# Ordinal spaces for GLB<sup>0</sup>

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Topological Semantics for  ${\rm GLP}_\Lambda$ A topological topological completeness proof Future work Provability logics Extending to ordinals The logic GLB

### ▶ The modal provability logic GL

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### The modal provability logic GL

characterizes provability for Σ<sub>1</sub>-sound theories

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## • $GLP_{\omega}$ has a modality [*i*] for each $i < \omega$

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Topological Semantics for  ${\rm GLP}_{\Lambda}$ A topological topological completeness proof Future work Provability logics Extending to ordinals The logic GLB

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- The closed fragment of  $GLP_{\omega}$ : no variables
- It has been used for an ordinal analysis of Peano Arithmetic
- Calculation carried out within the closed fragment
- Can this type of analysis be extended to stronger theories?

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 Introduce modalities [α] for each ordinal α satisfying the GLP axioms

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- $\blacktriangleright \varphi$  is provable from  ${\cal T}$  together with all true hyperarithmetical sentences of level  $\alpha$

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- Semantics based on ordinal spaces!

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Scattered spaces Existing results Drawbacks and new approach

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- $d_i$ : the derived set operator corresponding to  $\tau_i$
- ►  $x \in d_i(X) \iff \forall \mathcal{U} \in \tau_i \ (x \in \mathcal{U} \rightarrow \mathcal{U} \cap X \setminus \{x\} \neq \emptyset)$

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Scattered spaces Existing results Drawbacks and new approach

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- ► [Icard]  $GLP^0_{\omega}$  is complete w.r.t.  $\epsilon_0$  for specificly tailored topologies

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 The logics GLPA
 Scattered spaces

 Topological Semantics for GLPA
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- Our aim: a constructive definition with a purely topological completeness proof.

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Topological Semantics for $GLP_\Lambda$	Worms
A topological topological completeness proof	Soundness
Future work	Completeness

► Work in progress

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- Which ordinals can that be?

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- We define  $d^{\alpha}(X)$  for ordinals  $\alpha$

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$$d^0(X) = X$$

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Lemma

$$d_0^\alpha(X) = \{x \mid \mathsf{le}(x) \ge \alpha\}$$

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### ▶ **GLB** $\vdash \langle 1 \rangle \top \rightarrow \langle 0 \rangle^n \top$

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$$\tau_1 := \tau_0 \cup \{ \overline{\{x_a + \omega^a \mid a \in A \& x_a \in A'\}} \mid A \subseteq \text{Succ } \& A' \subseteq \text{On} \}$$

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• 
$$\overline{Y}$$
 denotes the closure of Y in  $\tau_0$ 

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► Modal formulas that consist only of ⊤ preceded by a (possibly empty) sequence of consistency operators are called worms.

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- ► Modal formulas that consist only of ⊤ preceded by a (possibly empty) sequence of consistency operators are called worms.
- We often write binary words a<sub>0</sub>a<sub>1</sub>...a<sub>n</sub> instead of ⟨a<sub>0</sub>⟩⟨a<sub>1</sub>⟩...⟨a<sub>n</sub>⟩⊤

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- Theorem Every closed formula of GLB is equivalent to a boolean combination of worms

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## ▶ Define $<_i$ on *S* by $\alpha <_i \beta \iff \mathbf{GLB} \vdash \beta \rightarrow \langle i \rangle \alpha$ for $i \in \{0, 1\}$

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- ▶ and  $o(0^m) = m$

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### • Theorem $([1, \kappa], \tau_0, \tau_1)$ is sound for **GLB**<sub>0</sub>.

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$\begin{tabular}{lllllllllllllllllllllllllllllllllll$	The main ideas Worms Soundness Completeness
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- Remaining axiom:  $\langle 0 \rangle \varphi \rightarrow [1] \langle 0 \rangle \varphi$ .

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#### $\blacktriangleright$ If $\alpha$ is some worm

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- If  $\alpha$  is some worm
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$$d(\alpha, X) := \{ x + \omega^{o(\beta)} \mid \mathsf{GLB} \vdash \beta \to \alpha \And x \in \mathsf{On} \}$$

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Proof: with hand and feet, case distinctions, etc

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• Corollary 
$$d(0\alpha, X) = d_0^{o(0\alpha)}(X)$$

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- Remaining axiom:  $\langle 0 \rangle \alpha \rightarrow [1] \langle 0 \rangle \alpha$ .
- $\blacktriangleright~\langle 0 \rangle \varphi$  is equivalent to a Boolean combination of words

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- Which is a relatively easy exercise

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#### Completeness is relatively easy

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- Completeness is relatively easy
- ▶ If GLB  $\nvdash \varphi$ , and GLB  $\nvdash \neg \varphi$ then

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The logics $GLP_{\Lambda}$	The main ideas
Topological Semantics for $GLP_{\Lambda}$	Worms
A topological topological completeness proof	Soundness
Future work	Completeness

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- Theorem ([1, κ], τ<sub>0</sub>, τ<sub>1</sub>) is sound and complete for GLB<sub>0</sub> whenever κ ≥ ω<sup>ω<sup>ω</sup></sup>.

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► 
$$\tau_1 := \tau_0 \cup \{ \overline{\{x_a + \omega^a \mid a \in A \& x_a \in A'\}} \mid A \subseteq \text{Succ } \& A' \subseteq \text{On} \}$$

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• 
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Only one inclusion missing to finish the entire proof

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- ▶  $GLP_{\Lambda}$  is Kripke incomplete (w.r.t. canonical frames) for  $\Lambda \ge 2$
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- Apply Icard's technique to this frame

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## Barcelona, April 16-20, 2011

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- Thank you