The unification type of Łukasiewicz logic is nullary

Based on a joint work with V. Marra

Luca Spada

Department of Mathematics
University of Salerno
`http://logica.dmi.unisa.it/lucaspada`

Topology, Algebra, and Categories in Logic
Marseilles, 28th July 2011.

Łukasiewicz logic

Łukasiewicz (infinite-valued propositional) logic is a non-classical system going back to the 1920's which may be axiomatised using the primitive connectives $\to$ (implication) and $\neg$ (negation)

Łukasiewicz logic

Łukasiewicz (infinite-valued propositional) logic is a non-classical system going back to the 1920's which may be axiomatised using the primitive connectives $\to$ (implication) and $\neg$ (negation) by the four axiom schemata:

1. $\alpha \to (\beta \to \alpha)\,,$
2. $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))\,,$
3. $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)\,,$
4. $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)\,,$

with *modus ponens* as the only deduction rule.

Semantics of Łukasiewicz logic

Łukasiewicz logic is a subsystem of classical logic and has a many-valued semantics: assignments μ to atomic formulæ range in the unit interval $[0, 1] \subseteq \mathbb{R}$.

Semantics of Łukasiewicz logic

Łukasiewicz logic is a subsystem of classical logic and has a many-valued semantics: assignments μ to atomic formulæ range in the unit interval $[0,1] \subseteq \mathbb{R}$.
They are extended compositionally to compound formulæ via

$$\mu(\neg\alpha) \ = \ 1 \ - \ \mu(\alpha)\,,$$
$$\mu(\alpha \to \beta) = \min\{1, 1 - \mu(\alpha) + \mu(\beta)\}$$

Semantics of Łukasiewicz logic

Łukasiewicz logic is a subsystem of classical logic and has a many-valued semantics: assignments μ to atomic formulæ range in the unit interval $[0, 1] \subseteq \mathbb{R}$.
They are extended compositionally to compound formulæ via

$$\mu(\neg\alpha) = 1 - \mu(\alpha)\,,$$
$$\mu(\alpha \to \beta) = \min\{1, 1 - \mu(\alpha) + \mu(\beta)\}$$

Tautologies are defined as those formulæ that evaluate to 1 under every such assignment.

MV-algebras

Chang first considered the Tarski-Lindenbaum algebras of Łukasiewicz logic, and called them MV-algebras.

Definition

An MV-algebra is a structure $\mathcal{A} = \langle A, \oplus, ^*, 0\rangle$ such that:

MV-algebras

Chang first considered the Tarski-Lindenbaum algebras of Łukasiewicz logic, and called them MV-algebras.

Definition

An MV-algebra is a structure $\mathcal{A} = \langle A, \oplus, ^*, 0 \rangle$ such that:

- $\mathcal{A} = \langle A, \oplus, 0 \rangle$ is a commutative monoid,

MV-algebras

Chang first considered the Tarski-Lindenbaum algebras of Łukasiewicz logic, and called them MV-algebras.

Definition

An MV-algebra is a structure $\mathcal{A} = \langle A, \oplus, ^*, 0\rangle$ such that:

- $\mathcal{A} = \langle A, \oplus, 0\rangle$ is a commutative monoid,
- * is an involution

MV-algebras

Chang first considered the Tarski-Lindenbaum algebras of Łukasiewicz logic, and called them MV-algebras.

Definition

An MV-algebra is a structure $\mathcal{A} = \langle A, \oplus, ^*, 0\rangle$ such that:

- $\mathcal{A} = \langle A, \oplus, 0\rangle$ is a commutative monoid,
- * is an involution
- the interaction between those two operations is described by the following two axioms:

MV-algebras

Chang first considered the Tarski-Lindenbaum algebras of Łukasiewicz logic, and called them MV-algebras.

Definition

An MV-algebra is a structure $\mathcal{A} = \langle A, \oplus, {}^*, 0\rangle$ such that:

- $\mathcal{A} = \langle A, \oplus, 0\rangle$ is a commutative monoid,
- * is an involution
- the interaction between those two operations is described by the following two axioms:
  - $x \oplus 0^* = 0^*$
  - $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$

Example 1: The standard MV-algebra

In modern terms one says that MV-algebras are the equivalent algebraic semantics of Łukasiewicz logic.

Example 1: The standard MV-algebra

In modern terms one says that MV-algebras are the equivalent algebraic semantics of Łukasiewicz logic.

Example

Consider the set of real number $[0,1]$ endowed with the following operation:

$$\neg x = 1 - x \text{ and } x \oplus y = \min\{1, x + y\} \text{ (truncated sum).}$$

Then $\langle [0,1], \oplus, \neg, 0 \rangle$ is an MV-algebra.

Example 1: The standard MV-algebra

In modern terms one says that MV-algebras are the equivalent algebraic semantics of Łukasiewicz logic.

