A Graphical Game Semantics for MLL

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UMR 5127

Context

- Theory of prog. languages
  dynamics, composition of programs.
  - categorical models: good for composition, hide dynamics,
  - game semantics:
    - handle both,
    - not expressed in a general structure (usual framework: SMCC, CCC).

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- Proof theory for LL.
- Graphical game semantics for LL.
- Graphical game semantics for MLL.

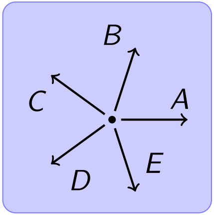
This attempt

- Graphical:
  - positions are graphs, moves transform graphs,
  - embedded notion of sites.
- Parallel composition as amalgamation in a stack
- Composition decomposed as amalgamation+hiding.

Graphical sequents

A sequent $\vdash A, B, C, D, E$:

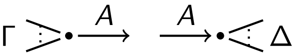
- A, B, C, D, E formulae: $A, B ::= \mathbb{A} \mid 1 \mid A \otimes B \mid \mathbb{A}^{\perp} \mid \perp \mid A \parr B$
- graphical representation.



Cut rule:

$$\frac{\Gamma \vdash A \qquad A \vdash \Delta}{\Gamma \vdash \Delta}$$

Idea: connect A output to A input:



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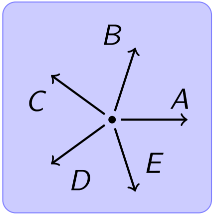

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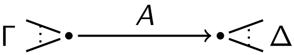
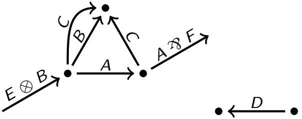


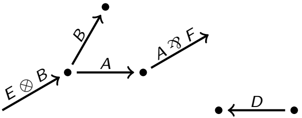
(Pre-)positions

Pre-position: graph which is:

- oriented and labelled by formulae, up to duality.
- partial: edges possibly without source/target.



Positions: topologically (i.e., forget orientation) acyclic:



Topology

See pre-positions as topological spaces:

- points: vertices and edges,
- open sets: for any vertex, contain all adjacent edges.

With neighbourhoods:

- neighbourhood of an edge: the edge itself,
- neighbourhood of a vertex: vertex plus adjacent edges.

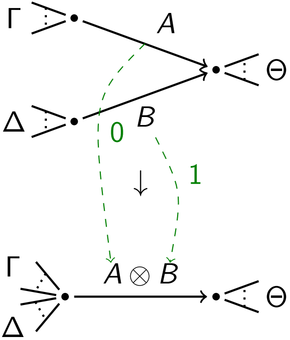
Morphisms

- Pre-positions are topological spaces plus edges and orientation.
- What are continuous maps?
  - almost graph morphisms, preserving paths,
  - may collapse an edge to a vertex,
  - and a few more things, which we will exclude.
- What about labels?
  - occurrences: formulae in the domain are sub-formulae of the image.
- Orientation: obeys duality.

Examples of morphisms

Morphism $f : U \to V$:

- f continuous $U \to V$,
- occurrences o_f: "where formulae in U come from."



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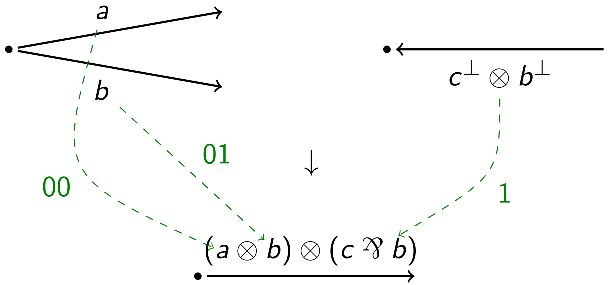

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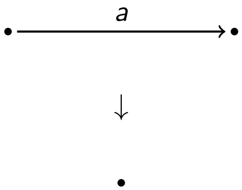

