Ordinal spaces for GLB^0

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- Calculation carried out within the closed fragment
- Can this type of analysis be extended to stronger theories?

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- More in particular:
- Semantics based on ordinal spaces!

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- $x \in d_i(X) \Leftrightarrow \forall \mathcal{U} \in \tau_i \, (x \in \mathcal{U} \rightarrow \mathcal{U} \cap X \setminus \{x\} \neq \varnothing)$

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- [Beklemishev] If V=L, then GLB is complete w.r.t. interval topology together with club topology for any ordinal $> \aleph_\omega$

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- [Icard] GLP^0_ω is complete w.r.t. ϵ_0 for specificly tailored topologies

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- Our aim: a constructive definition with a purely topological completeness proof.

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- Which ordinals can that be?

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- **Lemma**

$$d_0^\alpha(X) = \{x \mid \mathrm{le}(x) \geq \alpha\}$$

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- **Theorem** Every closed formula of GLB is equivalent to a boolean combination of worms

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- $o : S \rightarrow \omega^\omega$
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- and $o(0^m) = m$

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- Remaining axiom: $\langle 0\rangle\varphi \to [1]\langle 0\rangle\varphi$.

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- **Corollary** $d(0\alpha, X) = d_0^{o(0\alpha)}(X)$

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- **Theorem** $([1, \kappa], \tau_0, \tau_1)$ is sound and complete for $\mathbf{GLB}_0$ whenever $\kappa \geq \omega^{\omega^\omega}$.

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- Only one inclusion missing to finish the entire proof

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- Apply Icard's technique to this frame

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- Thank you