Example

Consider the set of real number $[0,1]$ endowed with the following operation:

$$\neg x = 1 - x \text{ and } x \oplus y = \min\{1, x + y\} \text{ (truncated sum).}$$

Then $\langle [0,1], \oplus, \neg, 0 \rangle$ is an MV-algebra.

Actually the above algebra generates the variety of all MV-algebras. So the equations that hold for any MV-algebra are exactly the ones that hold in [0,1].

Example 2: McNaughton functions

A McNaughton function is a function

$$f\colon [0,1]^n \to [0,1]$$

which is

Example 2: McNaughton functions

A McNaughton function is a function

$$f\colon [0,1]^n \to [0,1]$$

which is

continuous,
piece-wise linear,
and with integer coefficients.

Example 2: McNaughton functions

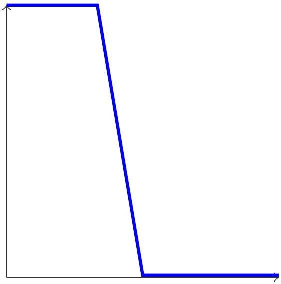
A McNaughton function is a function

$$f\colon [0,1]^n \to [0,1]$$

which is

continuous,
piece-wise linear,
and with integer coefficients.

Example:



Example 2: McNaughton functions

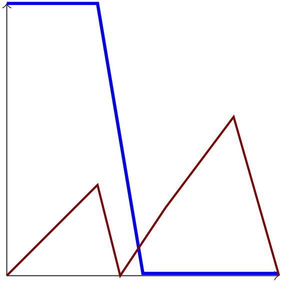
A McNaughton function is a function

$$f\colon [0,1]^n \to [0,1]$$

which is

continuous,
piece-wise linear,
and with integer coefficients.

Example:



The free MV-algebra

Example

One can endow the set of McNaughton functions in n variables with the structure of an MV-algebra by taking point-wise operation:

$$f \oplus g = \big(f \oplus g\big)(x) = f(x) \oplus g(x) \text{ (recall that } [0,1] \text{ is an MV-algebra)}$$

The free MV-algebra

Example

One can endow the set of McNaughton functions in n variables with the structure of an MV-algebra by taking point-wise operation:

$$f \oplus g = \big(f \oplus g\big)(x) = f(x) \oplus g(x) \text{ (recall that } [0,1] \text{ is an MV-algebra)}$$

These functions are named after McNaughton, who first found the following characterisation of free MV-algebras:

The free MV-algebra

Example

One can endow the set of McNaughton functions in n variables with the structure of an MV-algebra by taking point-wise operation:

$f \oplus g = \big(f \oplus g\big)(x) = f(x) \oplus g(x)$ (recall that $[0,1]$ is an MV-algebra)

These functions are named after McNaughton, who first found the following characterisation of free MV-algebras:

Theorem (McNaughton 1951)

The free MV-algebra over κ generators is isomorphic to the MV-algebra of McNaughton functions over $[0,1]^\kappa$.

Example 3: lattice ordered groups

A *$u\ell$-group* is a lattice-ordered group G with an element g such that

for any $g' \in G$ there exists $n \in \mathbb{N}$ such that $\underbrace{g + ... + g}_{n\text{-times}} \geq g'$.

Example 3: lattice ordered groups

A $u\ell$-group is a lattice-ordered group G with an element g such that

for any $g' \in G$ there exists $n \in \mathbb{N}$ such that $\underbrace{g + ... + g}_{n\text{-times}} \geq g'$.

If one truncates the operations of an $u\ell$-group to the interval $[0, g]$, the result is an MV-algebra.

Example 3: lattice ordered groups

A $u\ell$-group is a lattice-ordered group G with an element g such that

for any $g' \in G$ there exists $n \in \mathbb{N}$ such that $\underbrace{g + \ldots + g}_{n\text{-times}} \geq g'$.

If one truncates the operations of an $u\ell$-group to the interval $[0, g]$, the result is an MV-algebra.

Theorem (Mundici 1986)

The category of MV-algerbas is equivalent to the category of Abelian $u\ell$-groups (with ℓ-morphisms preserving the strong unit).

Unitarity of finite-valued Łukasiewicz logic

Ghilardi himself noticed that finite-valued Łukasiewicz logic has unitary type.

$\Bigg($ Finite-valued logics are obtained by restricting the possible values of evaluations to some subchain $\{0, \frac{1}{n}, .., \frac{n-1}{n}, 1\}$ of the [0,1] algebra. $\Bigg)$

Unitarity of finite-valued Łukasiewicz logic

Ghilardi himself noticed that finite-valued Łukasiewicz logic has unitary type.

This was re-proved explicitly and generalised to any finite-valued extension of Basic Logic by Dzik.

$$\left(\text{Finite-valued logics are obtained by restricting the possible values of evaluations to some subchain } \{0, \tfrac{1}{n}, .., \tfrac{n-1}{n}, 1\} \text{ of the [0,1] algebra.} \right)$$

Commutative lattice-ordered groups (ℓ-groups)

Theorem (Beynon 1977)

Finitely generated projective ℓ-groups are exactly the finitely presented ℓ-groups.

