		

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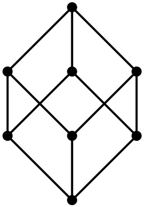

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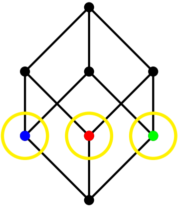

J^∞

Set

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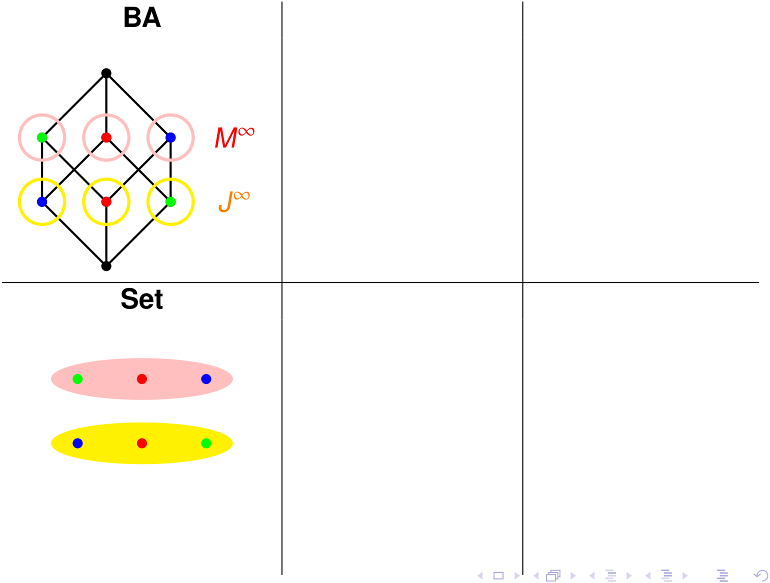

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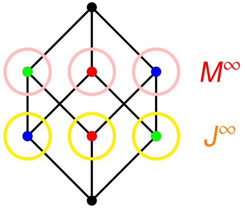		
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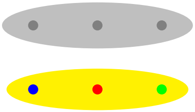		

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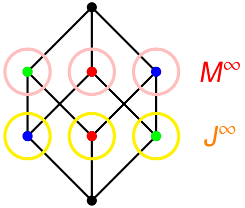		
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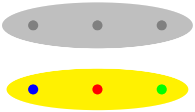		

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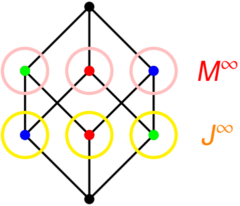
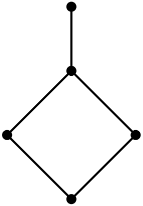


DLat



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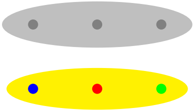

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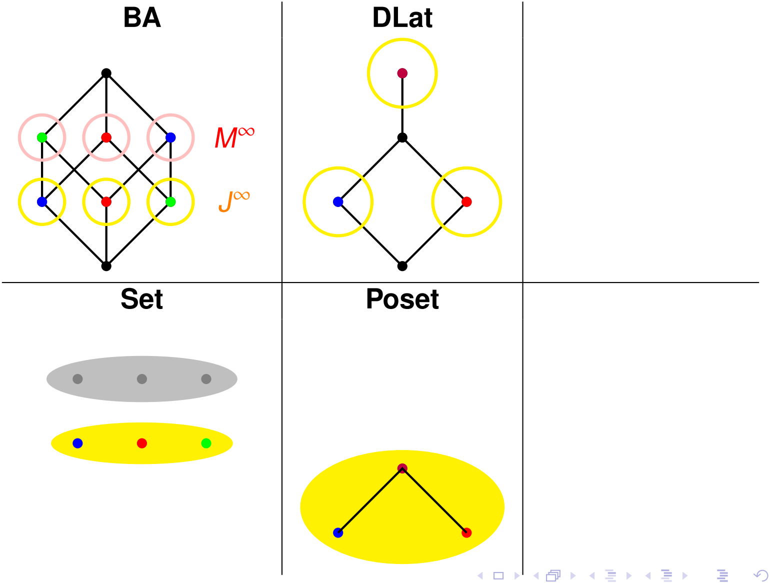

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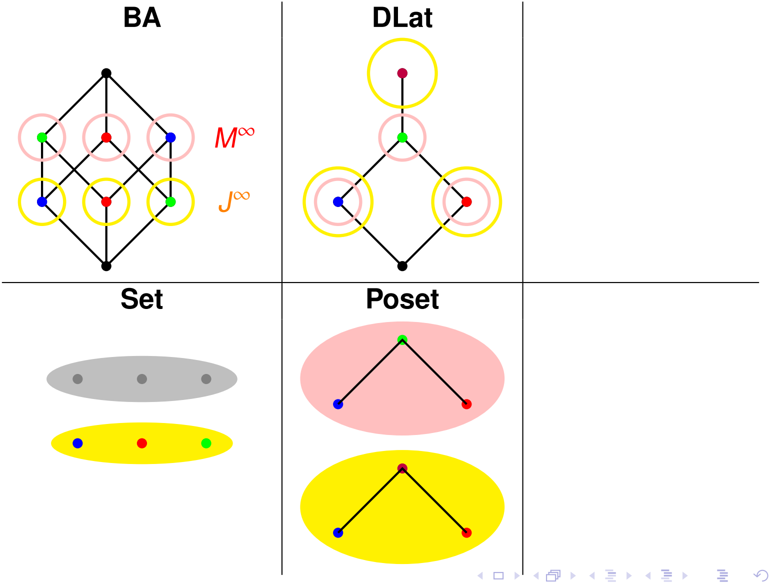

