Residuated Park theories

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An example: context-free languages

$$G: \quad X \rightarrow XX \mid aXb \mid bXa \mid \epsilon$$

$$L(G) = \{u \in \{a,b\}^* : |u|_a = |u|_b\}$$

Fact $L(G)$ is the least (pre-)fixed point of the map

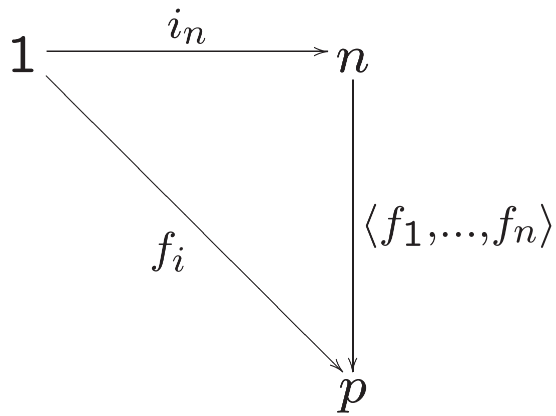
$$\begin{aligned} f_G : P(\{a,b\}^*) &\rightarrow P(\{a,b\}^*) \\ L &\mapsto LL \cup aLb \cup bLa \cup \{\epsilon\} \end{aligned}$$

Aim

To provide an axiomatic treatment of the least fixed operation in the framework of *Lawvere algebraic theories*.

Theories

A **Lawvere theory** is a small category whose objects are the nonnegative integers such that each integer n is the n-fold coproduct of object 1 with itself (i.e., each morphism $n \to p$ is uniquely determined by a sequence of n morphisms $1 \to p$).



Thus, 0 is initial obejct and we denote 0_p the **empty tupling** of morphisms $1 \to p$ (unique morphism $0 \to p$).

Sometimes theories are defined dually ...

Examples of theories

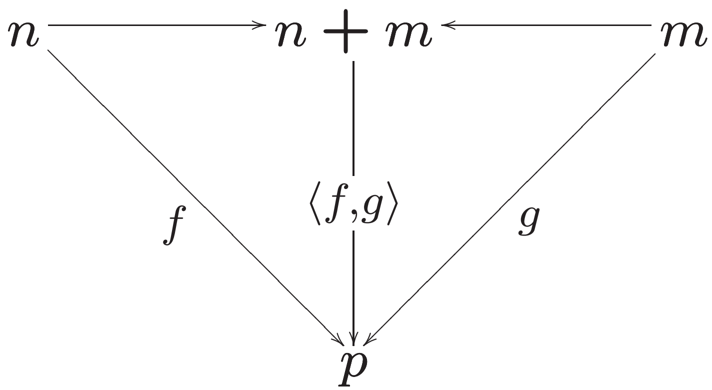
- $\mathbf{Lang}_A$: $L : 1 \to p \Leftrightarrow L \subseteq (A \cup \{X_1, \ldots, X_p\})^*$.
  $L : p \to q \Leftrightarrow L = (L_1, \ldots, L_p)$ with $L_i : 1 \to q$.

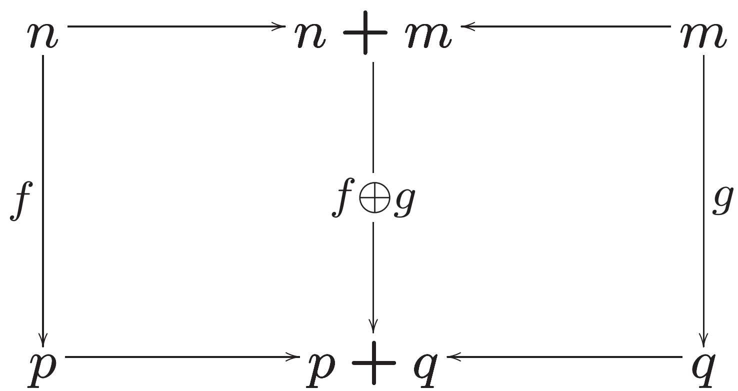
$$L \cdot (L_1, \ldots, L_p) = \{u_0 v_1 \ldots v_k u_k : u_0 X_{i_1} \ldots X_{i_k} v_k \in L, v_j \in L_{i_j}\}$$

- $\mathbf{Fun}_A$, A a set. $f : n \to p \Leftrightarrow f : A^p \to A^n$. Composition is function composition.

