The continuous weak Bruhat order

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The order on permutations and words

The continuous order, dimension 2

The continuous order, dimension > 2

Discrete geometry and combinatorics of words

Plan

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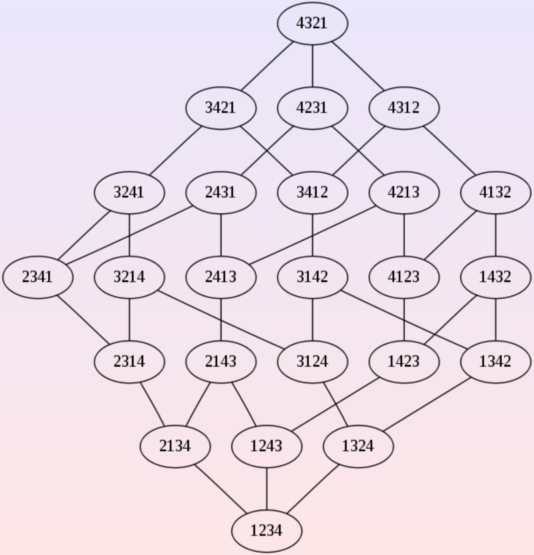
The permutohedron P(4)



Definition of the order on P(n)

Let

$$u_0 ij u_1 \prec u_0 ji u_1$$

if $i < j$.

Then:

$$\sigma \leq \sigma' \text{ iff } \sigma \prec^* \sigma' .$$

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Caracterisation of the order: inversions of a permutation

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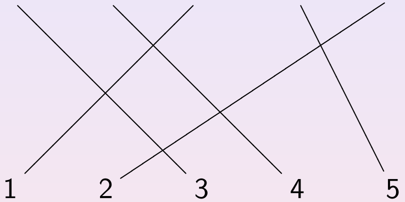

$$Inv(34152) = \{ (1,3), (1,4), (2,3), (2,4), (2,5) \}$$

Proposition

- $\sigma \leq \sigma'$ *iff* $Inv(\sigma) \subseteq Inv(\sigma')$,
- $(P(n), \leq)$ *is a lattice (for all* $n \geq 1$*).*

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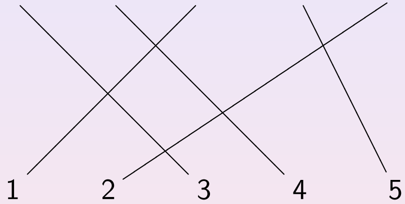

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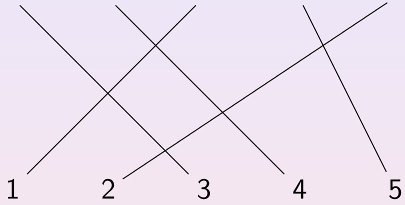

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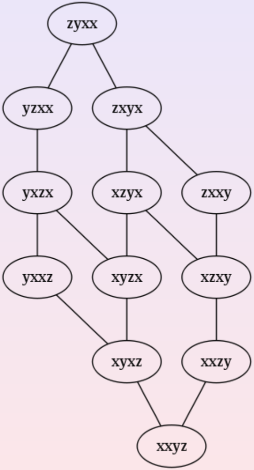
Another lattice, $L(2,1,1)$



The lattices $\mathsf{L}(v)$, $v \in \mathbb{N}^d$

$$\mathsf{L}(v_1, \ldots, v_d) = \{\, w \in \{\, x_1, \ldots, x_d \,\}^* \mid |w|_{x_i} = v_i \,\}$$

Order, as for $\mathsf{P}(n)$:

$$w \leq w' \text{ iff } w \prec^* w' \,,$$

$$\text{where } \; w = u_0 x_i x_j u_1 \prec u_0 x_j x_i u_1 = w' \;\text{ and }\; i < j \,.$$

Proposition

$(\mathsf{L}(v), \leq)$ *is a lattice (for all* $d \geq 1$ *and* $v \in \mathbb{N}^d$*).*

Geometry of the weak order

Let $\mathbb{I} = [0, 1]$.

$$\mathrm{L}(v_1, \ldots, v_d) = \text{discrete increasing paths in } \mathbb{I}^d \text{ from } (0, \ldots, 0) \text{ to } (1, \ldots, 1)\,.$$

Scan of letter x_i : move right of $\dfrac{1}{v_i}$ on the x_i axis.