Commutative lattice-ordered groups (ℓ-groups)

Theorem (Beynon 1977)

Finitely generated projective ℓ-groups are exactly the finitely presented ℓ-groups.

In the theory of ℓ-groups all system of equations are solvable. In the light of the Beynon's and Ghilardi's results, one easily gets:

Corollary

The unification type of the theory of ℓ-groups is unitary.

Commutative lattice-ordered groups (ℓ-groups)

Theorem (Beynon 1977)

Finitely generated projective ℓ-groups are exactly the finitely presented ℓ-groups.

In the theory of ℓ-groups all system of equations are solvable. In the light of the Beynon's and Ghilardi's results, one easily gets:

Corollary

The unification type of the theory of ℓ-groups is unitary.

In a forthcoming paper with V. Marra, we exploit a geometrical duality for ℓ-groups to give an algorithm that, taken any (system of) term in the language of ℓ-groups, outputs its most general unifier.

Non unitarity of the unification

Łukasiewicz logic has a weak disjunction property. Namely:

if $\varphi \vee \neg\varphi$ *is derivable then either* φ *or* $\neg\varphi$ *must be derivable.*
(In other words the rule $\frac{\varphi\vee\neg\varphi}{\varphi,\neg\varphi}$ is *admissible*.)

Non unitarity of the unification

Łukasiewicz logic has a weak disjunction property. Namely:

if $\varphi \vee \neg\varphi$ *is derivable then either* φ *or* $\neg\varphi$ *must be derivable*. (In other words the rule $\frac{\varphi\vee\neg\varphi}{\varphi,\neg\varphi}$ is *admissible*.)

(**Exercise**: prove this by using McNaughton representation)

This entails the unification type of Łukasiewicz logic to be at least not unitary.

Non unitarity of the unification

Łukasiewicz logic has a weak disjunction property. Namely:

if $\varphi \vee \neg\varphi$ is derivable then either φ or $\neg\varphi$ must be derivable. (In other words the rule $\frac{\varphi \vee \neg\varphi}{\varphi, \neg\varphi}$ is *admissible*.)

(**Exercise**: prove this by using McNaughton representation)

This entails the unification type of Łukasiewicz logic to be at least not unitary.

Indeed if σ is a unifier for $x \vee \neg x$, then it must unify either x (hence it is the substitution $x \mapsto 1$) or $\neg x$ (hence it must be the substitution $x \mapsto 0$).

Rational polyhedral geometry

McNaughton functions are only a first scratch on the surface of a stronger link between Łukasiewicz logic and Geometry.

Rational polyhedral geometry

McNaughton functions are only a first scratch on the surface of a stronger link between Łukasiewicz logic and Geometry.

Definition

A rational polytope is the convex hull of a finite set of **rational** points.

Rational polyhedral geometry

McNaughton functions are only a first scratch on the surface of a stronger link between Łukasiewicz logic and Geometry.

Definition

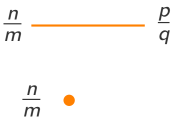
A rational polytope is the convex hull of a finite set of **rational** points.

Rational polyhedral geometry

McNaughton functions are only a first scratch on the surface of a stronger link between Łukasiewicz logic and Geometry.

Definition

A rational polytope is the convex hull of a finite set of **rational** points.

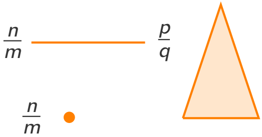


Rational polyhedral geometry

McNaughton functions are only a first scratch on the surface of a stronger link between Łukasiewicz logic and Geometry.

Definition

A rational polytope is the convex hull of a finite set of **rational** points.

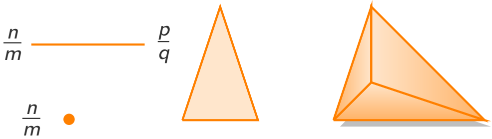


Rational polyhedral geometry

McNaughton functions are only a first scratch on the surface of a stronger link between Łukasiewicz logic and Geometry.

Definition

A rational polytope is the convex hull of a finite set of **rational** points.



Rational polyhedral geometry [Cont.d]

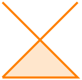
Definition

A rational polyhedron is the union of a finite number of rational polytopes.

Rational polyhedral geometry [Cont.d]

Definition

A rational polyhedron is the union of a finite number of rational polytopes.



Rational polyhedral geometry [Cont.d]

Definition

A rational polyhedron is the union of a finite number of rational polytopes.

Rational polyhedral geometry [Cont.d]

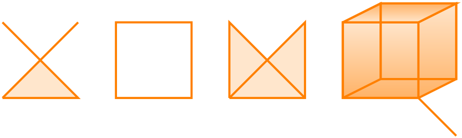
Definition

A rational polyhedron is the union of a finite number of rational polytopes.

Rational polyhedral geometry [Cont.d]

Definition

A rational polyhedron is the union of a finite number of rational polytopes.