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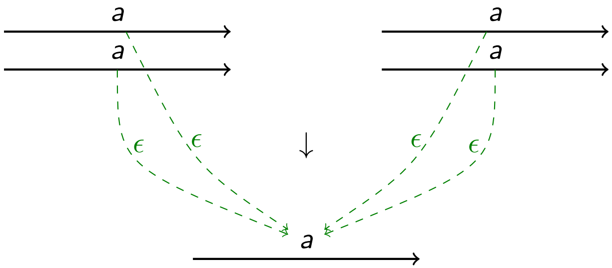
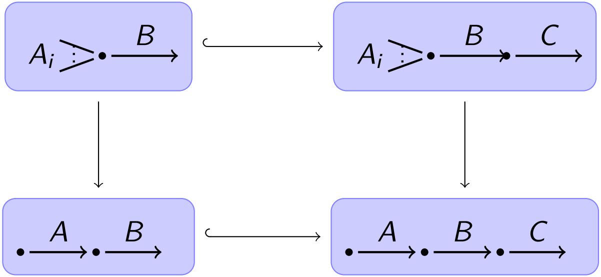


Pullbacks along embeddings

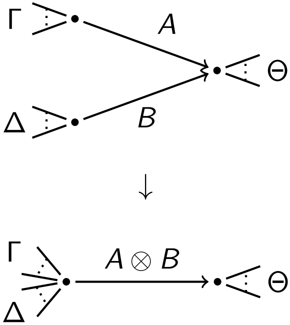
- An (open) embedding $U \hookrightarrow V$:
  - f open embedding: no edge added to vertices in U,
  - o_f constant, empty occurrence.
- Pullbacks along embeddings (= inverse image):



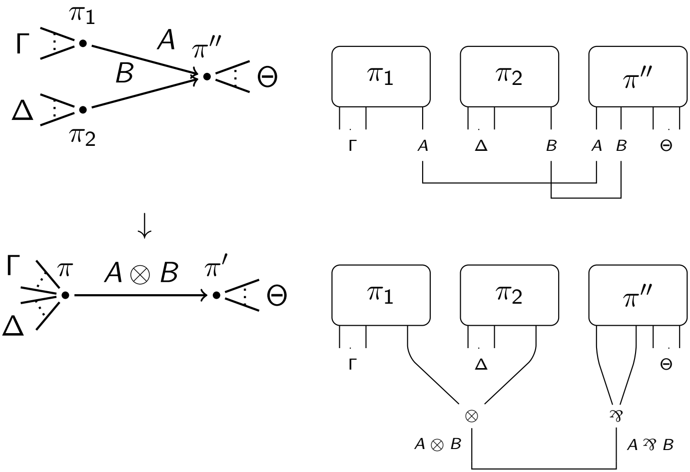
- Restriction = pullback along an embedding (⇝ defined up to iso).

Moves

- Move: transforms a position into another position.
- Basic move, tensor :



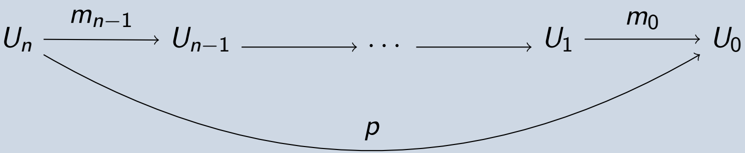
Moves and proof nets



Plays

Definition (Play)

Finite composite of moves:



- Guerini, Masini parsing criterion for proof nets

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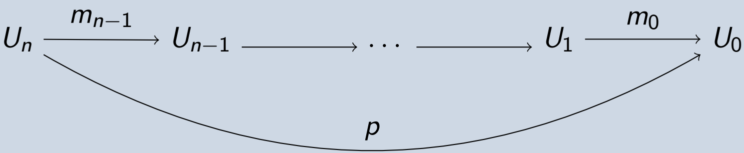

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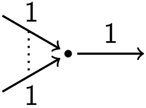
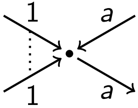
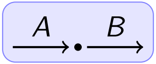
Won positions:

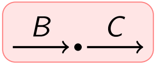




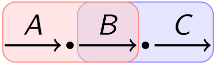
Amalgamating plays

Plays p and q over $A \vdash B$ and $B \vdash C$





Can we amalgamate them to a play r over $A \vdash B \vdash C$?