Goals:

- understanding the order by means of geometry,
- easy in dimension $d = 2$.

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$$\mathsf{L}(n,m) = \text{dicrete increasing paths on the plan from } (0,0) \text{ to } (1,1)\,,$$

scan of x : move right of $\frac{1}{n}$,

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Example, the word $xxyxxy \in \mathsf{L}(4,2)$:

Paths are step monotone functions CAD, the order is pointwise.

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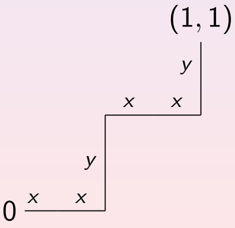

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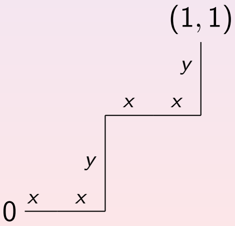

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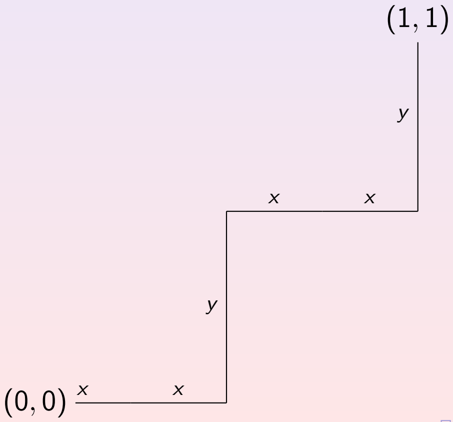

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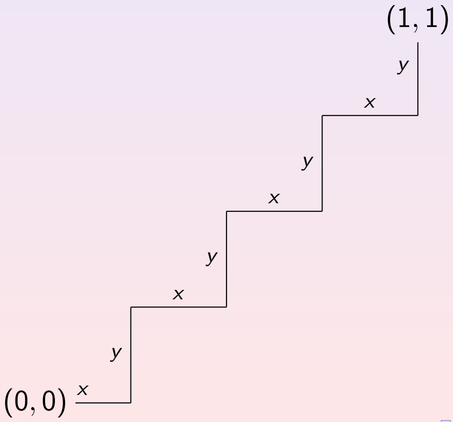

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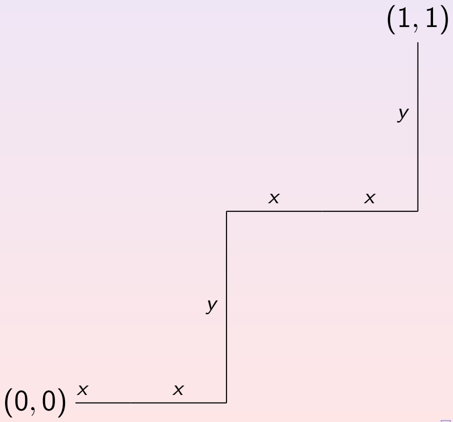

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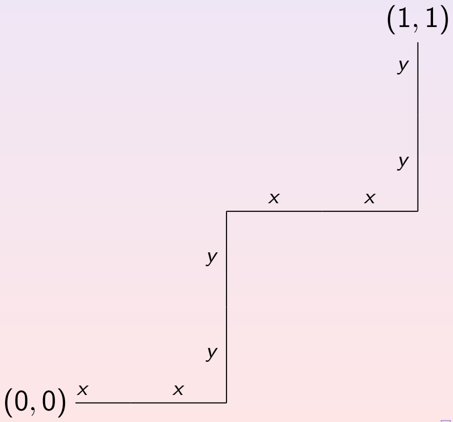

