Almost (MP)-based substructural logics

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Substructural logics

Non-associative Full Lambek Calculus $\mathbf{SL}$ [Galatos-Ono, APAL, 2010]

$$\vdash \varphi \searrow \varphi \qquad \varphi, \varphi \searrow \psi \vdash \psi \qquad \varphi \vdash (\varphi \searrow \psi) \searrow \psi$$

$$\varphi \searrow \psi \vdash (\psi \searrow \chi) \searrow (\varphi \searrow \chi) \qquad \psi \searrow \chi \vdash (\varphi \searrow \psi) \searrow (\varphi \searrow \chi)$$

$$\vdash \varphi \searrow ((\psi \swarrow \varphi) \searrow \psi) \qquad \varphi \searrow (\psi \searrow \chi) \vdash \psi \searrow (\chi \swarrow \varphi) \qquad \psi \swarrow \varphi \vdash \varphi \searrow \psi$$

$$\vdash \varphi \wedge \psi \searrow \varphi \qquad \vdash \varphi \wedge \psi \searrow \psi$$

$$\varphi, \psi \vdash \varphi \wedge \psi \qquad \vdash (\chi \searrow \varphi) \wedge (\chi \searrow \psi) \searrow (\chi \searrow \varphi \wedge \psi)$$

$$\vdash \varphi \searrow \varphi \vee \psi \qquad \vdash (\varphi \searrow \chi) \wedge (\psi \searrow \chi) \searrow (\varphi \vee \psi \searrow \chi)$$

$$\vdash \psi \searrow \varphi \vee \psi \qquad \vdash (\chi \swarrow \varphi) \wedge (\chi \swarrow \psi) \searrow (\chi \swarrow \varphi \vee \psi)$$

$$\vdash \psi \searrow (\varphi \searrow \varphi \,\&\, \psi) \qquad \psi \searrow (\varphi \searrow \chi) \vdash \varphi \,\&\, \psi \searrow \chi$$

$$\vdash \mathbf{1} \qquad \vdash \mathbf{1} \searrow (\varphi \searrow \varphi) \qquad \vdash \varphi \searrow (\mathbf{1} \searrow \varphi)$$

A convention

Convention

A logic L in a language $\mathcal{L}$ containing $\backslash$ or $/$ is substructural if

- L is an expansion of the $\mathcal{L} \cap \mathcal{L}_{\mathrm{SL}}$-fragment of SL.
- for each n, $i < n$, and each n-ary connective $c \in \mathcal{L} \setminus \mathcal{L}_{\mathrm{SL}}$:

$$\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} c(\chi_1, \dots \chi_i, \varphi, \dots, \chi_n) \to c(\chi_1, \dots \chi_i, \psi, \dots, \chi_n),$$

where $\to$ is any of the implications in $\mathcal{L}$.

Let us fix an one of the implications and denote it as $\to$.

Examples of substructural logics

- substructural logics in Ono's sense including e.g. monoidal logic, uninorm logic, psBL, GBL, BL, Intuitionistic logic, (variants of) relevance logics, Łukasiewicz logic;
- non-associative substructural logics recently developed by Buszkowski, Farulewski, Galatos, Ono, Halaš, Botur, etc.
- expansions by additional connectives, e.g. (classical) modalities, exponentials in (variants of) Linear Logic and Baaz's Delta in fuzzy logics;
- fragments to languages containing implication, e.g. BCK, BCI, psBCK, BCC, hoop logics, etc.;

A problem?

- Is the logic $\mathrm{BCK}_{\wedge}$ of BCK-semilattices substructural?
- It does not satisfy $(\chi \searrow \varphi) \wedge (\chi \searrow \psi) \searrow (\chi \searrow \varphi \wedge \psi)$.
- Solution: it can be considered a substructural logic in our sense if formulated in the language $\{\searrow, \overline{\wedge}, \dots\}$.