Stacks

- Stacks: weak version of sheaves.
- (U_i) conver of a site S
- Family of $p_i\colon V_i \to U_i$
- Compatible up to coherent iso:

$$\xi_{ij}\colon p_{i|j} \cong p_{j|i}$$

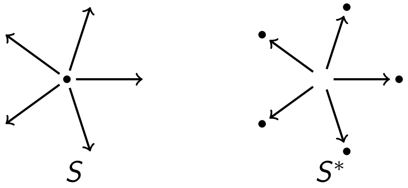
- cocycle condition:

$$\xi_{jk|i}\xi_{ij|k} = \xi_{ki|i}$$

- then there exists a unique (up to coherent iso) $p\colon V \to U$ s.t. $p_{|i} = p_i$.

Recasting DR criterion

A sequent, and its adverse position:

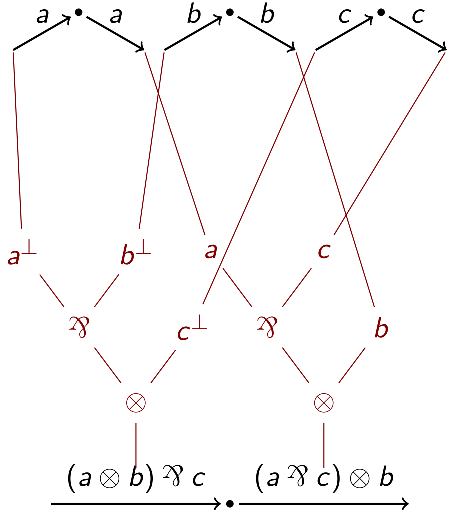


Theorem (Characterization of plays)

Well-balanced morphism p is a play iff, for any adverse play q compatible with p, $p|q$ is connected and acyclic.

- Adverse plays encode DR switchings of proof-structures,
- Implies that plays are a sub-stack of the stack of morphisms of pre-positions.

Example of a switching



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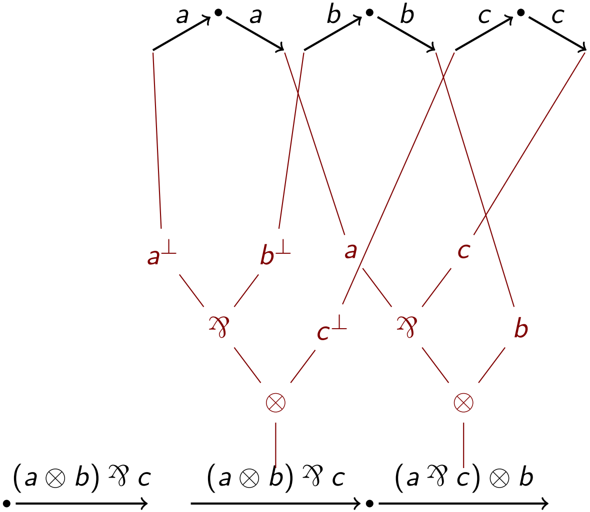

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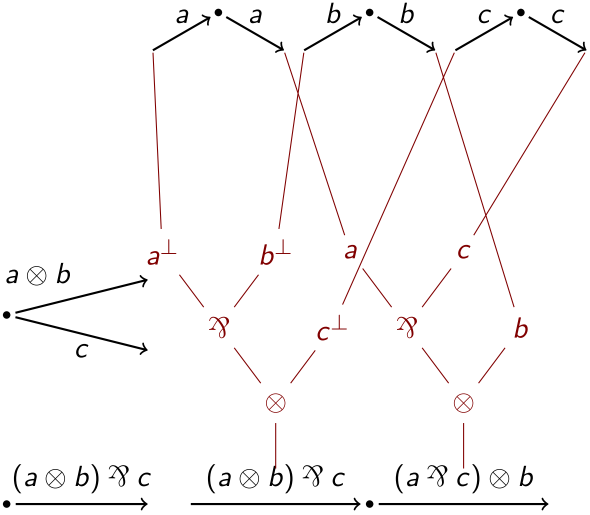