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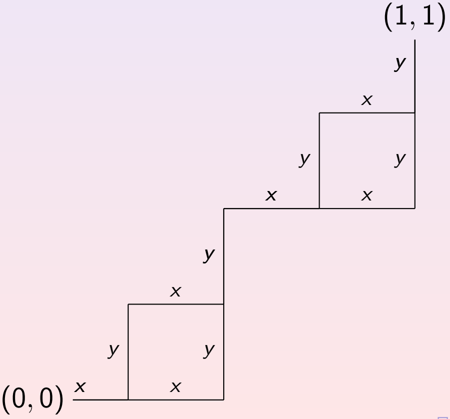

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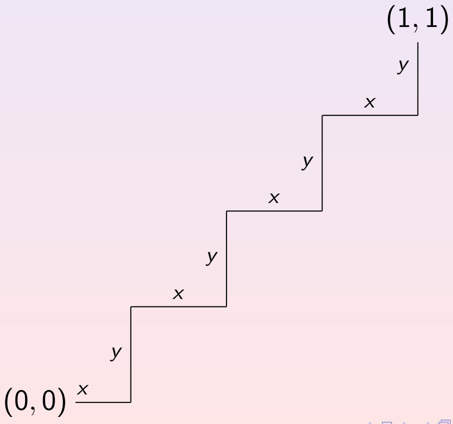

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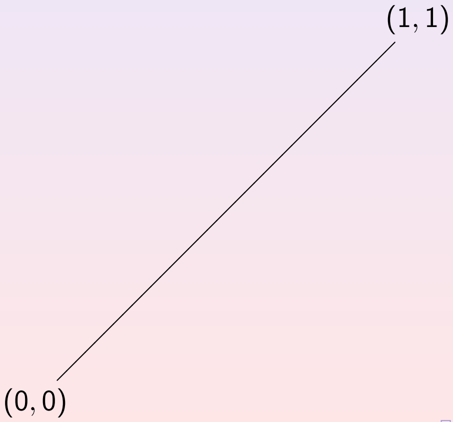

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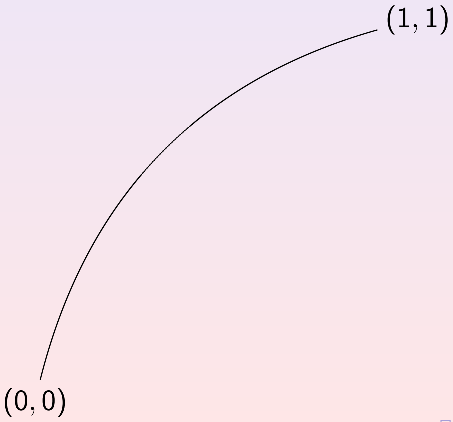

The lattice $\mathsf{L}(\mathbb{I}^2)$

$$\begin{aligned}
\mathsf{L}(\mathbb{I}^2) &= \{\, C \subseteq \mathbb{I} \times \mathbb{I} \mid C \text{ chain, dense, complete} \,\}\,, \\
&\simeq \{\, f : \mathbb{I} \to \mathbb{I} \mid f \text{ is CaD} \,\}\,, \\
&\simeq \{\, f : \mathbb{I} \to \mathbb{I} \mid f \text{ is CaG} \,\}\,,
\end{aligned}$$

order: pointwise .

Proposition

- $\mathsf{L}(\mathbb{I}^2)$ *is a complete distributive lattice,*
- $\mathsf{L}(\mathbb{I}^2)$ *contains all the* $\mathsf{L}(v)$,
- *every* $f \in \mathsf{L}(\mathbb{I}^2)$ *is the* $\bigwedge$ *and* $\bigvee$ *of monotone rational functions CAD.*

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Higher dimensions ?

Goal:

for $d \geq 3$, find a lattice $\mathsf{L}(\mathbb{I}^d)$ whose elements are (images of) monotone contintinuous paths

$$\pi : \mathbb{I} \longrightarrow \mathbb{I}^d$$

such that $\pi(0) = (0, \ldots, 0)$ and $\pi(1) = (1, \ldots, 1)$.

Two paths

Two possibilities:

- Lift characterizations from dimension 2 to higher dimensions, e.g. consider

$$\bigcup_{d \geq 0, v \in \mathbb{N}^d} \mathsf{L}(v)\,,$$

the colimit of the lattices $\mathsf{L}(v)$, $v \in \mathbb{N}^d$,

and then its Dedekind-MacNeille completion.

