Classifying Unification Problems in Distributive Lattices and Kleene Algebras

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(joint work with Simone Bova)

Preliminaries

Algebraic Unifiers

[1] S. Ghilardi. Unification through Projectivity. Journal Logic and computation 7(4), 1997.

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Given an algebraic language $\mathcal{L}$, a **unification problem** in the language $\mathcal{L}$ is a finite set of equations $S = \{(s_1, t_1), \ldots, (s_n, t_n)\} \subseteq \mathrm{Term}^2_{\mathcal{L}}$.

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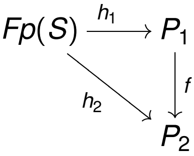
Given a unification problem S and an equational theory E, an **algebraic E-unifier** for S is pair (h, P) where P is a projective algebra in the equational class determined by E and $h\colon Fp(S) \to P$ is a homomorphism.

Preliminaries

Algebraic Unifiers

If $(h_1, P_1), (h_2, P_2)$ are algebraic E-unifiers for S, we say that (h_1, P_1) is **more general** than (h_2, P_2) $((h_2, P_2) \preccurlyeq (h_1, P_1))$ if there exists a homomorphism $f\colon P_1 \to P_2$ such that



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$$\begin{array}{ccc} Fp(S) & \xrightarrow{h_1} & P_1 \\ & {\scriptstyle h_2}\searrow & \downarrow {\scriptstyle f} \\ & & P_2 \end{array}$$

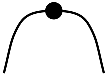
We denote by $\mathcal{U}_E(S)$ the pre-ordered set of algebraic E-unifiers for S.

Preliminaries

Unification types

A unification problem S in an equational theory E is said to have type:

$$\mathcal{U}_E(S)$$

1 

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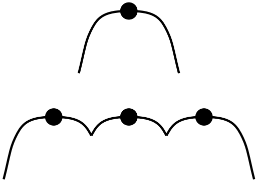

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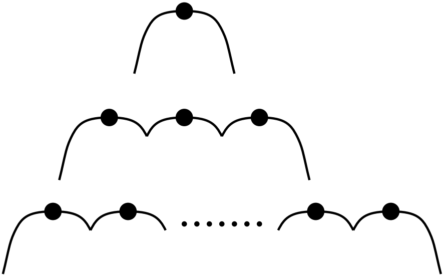

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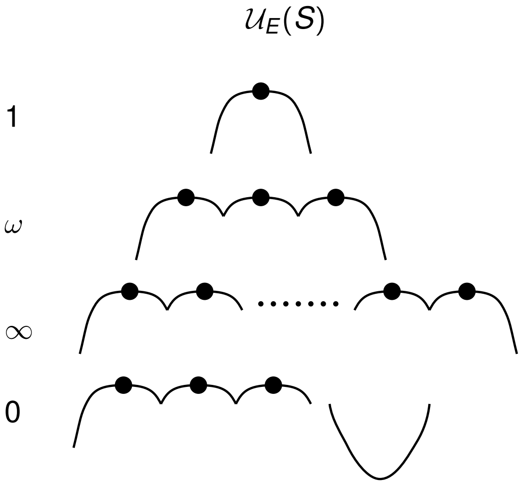

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Unification types

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- 1 if every unification problem S has type 1,
- ω if every unification problem S has type ω and at least one S has not unification type 1,
- ∞ if every unification problem S has type 1, ω or ∞ and at least one S has unification type ∞,
- 0 if at least one S has unification type 0.

Unifiers through duality

Duals of Unifiers

Unifiers through duality

Duals of Unifiers

$$Fp(S) \xrightarrow{u} P \qquad \leftrightsquigarrow \qquad I \xrightarrow{f} D(Fp(S))$$

(I) Description of finitely generated projective algebras. *Injective objects*.

Unifiers through duality

Working strategy

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(II) Analysis of the unification type. *Examples*

Unifiers through duality

Working strategy

(I) Description of finitely generated projective algebras. *Injective objects*.

(II) Analysis of the unification type. *Examples*

(III) Classification of a given unification problem. *Analysis of the examples*

Bounded Distributive Lattices

Bounded Distributive Lattices

(I) Description of finitely generated projectives

A finite bounded distributive lattice L is projective if and only if $\langle J(L), \leq \rangle$ is a lattice.

Bounded Distributive Lattices

(II) Analysis of the unification type

The unification problem $S = \{x \wedge y \approx z \vee t\}$ has nullary unification type.

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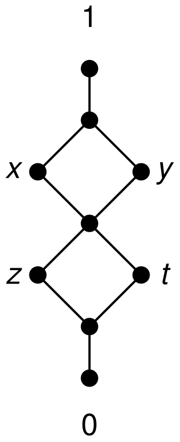

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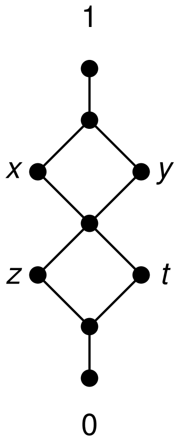

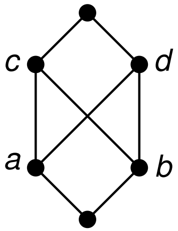

Bounded Distributive Lattices

(III): Causes of nullarity

Lemma

Let S be a unification problem in the language of bounded lattices. If there exist $x, a, b, c, d, y \in J(F_p(S))$ satisfying:

(i) $x \leq a, b \leq c, d \leq y$, *and*

(ii) *it does not exist* $e \in J(F_p(S))$ *such that* $a, b \leq e \leq c, d$,

then the unification type of S in the equational theory of distributive lattices is nullary.