- $\mathbf{Mon}_P$, P a poset. $f : n \to p \Leftrightarrow f : P^p \to P^n$ is a *monotone* function.

Theories





Park theories

A **Park theory** is an *ordered theory* T, ordered by $\leq$, which is equipped with an operation

$$\begin{array}{rcl} {}^\dagger : T(n, n+p) & \to & T(n,p), \quad n,p \geq 0 \\ f & \mapsto & f^\dagger \end{array}$$

such that for each $f : n \to n+p$, $f^\dagger$ is the **least solution** to the inequation

$$f \cdot \langle \xi, \mathbf{1}_p \rangle \ \leq \ \xi$$

in the variable $\xi : n \to p$. Moreover:

$$f^\dagger \cdot g \ \leq \ (f \cdot (\mathbf{1}_n \oplus g))^\dagger, \quad f : n \to n+p,\ g : p \to q$$

A **semilattice ordered Park theory** is a Park theory whose partial order is a semilattice order and thus comes with a *supremum operation* $\vee$, moreover:

$$(f \vee g) \cdot h \ \leq \ f \cdot h \vee g \cdot h, \quad f, g : n \to p,\ g : p \to q$$

Examples Park theories

- $\mathbf{Mon}_P$, P a dcpo, or a complete (semi)lattice.

$$f : n \to n + p \;\mapsto\; f^\dagger : n \to p, \quad \text{i.e.,}$$
$$f : P^{n+p} \to P^n \;\mapsto\; f^\dagger : P^p \to P^n$$

For any $y \in P^p$, $f^\dagger(y)$ is the least $x \in P^n$ with

$$f(x, y) \leq x$$

- $\mathbf{Lang}_A$, semilattice ordered.

Some properties of dagger

$$\begin{aligned} f \cdot \langle f^\dagger, \mathbf{1}_p \rangle &= f^\dagger, \quad f : n \to n+p \\ (f \cdot (\mathbf{1}_n \oplus g))^\dagger &= f^\dagger \cdot g, \quad f : n \to n+p,\ g : p \to q \\ \langle f, g \rangle^\dagger &= \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle,\ h^\dagger \rangle, \end{aligned}$$

where $f : n \to n+m+p$, $g : m \to n+m+p$ and

$$h = g \cdot \langle f^\dagger, \mathbf{1}_{m+p} \rangle : m \to m+p.$$

These are called the **fixed point equation**, the **parameter equation** and the **Bekić equation**.

Fixed point induction:

$$f \cdot \langle g, \mathbf{1}_p \rangle \leq g \ \Rightarrow\ f^\dagger \leq g$$

where $f : n \to n+p$, $g : n \to p$.

Completeness

Theorem (ZE) The following are equivalent for an equation $t = t'$ between terms t, t' in the language of theories equipped with a dagger operation:

- $t = t'$ holds in all theories $\mathbf{Mon}_P$, where P is a dcpo.
- $t = t'$ holds in all theories $\mathbf{Mon}_L$, where L is a complete lattice.
- $t = t'$ holds in all "continuous theories" or "continuous semilattice ordered theories" such as the theories $\mathbf{Lang}_A$.
- $t = t'$ holds in all Park theories.

Theorem (ZE) The following are equivalent for an equation $t = t'$ between terms t, t' in the language of theories equipped with a $\vee$ and a dagger operation:

- $t = t'$ holds in all theories $\mathbf{Mon}_L$, where L is a complete lattice.
- $t = t'$ holds in all continuous semilattice ordered theories such as the theories $\mathbf{Lang}_A$.
- $t = t'$ holds in all semilattice ordered Park theories.

Equational axiomatization

There is **no finite equational axiomatization** (involving only the theory operations, †, and possibly $\vee$). (Bloom-ZE)

Infinite equational axiomatization: axioms of **Iteration Theories** (Bloom–Elgot–Wright, ZE)

Theorem (ZE) The following set of equations is complete: equations defining theories +

$$\begin{aligned}(f \cdot (\mathbf{1}_n \oplus g))^\dagger &= f^\dagger \cdot g, \quad f : n \to n+p,\ g : p \to q \\ (f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^\dagger &= f^{\dagger\dagger}, \quad f : n \to n+n+p \\ (f \cdot \langle g, 0_n \oplus \mathbf{1}_p \rangle)^\dagger &= f \cdot \langle (g \cdot \langle f, 0_m \oplus \mathbf{1}_p \rangle)^\dagger, \mathbf{1}_p \rangle, \quad f : n \to m+p,\end{aligned}$$

$g : m \to n+p$, and an equation associated with **each finite (simple) group**.