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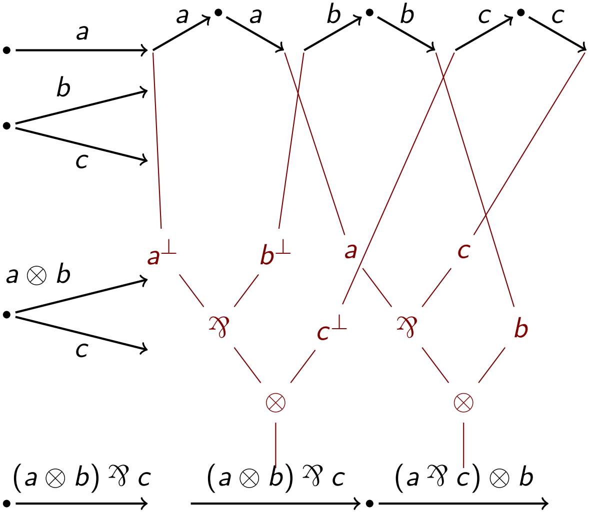

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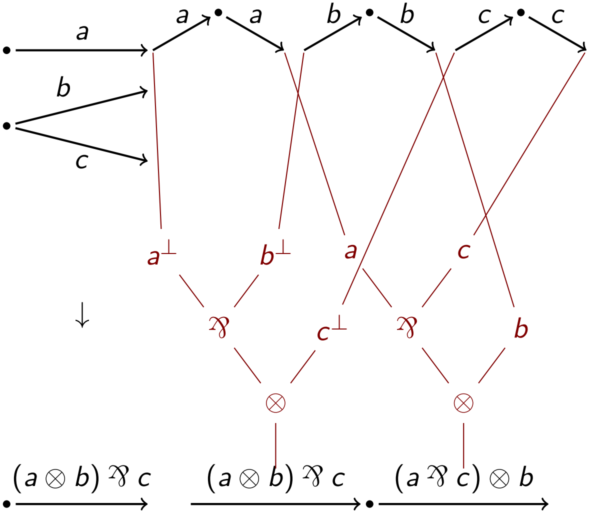

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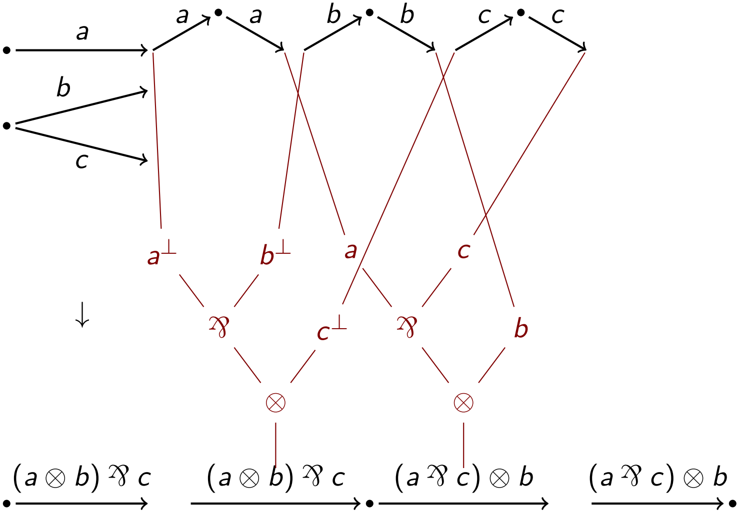