$\mathbb{Z}$-maps

Another way of looking at the McNaughton theorem is as a characterisation of definable functions in the language of MV-algebras.

$\mathbb{Z}$-maps

Another way of looking at the McNaughton theorem is as a characterisation of definable functions in the language of MV-algebras.
The only difference here being the fact that we can operate form the m-dimensional $\mathbb{R}^m$ spaces to the n-dimensional $\mathbb{R}^n$ spaces.

$\mathbb{Z}$-maps

Another way of looking at the McNaughton theorem is as a characterisation of definable functions in the language of MV-algebras.
The only difference here being the fact that we can operate form the m-dimensional $\mathbb{R}^m$ spaces to the n-dimensional $\mathbb{R}^n$ spaces.

Definition

A $\mathbb{Z}$-map is a continuous piecewise linear function with integer coefficients.

Rational polyhedra and MV-algebras

Let $\mathcal{MV}_{fp}$ be the category of f.p. MV-algebras with their homomorphisms.

Rational polyhedra and MV-algebras

Let $\mathcal{MV}_{fp}$ be the category of f.p. MV-algebras with their homomorphisms.

Let $\mathcal{P}_{\mathbb{Z}}$ be the category of rational polyhedra and $\mathbb{Z}$-maps between them.

Rational polyhedra and MV-algebras

Let $\mathcal{MV}_{fp}$ be the category of f.p. MV-algebras with their homomorphisms.

Let $\mathcal{P}_{\mathbb{Z}}$ be the category of rational polyhedra and $\mathbb{Z}$-maps between them.

I will define a pair of (contravariant) functors:

$$\mathcal{V}\colon \mathcal{MV}_{fp} \to \mathcal{P}_{\mathbb{Z}} \qquad \text{and} \qquad \mathcal{I}\colon \mathcal{P}_{\mathbb{Z}} \to \mathcal{MV}_{fp}\,.$$

Rational polyhedra and MV-algebras

Let $\mathcal{MV}_{fp}$ be the category of f.p. MV-algebras with their homomorphisms.

Let $\mathcal{P}_{\mathbb{Z}}$ be the category of rational polyhedra and $\mathbb{Z}$-maps between them.

I will define a pair of (contravariant) functors:

$$\mathcal{V} \colon \mathcal{MV}_{fp} \to \mathcal{P}_{\mathbb{Z}} \qquad \text{and} \qquad \mathcal{I} \colon \mathcal{P}_{\mathbb{Z}} \to \mathcal{MV}_{fp}\,.$$

These functors operate very similarly to the classical ones in algebraic geometry that associate ideals with varieties.

F.p. MV-algebras and rational polyhedra: objects

Let $A = \dfrac{Free_n}{\theta} \in \mathcal{MV}_{fp}$.

F.p. MV-algebras and rational polyhedra: objects

$$\text{Let } A = \frac{Free_n}{\theta} \in \mathcal{MV}_{fp}.$$

Let $V(\theta)$ be the collection of all real points p in $[0,1]^n$ such that

$$s(p) = t(p) \text{ for all } (s,t) \in \theta$$

F.p. MV-algebras and rational polyhedra: objects

$$\text{Let } A = \frac{Free_n}{\theta} \in \mathcal{MV}_{fp}.$$

Let $V(\theta)$ be the collection of all real points p in $[0,1]^n$ such that

$$s(p) = t(p) \text{ for all } (s,t) \in \theta$$

The set $V(\theta)$ is a rational polyhedron, so we set

$$\mathcal{V}(A) = V(\theta)\,.$$

F.p. MV-algebras and rational polyhedra: objects

Let $P \in \mathcal{P}_{\mathbb{Z}}$.

F.p. MV-algebras and rational polyhedra: objects

Let $P \in \mathcal{P}_{\mathbb{Z}}$.

Let $\imath(P)$ be the collection of all pair MV-terms (s, t) such that

$$s(x) = t(x) \text{ for all } x \in P$$

F.p. MV-algebras and rational polyhedra: objects

Let $P \in \mathcal{P}_{\mathbb{Z}}$.

Let $\imath(P)$ be the collection of all pair MV-terms (s, t) such that

$$s(x) = t(x) \text{ for all } x \in P$$

$\imath(P)$ is a congruence of the free MV-algebras on n generators, so it makes sense to set

$$\mathcal{I}(P) = \frac{Free_n}{\imath(P)}.$$

▸ Skip arrows

F.p. MV-algebras and rational polyhedra: arrows

Let $h\colon A \to B$ be a diagram in $\mathcal{MV}_{fp}$.

Suppose that h sends the generators of A into the elements $\{t_i\}_{i\in I}$ of B, then define

$$\mathcal{V}(h)\colon \mathcal{V}(B) \to \mathcal{V}(A)$$

F.p. MV-algebras and rational polyhedra: arrows

Let $h\colon A \to B$ be a diagram in $\mathcal{MV}_{fp}$.