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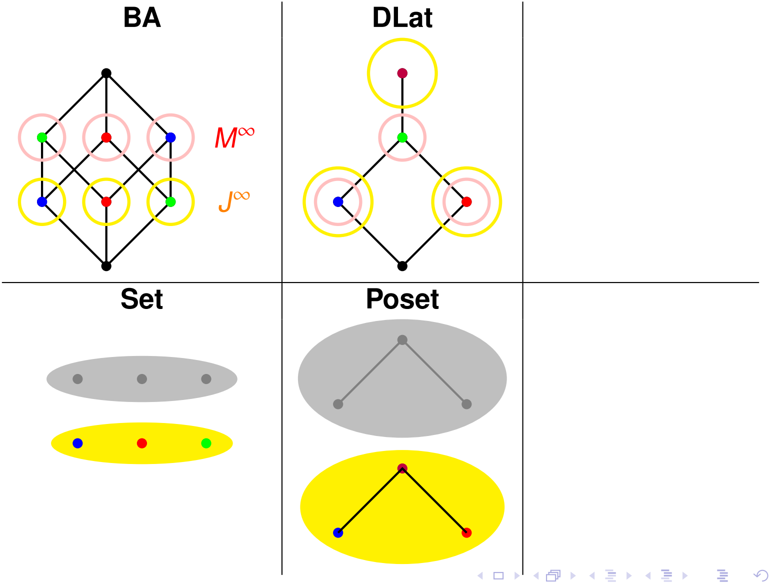

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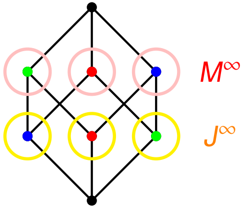	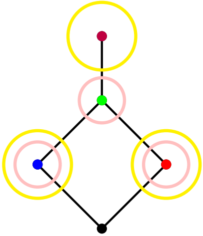	
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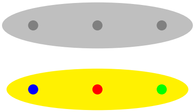	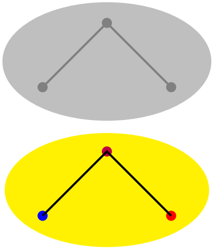	

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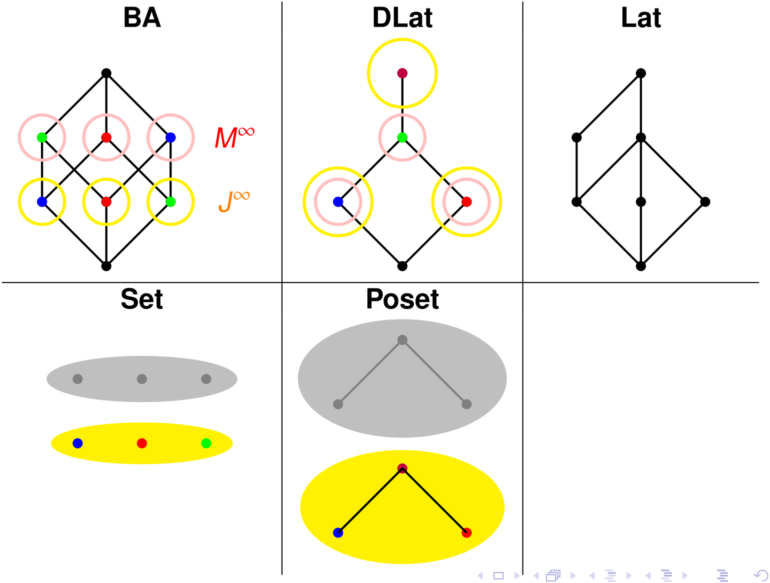

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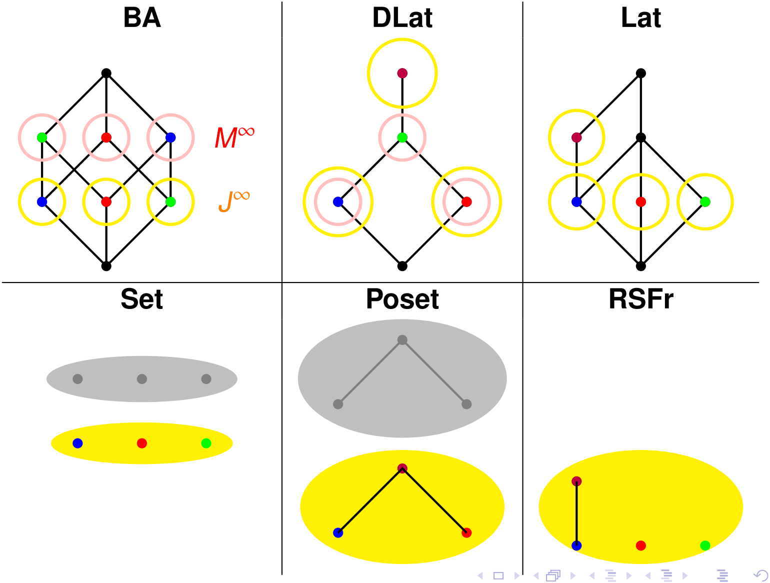

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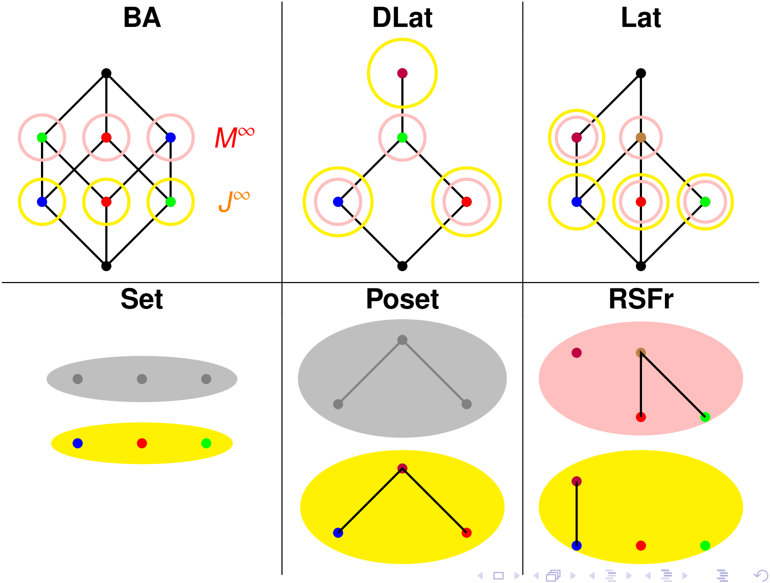