Definition

FL is the extension of SL by

- $\vdash_{\mathrm{L}} \varphi \,\&\, (\psi \,\&\, \chi) \to (\varphi \,\&\, \psi) \,\&\, \chi$
- $\vdash_{\mathrm{L}} (\varphi \,\&\, \psi) \,\&\, \chi \to \varphi \,\&\, (\psi \,\&\, \chi)$

Axiomatic extensions of SL and FL

usual name	**s**	**$\&$-form**	**$\to$-form**
exchange	e	$\varphi \,\&\, \psi \to \psi \,\&\, \varphi$	$\varphi \to (\psi \to \chi) \vdash \psi \to (\varphi \to \chi)$
contraction	c	$\varphi \to \varphi \,\&\, \varphi$	$\varphi \to (\varphi \to \psi) \vdash \varphi \to \psi$
weakening	w	i + o	
$\Downarrow$			
left-weak.	i	$\varphi \,\&\, \psi \to \psi$	$\psi \to (\varphi \to \psi)$
right-weak.	o		$\mathbf{0} \to \varphi$

Conjugation and axiomatic systems of FL and FL_e

Definition

- *a left conjugate* of φ is $\lambda_\alpha(\varphi) = (\alpha \backslash \varphi \,\&\, \alpha) \wedge \mathbf{1}$
- *a right conjugate* of φ is $\rho_\alpha(\varphi) = (\alpha \,\&\, \varphi \,/\, \alpha) \wedge \mathbf{1}$
- *an iterated conjugate* of φ is $\gamma_{\alpha_1}(\gamma_{\alpha_2} \cdots \gamma_{\alpha_n}(\varphi) \ldots))$
  where $\gamma_{\alpha_i} = \lambda_{\alpha_i}$ or $\gamma_{\alpha_i} = \rho_{\alpha_i}$

Let us consider the following rules:

(MP)	$\varphi, \varphi \backslash \psi \vdash \psi$		modus ponens
(Adj)	$\varphi \vdash \varphi \wedge \mathbf{1}$		unit adjunction
(PN)	$\varphi \vdash \lambda_\alpha(\varphi)$	$\varphi \vdash \rho_\alpha(\varphi)$	product normality

Theorem

Logic	The only rules needed in its axiomatization
$\mathrm{FL}_{\mathrm{ew}}$	*modus ponens*
FL_{e}	*modus ponens and unit adjunction*
FL	*modus ponens and product normality*

Almost (MP)-based logics

Main definition

We fix

- a substructural logic L in language *with* $\rightarrow, \&,$ *and* $\mathbf{1}$
- a propositional variable p, the meaning of $\delta(\varphi)$ is obvious

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Definition (Almost (MP)-based substructural logic)

L is *almost* (MP)*-based* w.r.t. a set of *basic deduction terms* bDT if it has an axiomatic system where

- there are no rules with *three or more premises*
- there is only one rule with *two premises*: modus ponens
- the remaining rules are $\{\varphi \vdash \chi(\varphi) \mid \varphi \in \mathrm{Fm}, \chi \in \mathrm{bDT}\}$
- for each $\beta \in \mathrm{bDT}$ and each φ, ψ, there are $\beta_1, \beta_2 \in \mathrm{bDT}$ s.t.:

$$\vdash_{\mathrm{L}} \beta_1(\varphi \rightarrow \psi) \rightarrow (\beta_2(\varphi) \rightarrow \beta(\psi)).$$

Examples and conventions

Example

almost (MP)-based logics	basic deduction terms
$\mathrm{FL_{ew}}$	$\emptyset$
$\mathrm{FL_e}$	$\{p \wedge \mathbf{1}\}$
FL	$\{\lambda_\alpha(p), \rho_\alpha(p) \mid \alpha$ a formula$\}$
K	$\{\Box p\}$

Examples and conventions

Example

almost (MP)-based logics	basic deduction terms
$\mathrm{FL_{ew}}$	$\emptyset$
$\mathrm{FL_e}$	$\{p \wedge \mathbf{1}\}$
FL	$\{\lambda_\alpha(p), \rho_\alpha(p) \mid \alpha$ a formula$\}$
K	$\{\Box p\}$

Definition (Iterated and conjuncted Γ-formulae)

Let Γ be a set of formulae. We define the sets of:

- *iterated* Γ-formulae Γ^* as the smallest set s.t.
  - $p \in \Gamma^*$,
  - $\delta(\chi) \in \Gamma^*$ for each $\delta(p) \in \Gamma$ and each $\chi \in \Gamma^*$.
- *conjuncted* Γ-formulae $\Pi(\Gamma)$ as the smallest set containing $\Gamma \cup \{\mathbf{1}\}$ and closed under $\&$.