(Abstract path)

- Lift to the continuous case

the structure of the lattices $\mathsf{L}(v)$, $v \in \mathbb{N}^d$.

(Concrete path)

Back to $\mathrm{L}(v)$

By exemple, we work on $\mathrm{L}(3,2,4)$.

We have:

$$xzxzxyzyz \leq xzzyzxxzy$$

since

$$\begin{aligned} xxxyy &\leq xyzxxy && \text{in } \mathrm{L}(3,2)\,, \\ zzyzyz &\leq zzyzzy && \text{in } \mathrm{L}(2,4)\,, \\ xzxzxzz &\leq xzzzxxz && \text{in } \mathrm{L}(3,4)\,. \end{aligned}$$

Word reconstuction problem . . .

Let:

$$w_{1,2} \in \{\, x, y \,\}^* \cap \mathrm{L}(3,2)\,,$$
$$w_{2,3} \in \{\, y, z \,\}^* \cap \mathrm{L}(2,4)\,,$$
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Find a word $w \in \{\, x, y, z \,\}^* \cap \mathrm{L}(3,2,4)$ such that:

$$w_{1,2} = w \setminus \{\, z \,\}\,,$$
$$w_{2,3} = w \setminus \{\, x \,\}\,,$$
$$w_{1,3} = w \setminus \{\, y \,\}\,.$$

. . . and its solution

For $u \in \{a, b\}^*$ write

$$(a, i) <_u (b, j)$$

for

the i-th occurence of a precedes the j-th occurence of b in u.

A word exists (and then is unique) iff $(w_{1,2}, w_{2,3}, w_{1,3})$ is

1. closed:

$$(z, k) <_{w_{2,3}} (y, j) <_{w_{1,2}} (x, i) \quad \text{implies} \quad (z, k) <_{w_{1,3}} (x, i),$$

2. open:

$$(x, i) <_{w_{1,2}} (y, j) <_{w_{2,3}} (z, k) \quad \text{implies} \quad (x, i) <_{w_{1,3}} (z, k).$$

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Some algebraic properties

- $\mathsf{L}(3,2) \times \mathsf{L}(2,4) \times \mathsf{L}(3,4)$ is a lattice,
- closed tuples are stable under infs,
- open tuples are stable under sups.

For $w \in \mathsf{L}(3,2) \times \mathsf{L}(2,4) \times \mathsf{L}(3,4)$, let

$$\overline{w} = \bigwedge \{ u \text{ closed} \mid w \leq u \}, \qquad \text{(closure of } w\text{)},$$

$$w^\circ = \bigvee \{ u \text{ open} \mid w \leq u \}, \qquad \text{(interior of } w\text{)}.$$

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The clopens

Let

$$\mathsf{CO}(3,2,4) = \{\, w \in \mathsf{L}(3,2)\times\mathsf{L}(2,4)\times\mathsf{L}(3,4) \mid w \text{ is closed and open } \}\,.$$

We have an order isomorphism

$$\mathsf{L}(3,2,4) \simeq \mathsf{CO}(3,2,4)\,.$$

$\mathsf{CO}(3,2,4)$ is a lattice:

$$w \vee_{\mathsf{CO}(3,2,4)} u = \overline{(w \vee_{\mathsf{L}(3,2)\times\mathsf{L}(2,4)\times\mathsf{L}(3,4)} u)}\,,$$
$$w \wedge_{\mathsf{CO}(3,2,4)} u = (w \wedge_{\mathsf{L}(3,2)\times\mathsf{L}(2,4)\times\mathsf{L}(3,4)} u)^{\circ}\,.$$

Whence:

Proposition

$\mathsf{L}(3,2,4)$ *is a lattice.*

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The continuous case

Let $f = (f_{1,2}, f_{2,3}, f_{1,3}) \in \mathsf{L}(\mathbb{I}^2) \times \mathsf{L}(\mathbb{I}^2) \times \mathsf{L}(\mathbb{I}^2)$.