Bounded Distributive Lattices

Classification

Theorem

Let S be a unification problem in the language of bounded lattices. Then the unification type of S is:

Unitary *if and only if $J(F_p(S))$ is a lattice,*

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- **Unitary** *if and only if* $J(F_p(S))$ *is a lattice,*
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Nullary *otherwise.*

Kleene Algebras

Kleene Algebras

Definition

A **Kleene** algebra $\mathbf{A} = (A, \wedge, \vee, \neg, 0, 1)$ is a bounded distributive lattice equipped with a unary operation, $\neg x$, satisfying:

$$x = \neg\neg x,$$

$$x \wedge y = \neg(\neg x \vee \neg y),$$

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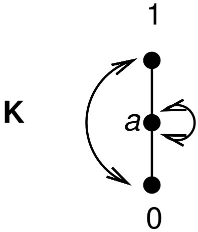

Kleene Algebras

Natural duality

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A structure $\mathbf{X} = \langle X, \leq, \sim, Y, \tau \rangle$ is called a **Kleene space** if it satisfies the following conditions:

(i) $\langle X, \leq, \tau \rangle$ is a Priestley space,

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(ii) $\sim$ is a closed binary relation, i.e., $\sim$ is a closed subset of X^2,

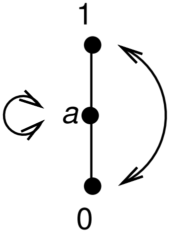
(iii) Y is a closed subset of X, and

(iv) for every $x, y, z \in X$:

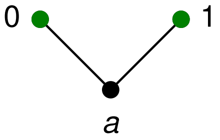
(a) $x \sim x$,
(b) if $x \sim y$ and $x \in Y$, then $x \leq y$,
(c) if $x \sim y$ and $y \leq z$, then $z \sim x$.

Kleene Algebras

Natural duality



K



$\tilde{\mathbf{K}}$

$\{\leq, \sim, Y\}$

Theorem

Let A be a finite Kleene Algebra. Then the following statements are equivalent:

(i) *A is projective,*

(ii) *$XK(A) = \{X_A, \leq_A, \sim_A, Y_A, \tau_A\}$ satisfies the following conditions:*

  (a) *$\langle X_A, \leq_A \rangle$ is a meet semi-lattice,*

  (b) *$Y_A = Max(\langle X_A, \leq_A \rangle)$,*

  (c) *X_A is 2-conditionally complete,*

  (d) *$x \sim_A y$ if and only if there exists $z \in X_A$ such that $x, y \leq z$.*

Kleene Algebras

(II): Nullarity

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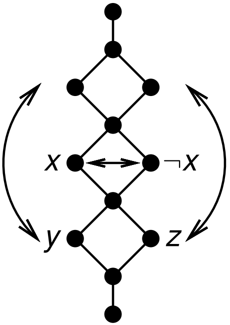

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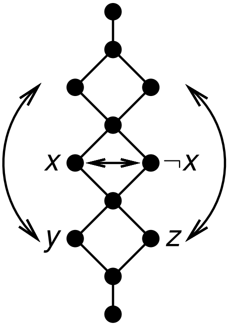

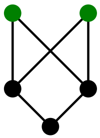

Kleene Algebras

(III): Necessary Conditions for Nullarity

Lemma

Let S be a unification problem in the language of kleene algebras. If there exist $x, a, b, c, d, y, z \in XK(F_p(S))$ satisfying:

(i) $x \leq a, b \leq c, d$, $c \leq y$ *and* $d \leq z$,

(ii) $y, z \in Y$, *and*

(iii) *it does not exist* $e \in XK(F_p(S))$ *such that* $a, b \leq e \leq c, d$,

then the unification type of S is nullary.

Kleene Algebras

(III): Necessary Conditions for Nullarity

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Let S be a unification problem in the language of kleene algebras with. If there exist $w, a, b, c, d, e, f, x, y, z \in XK(F_p(S))$ satisfying:

(i) $w \leq a, b, c$; $a \leq d, e$; $b \leq d, f$; $c \leq e, f$; $d \leq x$; $e \leq y$; *and* $f \leq z$,

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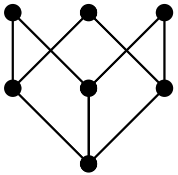

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$$K = \{x \in XK(Fp(S)) \mid \exists y \in Y, x \leq y\}$$

is a 2-conditionally complete meet semilattice,

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- **nullary** *otherwise.*

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Thank you for your attention!

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