Suppose that h sends the generators of A into the elements $\{t_i\}_{i\in I}$ of B, then define

$$\mathcal{V}(h)\colon \mathcal{V}(B) \to \mathcal{V}(A)$$

as

$$p \in \mathcal{V}(A) \xmapsto{\mathcal{V}(h)} \langle t_i(p)\rangle_{i\in I} \in \mathcal{V}(A).$$

F.p. MV-algebras and rational polyhedra: arrows

Let $h\colon A \to B$ be a diagram in $\mathcal{MV}_{fp}$.

Suppose that h sends the generators of A into the elements $\{t_i\}_{i\in I}$ of B, then define

$$\mathcal{V}(h)\colon \mathcal{V}(B) \to \mathcal{V}(A)$$

as

$$p \in \mathcal{V}(A) \xmapsto{\mathcal{V}(h)} \langle t_i(p)\rangle_{i\in I} \in \mathcal{V}(A).$$

Then, the function $\mathcal{V}(h)\colon \mathcal{V}(B) \to \mathcal{V}(A)$ is a $\mathbb{Z}$-map.

F.p. MV-algebras and rational polyhedra: arrows

Let $\zeta\colon P \to Q$ be a diagram in $\mathcal{P}_{\mathbb{Z}}$.

Define

$$\mathcal{I}(\zeta)\colon \mathcal{I}(Q) \to \mathcal{I}(P)$$

F.p. MV-algebras and rational polyhedra: arrows

Let $\zeta\colon P \to Q$ be a diagram in $\mathcal{P}_{\mathbb{Z}}$.

Define

$$\mathcal{I}(\zeta)\colon \mathcal{I}(Q) \to \mathcal{I}(P)$$

as

$$f \in \mathcal{I}(Q) \xmapsto{\mathcal{I}(\zeta)} f \circ \zeta \in \mathcal{I}(P).$$

F.p. MV-algebras and rational polyhedra: arrows

Let $\zeta \colon P \to Q$ be a diagram in $\mathcal{P}_{\mathbb{Z}}$.

Define

$$\mathcal{I}(\zeta) \colon \mathcal{I}(Q) \to \mathcal{I}(P)$$

as

$$f \in \mathcal{I}(Q) \xmapsto{\mathcal{I}(\zeta)} f \circ \zeta \in \mathcal{I}(P).$$

Then, the function $\mathcal{I}(\zeta)$ is a homomorphism of MV-algebras.

Duality for finitely presented MV-algebras

Theorem (Folklore)

The pair of functors

$$\mathcal{I}\colon \mathcal{MV}_{fp} \to \mathcal{P}_{\mathbb{Z}} \qquad \text{and} \qquad \mathcal{V}\colon \mathcal{P}_{\mathbb{Z}} \to \mathcal{MV}_{fp}\,.$$

constitutes a contravariant equivalence between the two categories.

Corollaries

As a corollaries of the above duality one immediately gets

Corollary

The rational polyhedron associated to the free algebra over n generators is the n-dimensional cube $[0,1]^n$.

Corollaries

As a corollaries of the above duality one immediately gets

Corollary

The rational polyhedron associated to the free algebra over n generators is the n-dimensional cube $[0,1]^n$.

Corollary

The rational polyhedron associated to any n-generated projective MV-algebra is a retraction of the n-dimensional cube $[0,1]^n$.

Corollaries

As a corollaries of the above duality one immediately gets

Corollary

The rational polyhedron associated to the free algebra over n generators is the n-dimensional cube $[0, 1]^n$.

Corollary

The rational polyhedron associated to any n-generated projective MV-algebra is a retraction of the n-dimensional cube $[0, 1]^n$.

Corollaries [Con.d]

Corollary

The fundamental group of any injective rational polyhedra is trivial.

Corollaries [Con.d]

Corollary

The fundamental group of any injective rational polyhedra is trivial.

Proof.

Let P an injective rational polyhedron corresponding to a n-generated MV-algebra, then

$$\begin{tikzcd} \pi_1(P) \arrow[r, tail] & \{\pi_1([0,1]^n)\} \arrow[r, two heads] & \pi_1(P) \end{tikzcd}$$

□

Corollaries [Con.d]

Corollary

The fundamental group of any injective rational polyhedra is trivial.

Proof.

Let P an injective rational polyhedron corresponding to a n-generated MV-algebra, then

$$\begin{tikzcd}
\pi_1(P) \arrow[r, tail] \arrow[rr, bend right, "\mathbf{id}"'] & \{\pi_1([0,1]^n)\} \arrow[r, two heads] & \pi_1(P)
\end{tikzcd}$$

□

Corollaries [Con.d]

Corollary

The fundamental group of any injective rational polyhedra is trivial.

Proof.