Say that f is:

- closed si

$$r < f_{2,3}(q) \text{ and } q < f_{1,2}(p) \text{ implies } r < f_{1,3}(p)\,,$$

  for all $p, q, r \in \mathbb{I}$,

- open if

$$p < f_{2,1}(q) \text{ and } q < f_{3,2}(r) \text{ implies } p < f_{3,1}(r)\,,$$

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Proposition

1. *the closure of an open is open,*
2. *the interior of a closed is closed,*
3. $\mathsf{L}(\mathbb{I}^3)$, *with the order inherited from* $\mathsf{L}(\mathbb{I}^2)^3$, *is a (complete) lattice,*
4. *every* $\mathsf{L}(n,m,l)$ *embeds into* $\mathsf{L}(\mathbb{I}^3)$,
5. *every element of* $\mathsf{L}(\mathbb{I}^3)$ *is both a inf and a sup of elements from* $\bigcup_{(n,m,l)\in\mathbb{N}^3} \mathsf{L}(n,m,l)$.
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Main results

Let:

$$\mathsf{L}(\mathbb{I}^3) := \{\, f \in \mathsf{L}(\mathbb{I}^2) \times \mathsf{L}(\mathbb{I}^2) \times \mathsf{L}(\mathbb{I}^2) \mid f \text{ is closed and open} \,\}\,.$$

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Open problem(s)

- Is there a bijective correspondence

$$\begin{aligned} \mathsf{L}(\mathbb{I}^3) &\simeq \{\, C \subseteq \mathbb{I}^3 \mid C \text{ chain, dense, complete} \,\} \\ &= \{\, \text{images of continous paths } \pi : \mathbb{I} \longrightarrow \mathbb{I}^3 \,\} \qquad ? \end{aligned}$$

Plan

The order on permutations and words

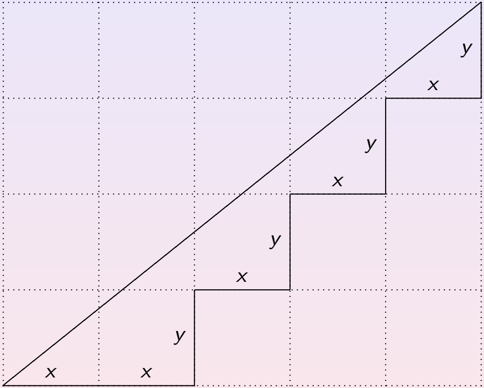
The continuous order, dimension 2

The continuous order, dimension > 2

Discrete geometry and combinatorics of words

The Christoffel words $\mathcal{C}_{n,m}$

Best lower approximation of a straight lines of slope $\frac{m}{n}$:



That is:

$$\mathcal{C}_{5,4} = \bigvee \{ w \in \mathsf{L}(5,4) \mid w \leq \mathtt{id} \in \mathsf{L}(\mathbb{I}^2) \} .$$

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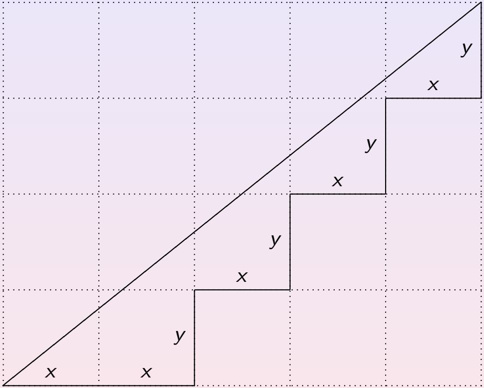

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Christoffel words in higher dimension?

- open question in combimnatorics of words ...
- Sturmian words in higher dimension?

Natural generalisation: for $v \in \mathbb{N}^d$, define

$$\mathcal{C}_v = \bigvee \{ w \in \mathsf{L}(v) \mid i(w) \leq \Delta \}$$

where

- i is the embedding of $\mathsf{L}(v)$ into $\mathsf{L}(\mathbb{I}^d)$,
- $\Delta \in \mathsf{L}(\mathbb{I}^d)$ codes the path $t \mapsto (t, \ldots, t)$.

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