When $\vee$ is present, one needs to add axioms for semilattice ordered theories and a few more equations.

Residuation

Definition A **residuated (semilattice ordered) theory** is a (semilattice) ordered theory T equipped with a binary operation $\Leftarrow$:

$$\begin{array}{rcl} T(n,q)\times T(p,q) & \to & T(n,p) \\ h:n\to q,\ \ g:p\to q & \mapsto & (h{\Leftarrow}g):n\to p \\ f\cdot g\le h & \Leftrightarrow & f\le (h{\Leftarrow}g), \quad \text{all } f:n\to p \end{array}$$

Alternative axiomatization in the semilattice ordered case:

$$\begin{array}{cl} (h{\Leftarrow}g)\cdot g\le h & \qquad g:p\to q,\ \ h:n\to q \\ f\le (f\cdot g){\Leftarrow}g & \qquad f:n\to p,\ \ g:p\to q \\ h{\Leftarrow}g\le (h\vee h'){\Leftarrow}g & \qquad g:p\to q,\ \ h,h':n\to p \end{array}$$

Fact Any residuated semilattice ordered theory is right distributive.

Residuation

Definition A **residuated (semilattice ordered) Park theory** is any residuated (semilattice ordered) theory which is a Park theory.

Example $\mathbf{Mon}_L$, where L a complete lattice, $\mathbf{Lang}_A$ where A is an alphabet.

Theorem (ZE) Residuated semilattice ordered Park theories can be axiomatized by equations (and are thus closed under quotients).

An equational axiomatization consists of: equations defining residuated semilattice ordered theories, the fixed point equation, the parameter equation and

$$\begin{array}{ll} f^\dagger \leq (f \vee g)^\dagger & f, g : n \to n + p \\ (g \Leftarrow \langle g, \mathbf{1}_p \rangle)^\dagger \leq g, & g : n \to p \end{array}$$

where the second equation is called **pure induction**.

Proof uses ideas of Pratt and Santocanale.

Completeness, again

Theorem (ZE) An equation between terms involving the theory operations and dagger holds in all theories $\mathbf{Mon}_P$, where P is a dcpo or complete lattice, iff it holds in all residuated Park theories.

An equation between terms involving the theory operations, dagger and $\vee$ holds in all theories $\mathbf{Mon}_L$, where L is a complete lattice, iff it holds in all residuated semilattice ordered Park theories.

Variations

There is a similar treatment using:

Star operation

$$f : n \to n{+}p \quad \mapsto \quad f^* := (f^\tau)^\dagger = (f\cdot(\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) \vee (0_n \oplus \mathbf{1}_n \oplus 0_p))^\dagger : n \to n{+}p$$

Star fixed point equation

$$f \cdot \langle f^*, 0_n \oplus \mathbf{1}_p \rangle \vee (\mathbf{1}_n \oplus 0_p) = f^*, \quad f : n \to n + p$$

Star least fixed point rule

$$f \cdot \langle g, 0_n \oplus \mathbf{1}_p \rangle \vee h \leq g \;\Rightarrow\; f^* \cdot \langle h, 0_n \oplus \mathbf{1}_p \rangle \leq g, \quad f, g, h : n \to n + p$$

Star pure induction

$$(g \Leftarrow \langle g, 0_n \oplus \mathbf{1}_p \rangle)^* \;\leq\; (g \Leftarrow \langle g, 0_n \oplus \mathbf{1}_p \rangle), \quad g : n \to n + p$$

Variations

Scalar dagger and **scalar star**

Theorem (Bekić, DeBakker-Scott) In a Park theory,

$$\langle f, g\rangle^\dagger \;=\; \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle, h^\dagger\rangle, \quad f : n \to n+m+p,\; g : m \to n+m+p$$

where

$$h \;=\; g \cdot \langle f^\dagger, \mathbf{1}_{m+p}\rangle : m \to m+p$$

Functional languages: μ-expressions, *-expressions, “letrec” expressions, etc ...

Applications

- Complete axiomatization of
  - Strong and weak behavior of flowchart schemes (cyclic programs)
  - Regular tree laguages and word languages
  - Rational power series and tree series
  - Process behaviors, etc.

- Axiomatic foundation of automata theory.

- Applications to programming logics (e.g. soundness and relative completeness of Hoare logic), cyclic term rewriting, domain equations, . . .

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