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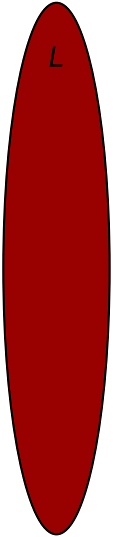

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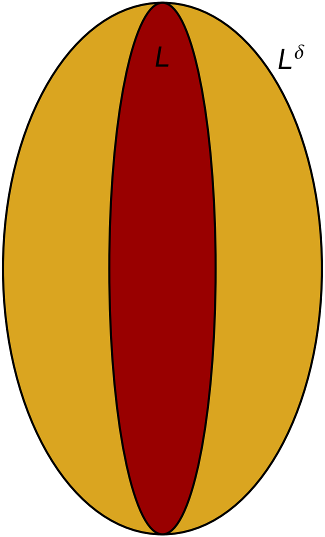

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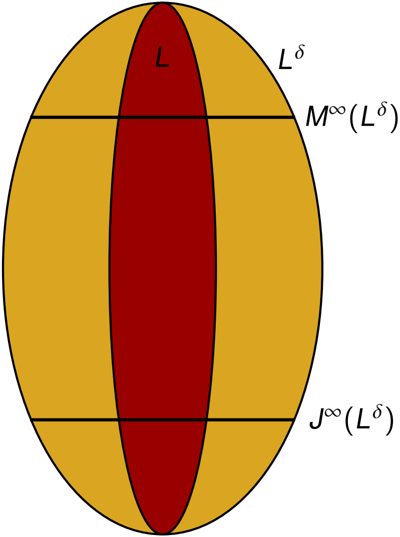

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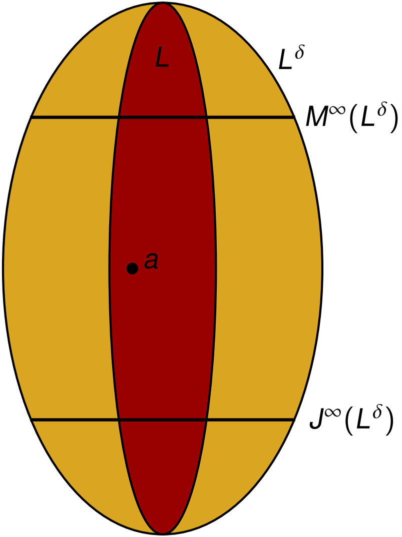

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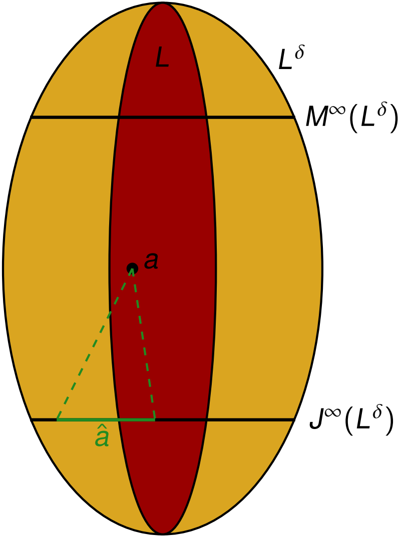

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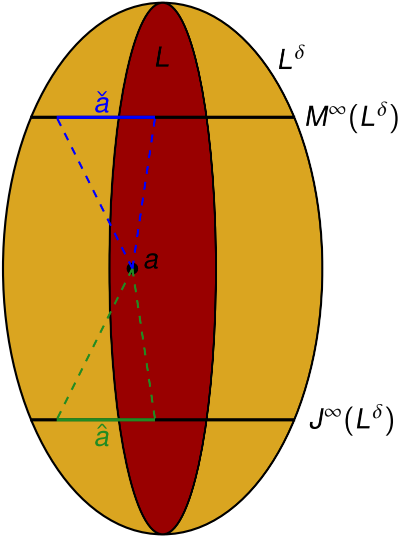

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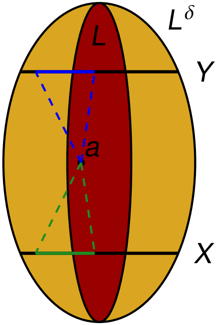

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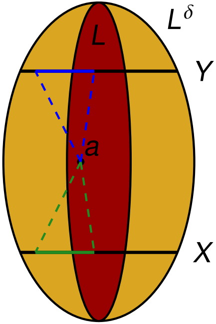

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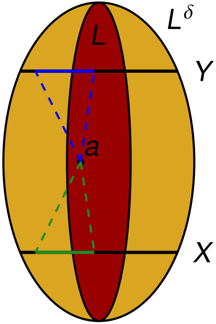

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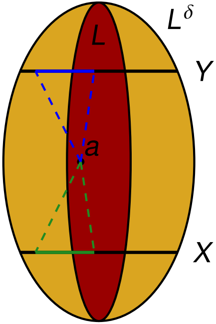

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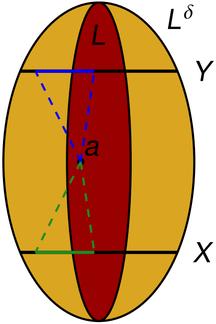