Let P an injective rational polyhedron corresponding to a n-generated MV-algebra, then

$$\begin{tikzcd} & \{*\} \arrow[d, equal] & \\ \pi_1(P) \arrow[r, tail] \arrow[rr, bend right, "\mathbf{id}"'] & \{\pi_1([0,1]^n)\} \arrow[r, two heads] & \pi_1(P) \end{tikzcd}$$

□

Co-unification

Since Ghilardi's approach is purely categorical one can speak of co-unification to refer to the dual problem in the dual category.

Co-unification

Since Ghilardi's approach is purely categorical one can speak of co-unification to refer to the dual problem in the dual category.
As we have seen that rational polyhedra are equivalent to the dual category of finitely presented MV-algebras, it makes sense to define:

Co-unification

Since Ghilardi's approach is purely categorical one can speak of co-unification to refer to the dual problem in the dual category.
As we have seen that rational polyhedra are equivalent to the dual category of finitely presented MV-algebras, it makes sense to define:

- A co-unification problem as a rational polyedron Q.

Co-unification

Since Ghilardi's approach is purely categorical one can speak of co-unification to refer to the dual problem in the dual category.
As we have seen that rational polyhedra are equivalent to the dual category of finitely presented MV-algebras, it makes sense to define:

- A co-unification problem as a rational polyedron Q.
- An co-unifier for the problem Q as a pair (P, u) where
  1. P is an injective rational polyhedron,
  2. u is a $\mathbb{Z}$-map from P to Q, $u: P \longrightarrow Q$.

Finitarity result

Proposition

The unification type of the 1-variable fragment of Łukasiewicz logic is finitary.

Finitarity result

Proposition

The unification type of the 1-variable fragment of Łukasiewicz logic is finitary.

In particular the proof shows that there are at most two most general unifiers, for any given formula.

Finitarity result

Proposition

The unification type of the 1-variable fragment of Łukasiewicz logic is finitary.

In particular the proof shows that there are at most two most general unifiers, for any given formula.
Indeed the most general form of a co-unification problem is

With A or B possibly empty or restricted to a point.

From 1-variable to the full calculus

The reason of the absence of a good unification theory for Łukasiewicz logic has to be found in the following geometrical phenomenon:

From 1-variable to the full calculus

The reason of the absence of a good unification theory for Łukasiewicz logic has to be found in the following geometrical phenomenon:

From 1-variable to the full calculus

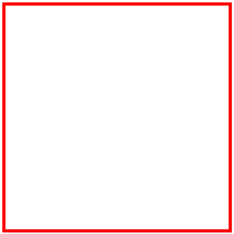
The reason of the absence of a good unification theory for Łukasiewicz logic has to be found in the following geometrical phenomenon:

From 1-variable to the full calculus

The reason of the absence of a good unification theory for Łukasiewicz logic has to be found in the following geometrical phenomenon:

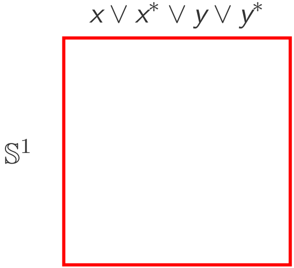
$x \vee x^*$

$x \vee x^* \vee y \vee y^*$



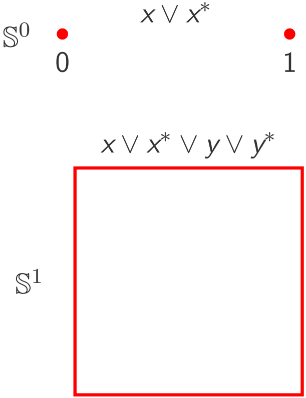
From 1-variable to the full calculus

The reason of the absence of a good unification theory for Łukasiewicz logic has to be found in the following geometrical phenomenon:



From 1-variable to the full calculus

The reason of the absence of a good unification theory for Łukasiewicz logic has to be found in the following geometrical phenomenon:



Nullarity of Łukasiewicz logic

Theorem

The full Łukasiewicz logic has nullary unification type.

Nullarity of Łukasiewicz logic

Theorem

The full Łukasiewicz logic has nullary unification type.

Proof. It is sufficient to exhibit one problem with nullary type.

Nullarity of Łukasiewicz logic

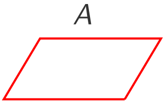
Theorem

The full Łukasiewicz logic has nullary unification type.

Proof. It is sufficient to exhibit one problem with nullary type. As seen above the co-unification problem associated to

$$(x \vee x^* \vee y \vee y^*, 1)$$

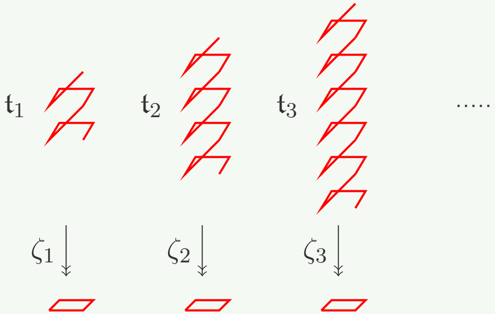
is the rational polyhedron



Proof Cont.'d

Step 1.