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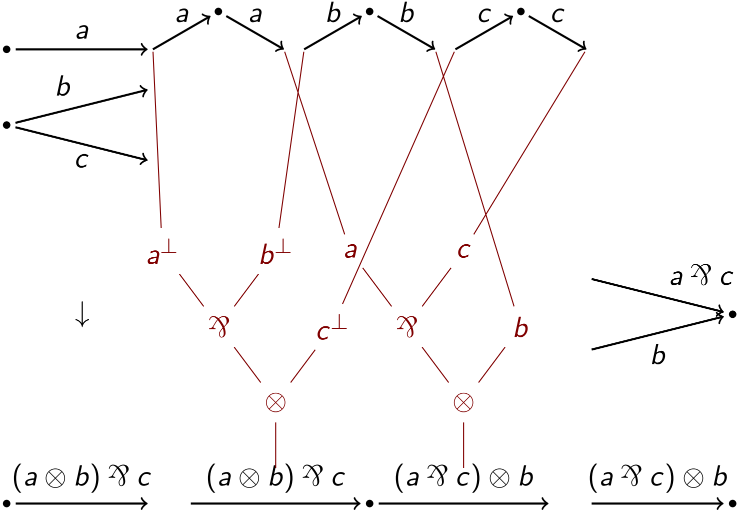

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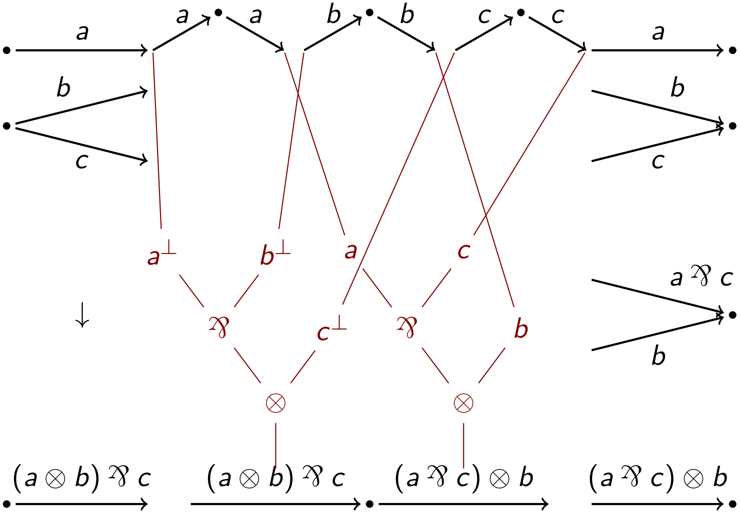

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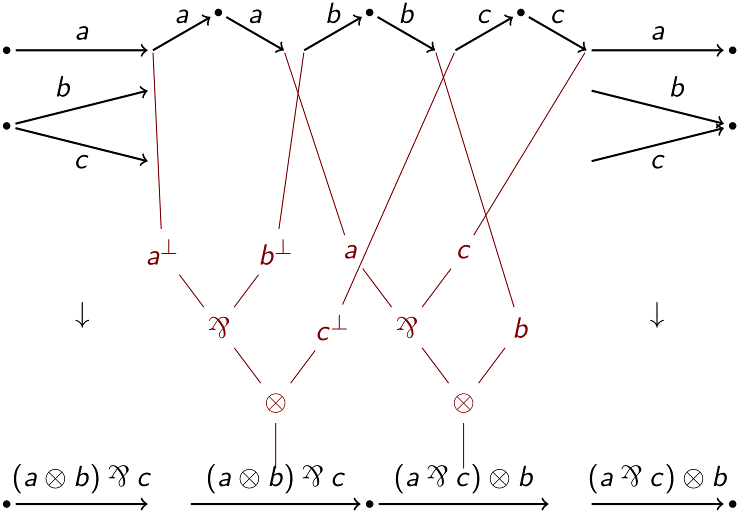