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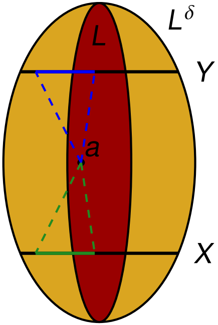

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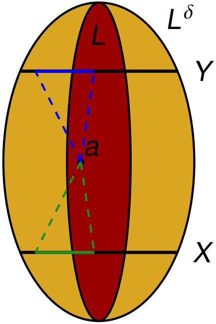

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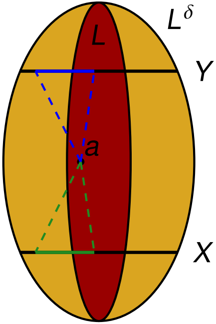

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- R: order of L^δ restricted to $X \times Y$.

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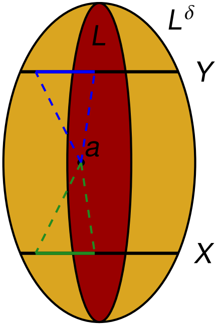

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- Topology on X: $\{\hat{a} : a \in L\}$ subbasis of closed sets,
- Topology on Y: $\{\check{a} : a \in L\}$ subbasis of closed sets.
- R: order of L^δ restricted to $X \times Y$.
- L distributive $\Rightarrow X \cong Y$ are spectral dual spaces of L.

Topological duality for lattices

Theorem (Hartung (1992))

The lattice L is isomorphic to the lattice of doubly closed, Galois-stable subsets of X (or Y).

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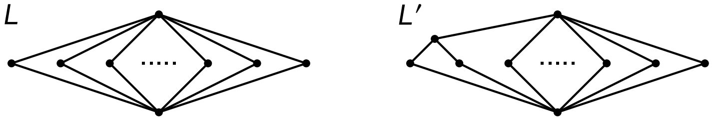
Properties of the dual space

Not always sober



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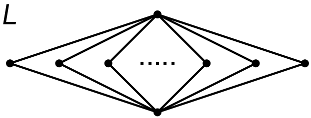

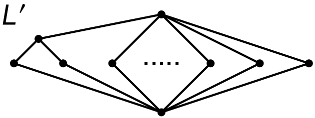

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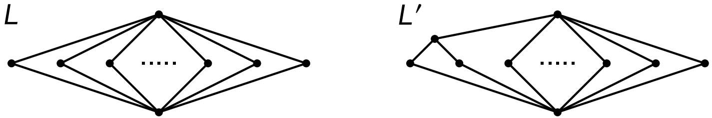
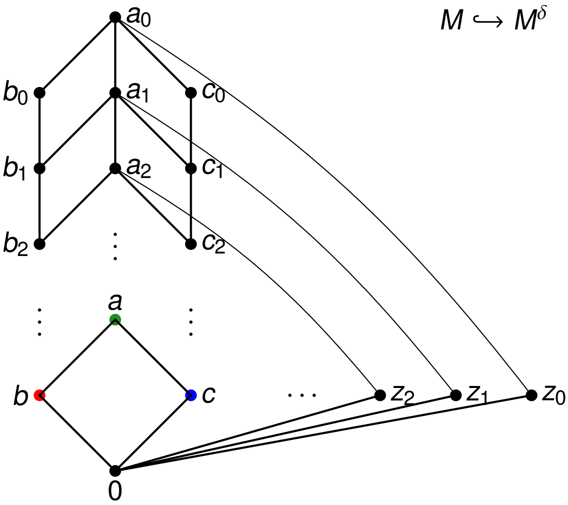


- Here, $L^\delta = L$, $(L')^\delta = L'$,
- $X_L \cong \mathbb{N} \cong X_{L'}$, where
- topology on X_L (and $X_{L'}$) is generated by taking singletons to be closed: cofinite topology (not sober).

Properties of the dual space

Compact-opens not always closed under ∩

$M \hookrightarrow M^\delta$



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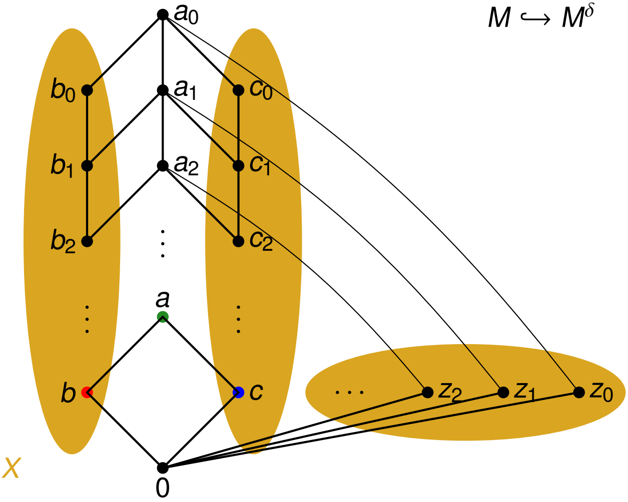

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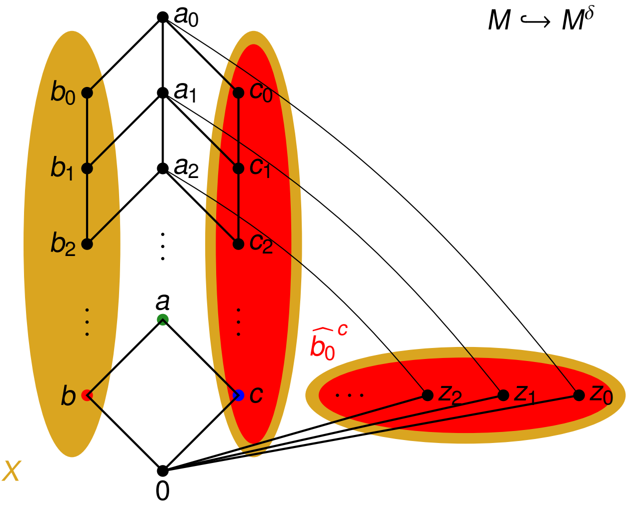