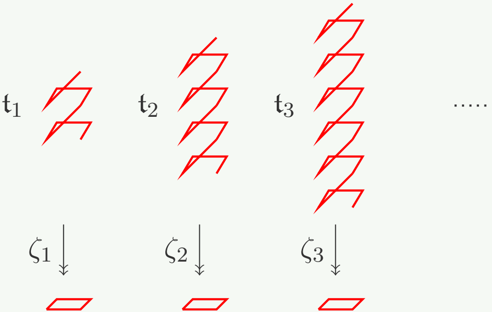
Consider the following sequence of pair of maps and rational polyhedra,



Proof Cont.'d

Step 1.

Consider the following sequence of pair of maps and rational polyhedra,

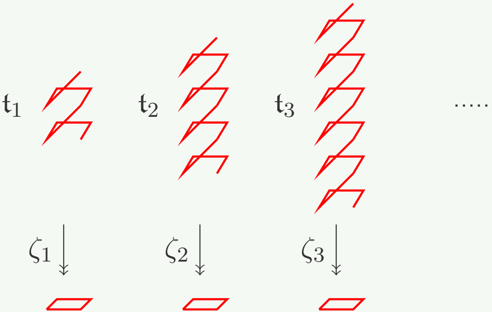


It can be proved (cfr. Cabrer and Mundici) that each $\mathfrak{t}_i$ is a retract of $[0,1]^n$ for some n,

Proof Cont.'d

Step 1.

Consider the following sequence of pair of maps and rational polyhedra,



It can be proved (cfr. Cabrer and Mundici) that each $\mathfrak{t}_i$ is a retract of $[0,1]^n$ for some n, so the pairs $(\mathfrak{t}_i, \zeta_i)$ are co-unifiers for A.

Proof Cont.'d

Step 2.

The sequence is increasing,

Proof Cont.'d

Step 2.

The sequence is increasing, i.e. for any $i < j$,
there exists ι_{ij} such that the following diagram commutes.

$$\begin{tikzcd}
 & \mathfrak{t}_j \arrow[dl, "\zeta_j"'] \\
A & \\
 & \mathfrak{t}_i \arrow[ul, "\zeta_i"] \arrow[uu, hook, "\iota_{ij}"']
\end{tikzcd}$$

Proof Cont.'d

Step 2.

The sequence is increasing, i.e. for any $i < j$, there exists ι_{ij} such that the following diagram commutes.

$$\begin{tikzcd}
 & \mathfrak{t}_j \arrow[dl, "\zeta_j"'] \\
A & \\
 & \mathfrak{t}_i \arrow[ul, "\zeta_i"] \arrow[uu, hook, "\iota_{ij}"']
\end{tikzcd}$$

Indeed ι_{ij} is just the embedding of $\mathfrak{t}_i$ in $\mathfrak{t}_j$

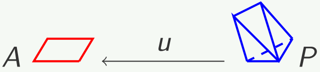
Proof Cont.'d

Step 3: The lifting of definable functions.

Proof Cont.'d

Step 3: The lifting of definable functions.

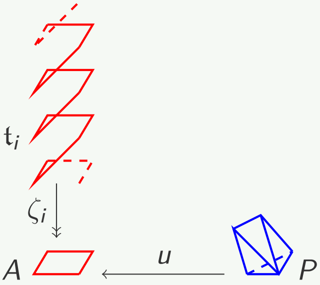
For **any** co-unifier (u, P) of A,



Proof Cont.'d

Step 3: The lifting of definable functions.

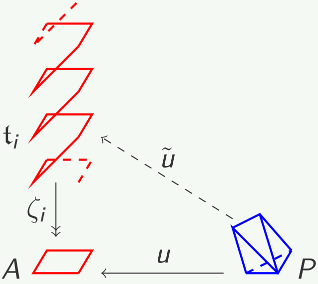
For **any** co-unifier (u, P) of A, there exists some $\mathfrak{t}_i$



Proof Cont.'d

Step 3: The lifting of definable functions.

For **any** co-unifier (u, P) of A, there exists some $\mathfrak{t}_i$ and an arrow $\tilde{u}$ (called the lift of u) making the following diagram commute.



$$\begin{tikzcd}
\mathfrak{t}_i \arrow[d, two heads, "\zeta_i"'] & \\
A & P \arrow[l, "u"'] \arrow[ul, dashed, "\tilde{u}"']
\end{tikzcd}$$

Considerations

The above lemma is the piecewise linear version of the "*Lifting of functions*" *Lemma*, widely used in algebraic topology.

Considerations

The above lemma is the piecewise linear version of the "*Lifting of functions*" *Lemma*, widely used in algebraic topology.

The crucial fact here is that we can always factorize through a finite portion of the *piece-wise linear cover*.

Considerations

The above lemma is the piecewise linear version of the "*Lifting of functions*" *Lemma*, widely used in algebraic topology.

The crucial fact here is that we can always factorize through a finite portion of the *piece-wise linear cover*.