Almost-Implicational Deduction Theorem

Theorem

Let L *be almost* (MP)*-based w.r.t. a set of basic deductive terms* bDT*. Then for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{L}} \delta(\varphi) \to \psi \textit{ for some } \delta \in \Pi(\mathrm{bDT}^*).$$

Almost-Implicational Deduction Theorem cont.

Definition

A logic L has the *Almost-Implicational Deduction Theorem* w.r.t. a set of *deductive terms* DT, if for each set $\Gamma \cup \{\varphi, \psi\}$ of formulae:

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad \text{iff} \qquad \Gamma \vdash_{\mathrm{L}} \delta(\varphi) \to \psi \text{ for some } \delta \in \mathrm{DT}.$$

Theorem

Let L *have the Almost-Implicational Deduction Theorem w.r.t.* DT.

- *If* L *is finitary, then it is almost* (MP)*-based w.r.t.*

$$\mathrm{bDT} = \{\sigma\delta \mid \delta \in \mathrm{DT}, \sigma \text{ a substitution such that } \sigma p = p\}.$$

- L *has the Almost-Implicational Deduction Theorem w.r.t.* $\mathrm{DT}' \subseteq \mathrm{DT}$ *IFF for every* $\chi \in \mathrm{DT}$ *there is* $\varphi \in \mathrm{DT}'$ *s.t.* $\vdash_{\mathrm{L}} \varphi \to \chi$.

Proof by cases

Theorem (Proof by Cases Property)

Let L *be almost* (MP)*-based w.r.t.* bDT *s.t.*

- *for each* $\beta \in \mathrm{bDT}$ *we have* $\vdash_{\mathrm{L}} \beta(p) \to \mathbf{1}$*,*
- *there is* $\beta_0 \in \mathrm{bDT}$ *such that* $\vdash_{\mathrm{L}} \beta_0(p) \to p$*.*

Then

$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \chi \qquad\qquad \Gamma, \psi \vdash_{\mathrm{L}} \chi}{\Gamma \cup \{\alpha(\varphi) \vee \beta(\psi) \mid \alpha, \beta \in \mathrm{bDT}^*\} \vdash_{\mathrm{L}} \chi}$$

Corollary (Proof by Cases Property for logics with weakening)

Let L *satisfy weakening and be almost* (MP)*-based w.r.t.* bDT*.*
Then

$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \chi \qquad\qquad \Gamma, \psi \vdash_{\mathrm{L}} \chi}{\Gamma \cup \{\alpha(\varphi) \vee \beta(\psi) \mid \alpha, \beta \in \mathrm{bDT}^*\} \vdash_{\mathrm{L}} \chi}$$

Proof by cases - examples

Corollary (Proof by cases in notable logics)

The following meta-rules are valid:

$$\frac{\Gamma, \varphi \vdash_{\mathrm{FL}} \chi \qquad \Gamma, \psi \vdash_{\mathrm{FL}} \chi}{\Gamma \cup \{\gamma_1(\varphi) \vee \gamma_2(\psi) \mid \gamma_1, \gamma_2 \textit{ iterated conjugates}\} \vdash_{\mathrm{FL}} \chi}$$

$$\frac{\Gamma, \varphi \vdash_{\mathrm{FL_e}} \chi \qquad \Gamma, \psi \vdash_{\mathrm{FL_e}} \chi}{\Gamma, (\varphi \wedge \mathbf{1}) \vee (\psi \wedge \mathbf{1}) \vdash_{\mathrm{FL_e}} \chi}$$