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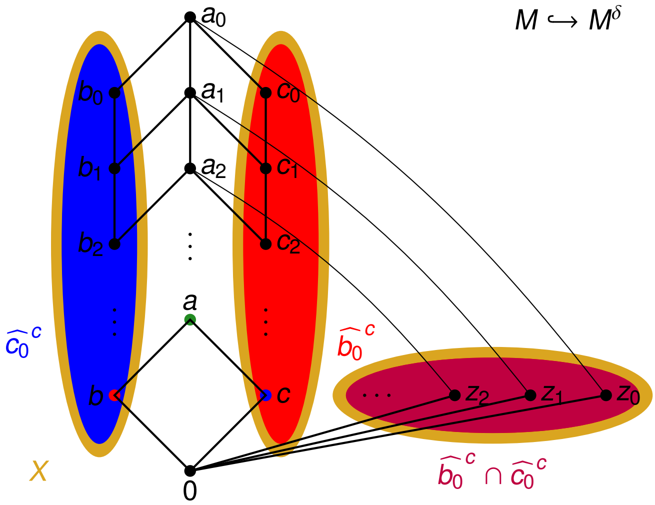

Topological duality for lattices

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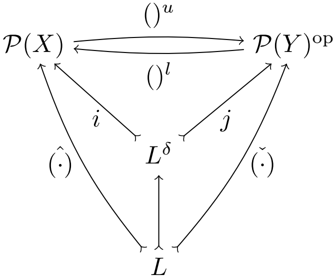
- On the other hand, by Hartung's results, a lattice L is represented by the bases of the spaces X and Y, which are distributive lattices.
- We now investigate an approach to topological duality for lattices which makes the connection to distributive lattices explicit.

Representing canonical extension

Back to distributive lattices



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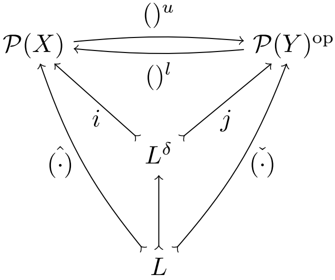

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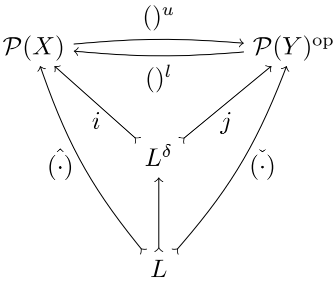

- We embed L into a perfect lattice L^δ.
- In turn, L^δ is represented as Galois-stable elements of $\mathcal{P}(X)$ and $\mathcal{P}(Y)^{\mathrm{op}}$.
- In particular, L embeds in $\mathcal{P}(X)$ and $\mathcal{P}(Y)^{\mathrm{op}}$ via $\hat{()}$ and $\check{()}$.

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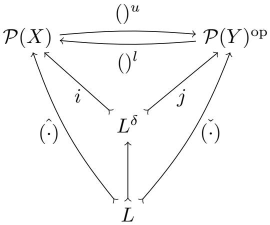

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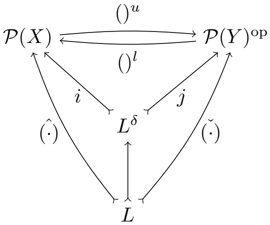

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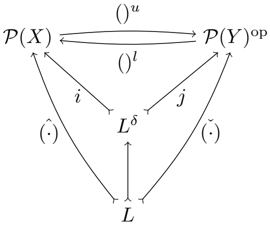

- Let $D^\wedge(L)$ and $D^\vee(L)$ sublattices of $\mathcal{P}(X)$ and $\mathcal{P}(Y)^{\mathrm{op}}$ generated by $\hat{L}$ and $\check{L}$.

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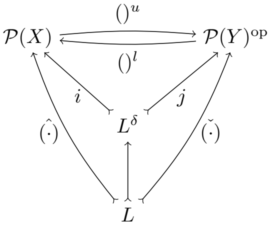

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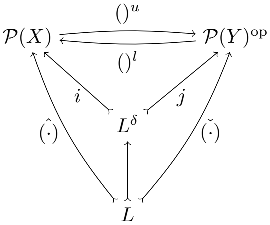

- Let $D^\wedge(L)$ and $D^\vee(L)$ sublattices of $\mathcal{P}(X)$ and $\mathcal{P}(Y)^{\mathrm{op}}$ generated by $\hat{L}$ and $\check{L}$.
- Note: $D^\wedge(L)$ and $D^\vee(L)$ are bases for closed sets of X_L and Y_L.
- Question: Algebraic description of $D^\wedge(L)$ and $D^\vee(L)$?

Distributive envelope

Definition

A finite subset $M \subseteq L$ is join-admissible if, for all $a \in L$,

$$a \wedge \bigvee M = \bigvee_{m \in M} a \wedge m.$$

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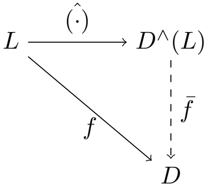
Theorem

The extension $\hat{()} : L \to D^{\wedge}(L)$ is the free distributive meet- and admissible-join-preserving extension of L:



The dual statement holds for $D^{\vee}(L)$.

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Algebraic construction

Definition (cf. Bruns & Lakser (1970))

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Theorem

The poset of finitely generated a-ideals, ordered by inclusion, is isomorphic to $D^{\wedge}(L)$ as a $(\wedge, a\vee)$-extension.