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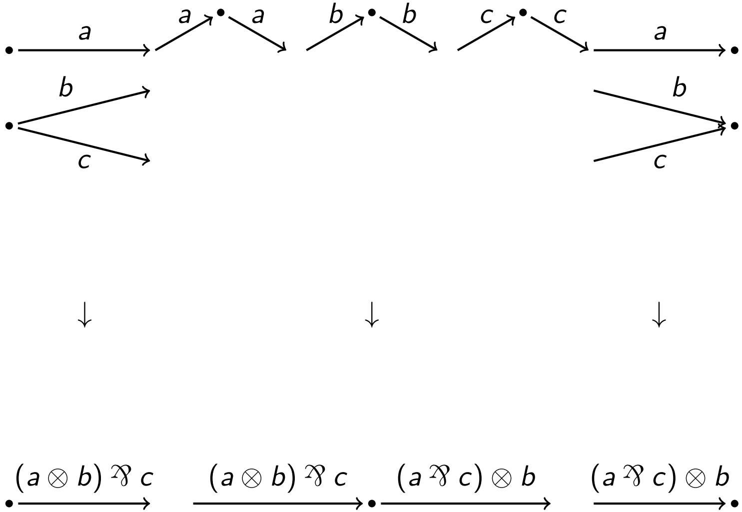

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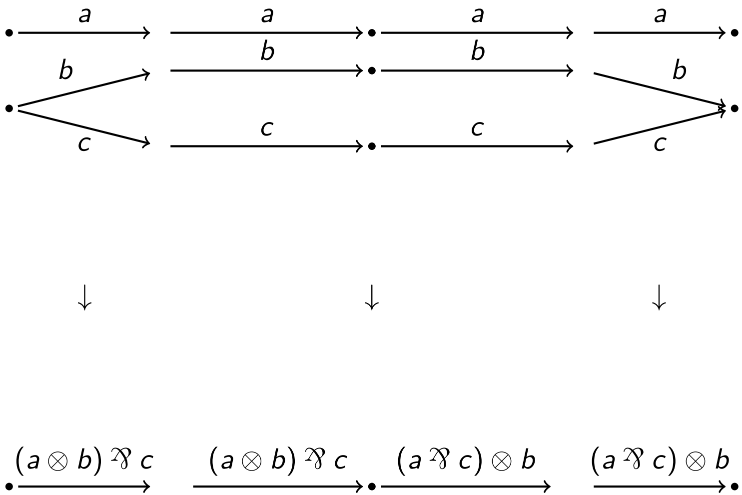


$\mathcal{G}(\mathbb{A})$

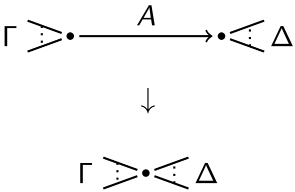
Define a category $\mathcal{G}(\mathbb{A})$:

- objects are MLL formulae,
- morphisms $A \to B$ are won plays over $A \vdash B$.
  - composition?

Composition in $\mathcal{G}(\mathbb{A})$

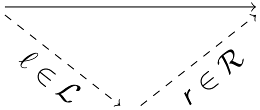
Amalgamation of plays produces morphisms over positions like:

Hiding the middle part with a sequence of cuts:



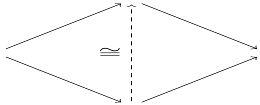
Factorising cuts

Every morphisms factors as $r \circ \ell$:



- $\mathcal{L}$: generalized cuts,
- $\mathcal{R}$: "cut-free" morphisms, i.e., never collapse edges.

Uniquely:



Split $\star$-autonomous category

Theorem

$\mathcal{G}(\mathbb{A})$ isomorphic to Hugues' free split $\star$-autonomous category generated by $\mathbb{A}$.

- Associativity of composition from diagram chasing using general properties of stacks and factorisation systems.

Further work

- Suitable notion of strategy to deal with Trimble's rewiring
  - free $\star$-autonomous, not only "split".
- Full LL:
  - provability ok,
  - proof theory?
- Restricted λ-calculus with contractions
  - Selinger response categories
- Positions as presheaves:
  - CCS,
  - implementation of cuts?