$$\frac{\Gamma, \varphi \vdash_{\mathrm{FL_{ew}}} \chi \qquad \Gamma, \psi \vdash_{\mathrm{FL_{ew}}} \chi}{\Gamma, \varphi \vee \psi \vdash_{\mathrm{FL_{ew}}} \chi}$$

$$\frac{\Gamma, \varphi \vdash_{K} \chi \qquad \Gamma, \psi \vdash_{K} \chi}{\Gamma \cup \{\Box^n(\varphi) \vee \Box^m(\psi) \mid n, m \geq 0\} \vdash_{K} \chi}$$

Proof of the Almost-Implicational Deduction Theorem

Theorem

Let L *be almost* (MP)*-based w.r.t. a set of basic deductive terms* bDT*. Then for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{L}} \delta(\varphi) \to \psi \textit{ for some } \delta \in \Pi(\mathrm{bDT}^*).$$

One direction: obvious from (MP) and $\varphi \vdash_{\mathrm{L}} \delta(\varphi)$ for $\delta \in \Pi(\mathrm{bDT}^*)$

The other direction: for each χ in the proof of ψ from $\Gamma \cup \{\varphi\}$ we find $\delta_\chi \in \Pi(\mathrm{bDT}^*)$ s.t. $\Gamma \vdash_{\mathrm{L}} \delta_\chi(\varphi) \to \chi$

- if $\chi = \varphi$, we set $\delta_\chi = p$;
- if $\chi \in \Gamma$ or it is an axiom, we set $\delta_\chi = \mathbf{1}$.
- if χ results from η and $\eta \to \chi$ by (MP). IH: $\Gamma \vdash_{\mathrm{L}} \delta_\eta(\varphi) \to \eta$ and $\Gamma \vdash_{\mathrm{L}} \delta_{\eta\to\chi}(\varphi) \to (\eta \to \chi)$. We set $\delta_\chi = \delta_\eta \mathbin{\&} \delta_{\eta\to\chi}$.

Proof of the Almost-Implicational Deduction Theorem

Theorem

Let L *be almost* (MP)*-based w.r.t. a set of basic deductive terms* bDT*. Then for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae:*

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  From the former we derive $\Gamma \vdash_{\mathrm{L}} (\eta \to \chi) \to (\delta_\eta(\varphi) \to \chi)$, and so, by using the latter, $\Gamma \vdash_{\mathrm{L}} \delta_{\eta\to\chi}(\varphi) \to (\delta_\eta(\varphi) \to \chi)$, and thus $\Gamma \vdash_{\mathrm{L}} \delta_\eta(\varphi) \mathbin{\&} \delta_{\eta\to\chi}(\varphi) \to \chi$.

Proof of the Almost-Implicational Deduction Theorem

if χ is obtained from η using the rule $\eta \vdash \chi$. Thus $\chi = \beta(\eta)$ for some $\beta \in \mathrm{bDT}$. Induction Hypothesis: $\Gamma \vdash_{\mathrm{L}} \delta_\eta(\varphi) \to \eta$.

Claim: for each $\beta \in \mathrm{bDT}$, $\delta \in \Pi(\mathrm{bDT}^*)$, and formulae φ, ψ

- $\varphi \to \psi \vdash_{\mathrm{L}} \beta'(\varphi) \to \beta(\psi)$ for some $\beta' \in \mathrm{bDT}$
- $\vdash_{\mathrm{L}} \delta'(\varphi) \to \beta(\delta(\varphi))$ for some $\delta' \in \Pi(\mathrm{bDT}^*)$

From $\Gamma \vdash_{\mathrm{L}} \delta_\eta(\varphi) \to \eta$ we get $\beta' \in \mathrm{bDT}$ s.t. $\Gamma \vdash_{\mathrm{L}} \beta'(\delta_\eta(\varphi)) \to \beta(\eta)$. Thus there is δ_χ s.t. $\Gamma \vdash_{\mathrm{L}} \delta_\chi(\varphi) \to \chi$.

Thank you for your attention!