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Use for duality

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- There is more to be said (not here), using uniform spaces.

Morphism correspondence

Finite distributive case, ∨-preserving maps

$$yR_f x \iff y \leq f(x)$$

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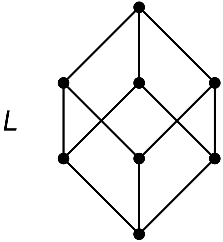

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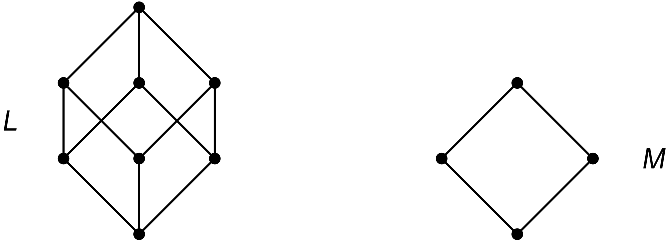

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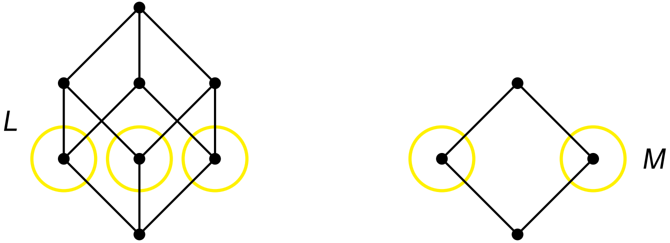

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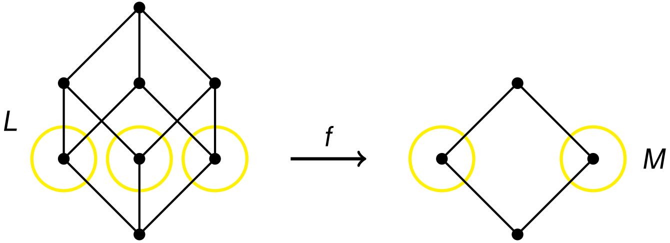

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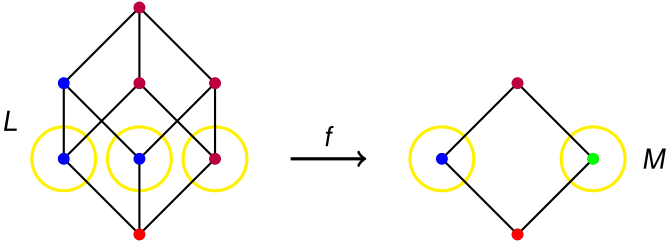

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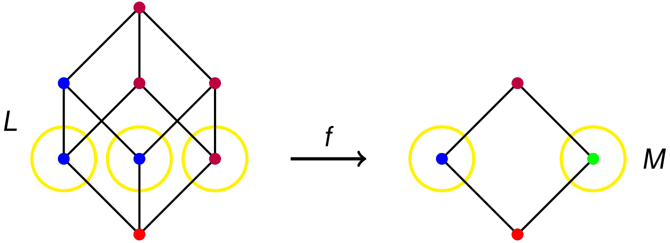

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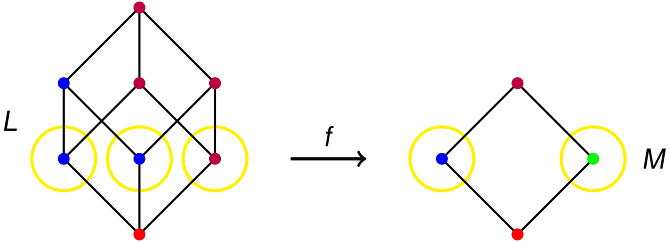

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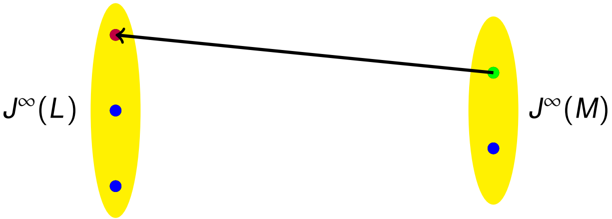

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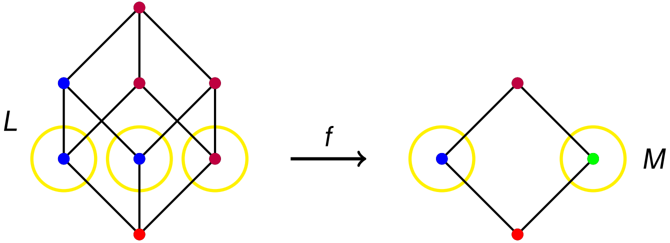

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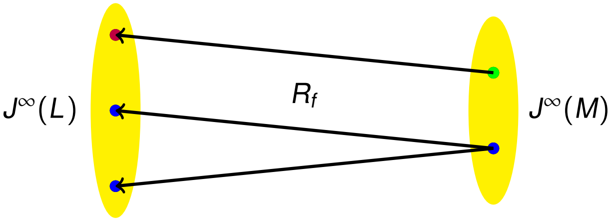

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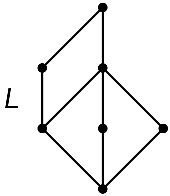

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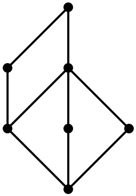
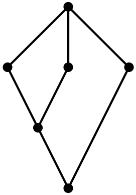
Finite lattice case, ∨-preserving maps