As a matter of fact, for general reasons when such a map exists is unique *up to translations*. So the fact that in our setting such a map is actually a $\mathbb{Z}$-map is a quite pleasant discovery.

Considerations

The above lemma is the piecewise linear version of the "*Lifting of functions*" *Lemma*, widely used in algebraic topology.

The crucial fact here is that we can always factorize through a finite portion of the *piece-wise linear cover*.

As a matter of fact, for general reasons when such a map exists is unique *up to translations*. So the fact that in our setting such a map is actually a $\mathbb{Z}$-map is a quite pleasant discovery.

The Definable Lifting Lemma has two important corollaries.

Proof Cont.'d

Step 4: Corollary 1.

Given any co-unifier (P, u) for A, there exists a co-unifier $(\mathfrak{t}_i, \zeta_i)$ in the above sequence such that (P, u) is less general than $(\mathfrak{t}_i, \zeta_i)$.
(The sequence is *cofinal* in the poset of co-unifiers.)

Proof Cont.'d

Step 4: Corollary 1.

Given any co-unifier (P, u) for A, there exists a co-unifier $(\mathfrak{t}_i, \zeta_i)$ in the above sequence such that (P, u) is less general than $(\mathfrak{t}_i, \zeta_i)$.
(The sequence is *cofinal* in the poset of co-unifiers.)

A lattice point is a vector with integer coordinates.

Step 5: Corollary 2.

If (P, u) is a co-unifier for A with strictly fewer lattice points than $\mathfrak{t}_i$,

Proof Cont.'d

Step 4: Corollary 1.

Given any co-unifier (P, u) for A, there exists a co-unifier $(\mathfrak{t}_i, \zeta_i)$ in the above sequence such that (P, u) is less general than $(\mathfrak{t}_i, \zeta_i)$.
(The sequence is *cofinal* in the poset of co-unifiers.)

A lattice point is a vector with integer coordinates.

Step 5: Corollary 2.

If (P, u) is a co-unifier for A with strictly fewer lattice points than $\mathfrak{t}_i$, then there is no arrow $v\colon \mathfrak{t}_i \to P$ making the following diagram commute.

$$\begin{tikzcd}
 & \mathfrak{t}_i \arrow[dl, "\zeta_i"'] \arrow[dd, "v"] \\
A & \\
 & P \arrow[ul, "u"]
\end{tikzcd}$$

Proof Cont.'d

Step 6.

As a consequence of the last corollary we obtain:

Proof Cont.'d

Step 6.

As a consequence of the last corollary we obtain:

1. The sequence of $\mathfrak{t}_i$ is strict; i.e. $\mathfrak{t}_i$ is not more general than $\mathfrak{t}_j$ if $i < j$.

Proof Cont.'d

Step 6.

As a consequence of the last corollary we obtain:

1. The sequence of $\mathfrak{t}_i$ is strict; i.e. $\mathfrak{t}_i$ is not more general than $\mathfrak{t}_j$ if $i < j$.
2. The sequence admits no bound with a finite number of lattice elements. Therefore, no rational polyhedra can bound the sequence of $\mathfrak{t}_i$.

End of the Proof

Conclusion

Summing up, we have found a strictly linearly ordered, cofinal sequence of unifiers for A.

End of the Proof

Conclusion

Summing up, we have found a strictly linearly ordered, cofinal sequence of unifiers for A.
Furthermore the sequence is unbounded, hence the co-unification type of rational polyhedra is nullary (in a stronger sense).

End of the Proof

Conclusion

Summing up, we have found a strictly linearly ordered, cofinal sequence of unifiers for A.
Furthermore the sequence is unbounded, hence the co-unification type of rational polyhedra is nullary (in a stronger sense).

This proves that the Łukasiewicz calculus (as well as the theory of MV-algebras and ℓ-groups with strong unit) has nullary unification type. ☐

References

WM Beynon.
Duality theorems for finitely generated vector lattices.
Proceedings of the London Mathematical Society, 3(1):114, 1975.

W.M. Beynon.
Applications of duality in the theory of finitely generated lattice-ordered abelian groups.
Canad. J. Math, 29(2):243–254, 1977.

L. Cabrer and D. Mundici.
Rational polyhedra and projective lattice-ordered abelian groups with order unit.
ArXiv preprint arXiv:0907.3064, 2009.

W. Dzik.
Unification in some substructural logics of BL-algebras and hoops.
Reports in Mathematical Logic, 43:73–83, 2008.

S. Ghilardi.
Unification through projectivity.
Journal of Logic and Computation, 7(6):733–752, 1997.

V. Marra, L. Spada.
Duality, projectivity, and unification in Łukasiewicz logic and MV-algebras.
Submitted.

V. Marra, L. Spada.
The dual adjunction between MV-algebras and Tychonoff spaces.
Submitted.

V. Marra, L. Spada.
Solving equations in Abelian ℓ-groups.
In preparation.