L



M



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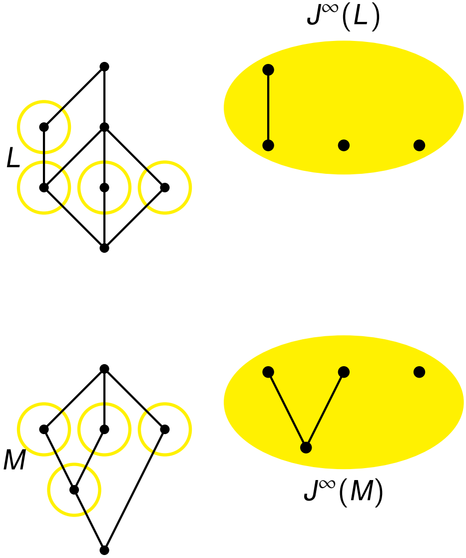

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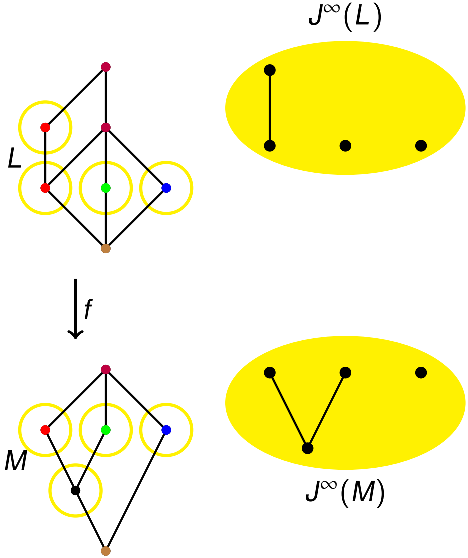

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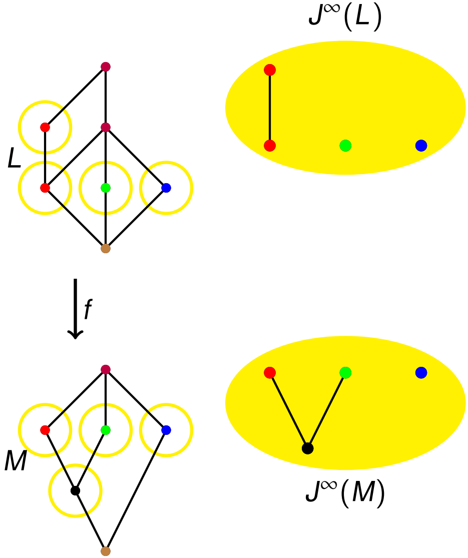

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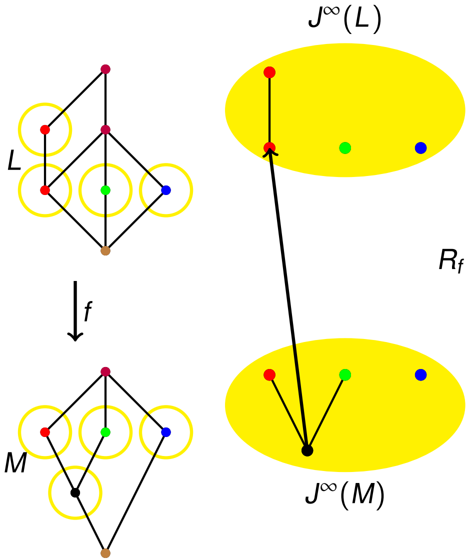

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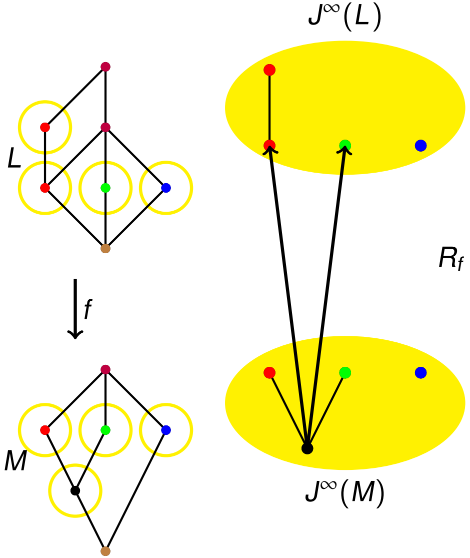

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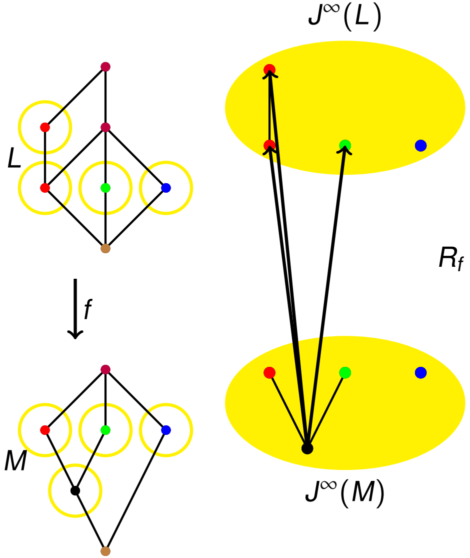

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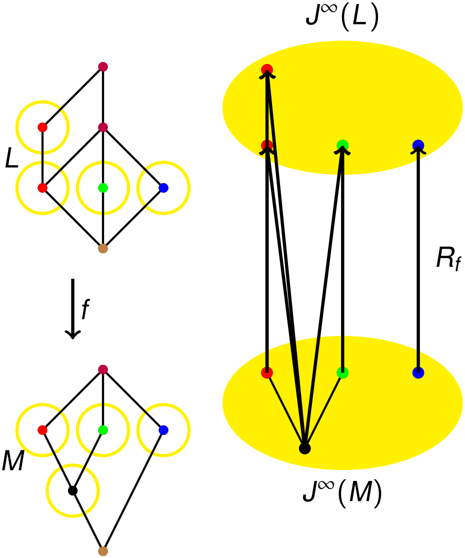

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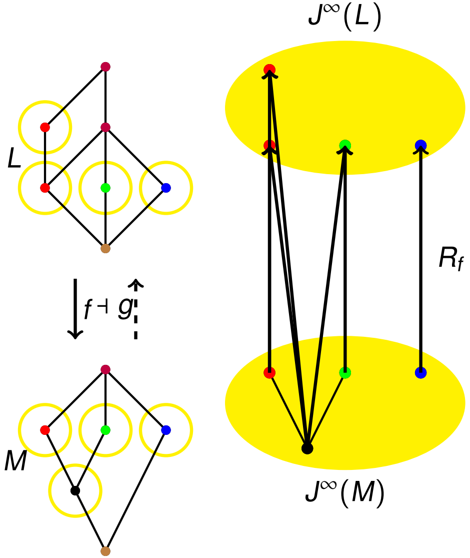

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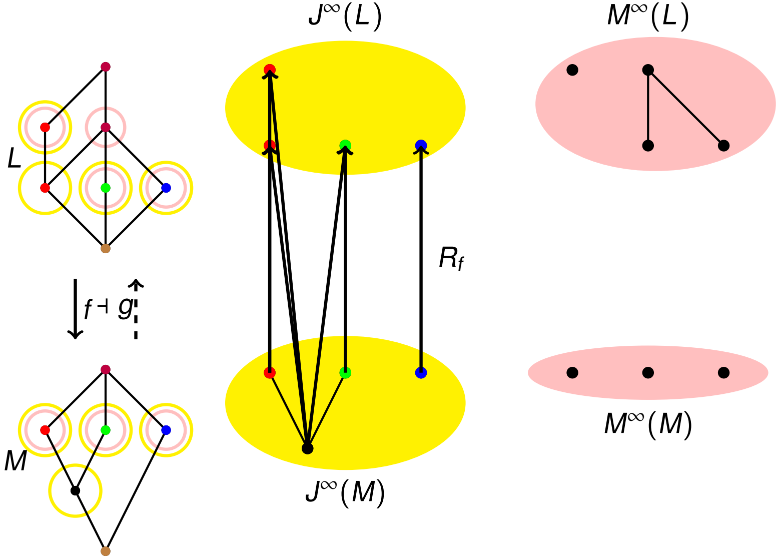

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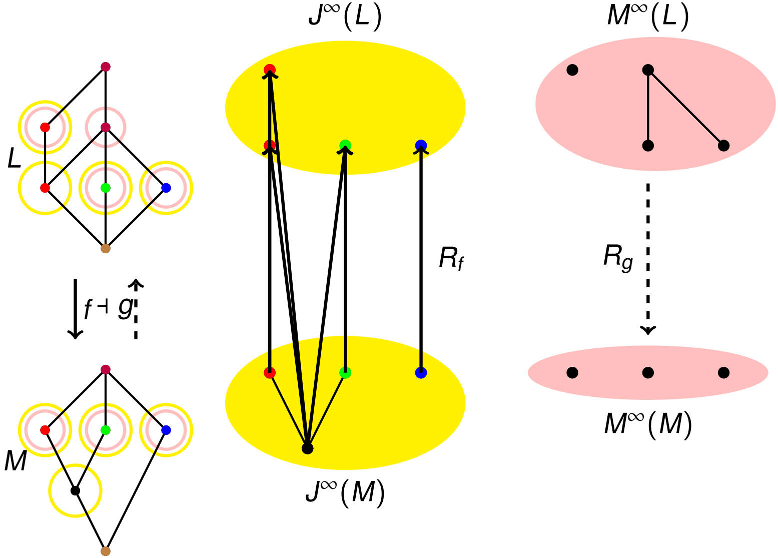

Morphism correspondence

Distributive lattice case

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Morphism correspondence
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Prop. f^δ is a $\bigvee$-homomorphism $\iff$ g sends $J^\infty(M^\delta)$ to $J^\infty(L^\delta)$.

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Prop. f^δ is a $\bigvee$-homomorphism $\iff$ g sends $J^\infty(M^\delta)$ to $J^\infty(L^\delta)$.

- Thus, if f is a lattice homomorphism, g restricts to a function $J^\infty(M^\delta) \to J^\infty(L^\delta)$: the usual dual morphism of f.

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Distributive lattice case

- For distributive lattices L and M:
  - If $f : L \to M$ a $\wedge$-homomorphism, let $f^\delta : L^\delta \to M^\delta$ its canonical extension.
  - Then f^δ is a $\bigwedge$-homomorphism, so let $g : M^\delta \to L^\delta$ its lower adjoint.

Prop. f^δ is a $\bigvee$-homomorphism $\iff$ g sends $J^\infty(M^\delta)$ to $J^\infty(L^\delta)$.

  - Thus, if f is a lattice homomorphism, g restricts to a function $J^\infty(M^\delta) \to J^\infty(L^\delta)$: the usual dual morphism of f.
- For arbitrary lattices, the Proposition may fail to hold.

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Arbitrary lattice case

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$$\text{If } g(x) \leq \bigvee T \text{ then } \forall y \in Y \text{ s.t. } x \not\leq y :$$
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- Duals of general lattice homomorphisms: given by relations rather than functions.

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- Because the spaces are not well-behaved, we are led to consider other options $\rightsquigarrow$ distributive envelope of a lattice.
- General morphisms remain hard to handle, but the canonical extension perspective allows for the use of correspondence methods.

Further work

(An incomplete wish list)

- The relation between the two candidates for dual objects: (X, Y, R) and (X_S, Y_S, R_S).

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- The relation between the two candidates for dual objects: (X, Y, R) and (X_S, Y_S, R_S).
- A nice(r) description of morphisms on the side of topological frames.
- Applications to obtain topological semantics for substructural logics which are not Kripke-complete.