Finite Representations for Finite Algebras of Binary Relations

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July 28, 2011

Algebras of Binary Relations

Subset of $\wp(X \times X)$ (some base X).
Relations: $=, \subseteq$.
Constants: $\emptyset, 1, Id_X$.
Functions: $\cup, \cap, \setminus, ^\smile, ;, \mathsf{dom}, \mathsf{rng}$
where

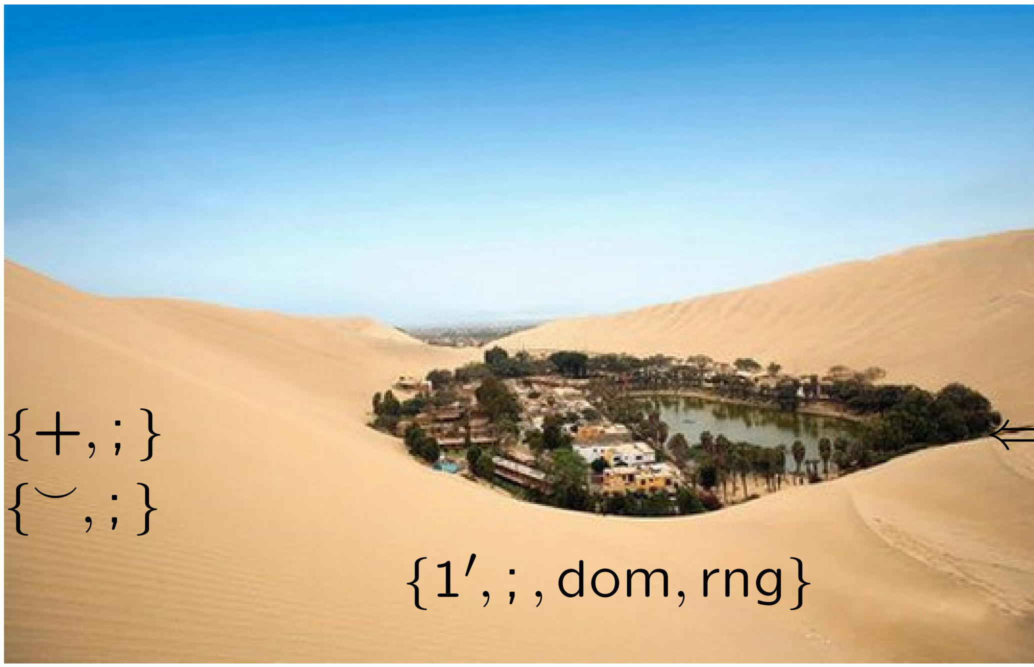
$$
\begin{aligned}
S^\smile &= \{(y,x) : (x,y) \in S\} \\
S;T &= \{(x,y) : \exists z\ (x,z) \in S \& (z,x) \in T\} \\
\mathsf{dom}(S) &= \{(x,x) : \exists y\ (x,y) \in S\} \\
\mathsf{rng}(S) &= \{(y,y) : \exists x\ (x,y) \in S\}
\end{aligned}
$$

Representation Classes

Signature $S \subseteq \{=, \leq, 0, 1, -, +, \cdot, 1', ^\smile, ;, \mathsf{dom}, \mathsf{rng}\}$.
R(S): the class of S-algebras isomorphic to sets of binary relations closed under S.
Signature S has finite representation property if every finite, representable S-algebra has a representation over a finite base.
We seek signatures where R(S) is finitely axiomatisable, also signatures with FMP.
There are 567 inequivalent signatures.

Signature $\mathcal{F}$	Fin. ax.	Ref.
$\mathbf{RA}$	$\times$	Monk, 1964
$; \notin \mathcal{F}$	$\surd$	Schein, 1991
$\{\leq, ;\}, \{\cdot, ;\}$	$\surd$	Schein, 1991
$\{\cdot, ;\}^{\uparrow}$	$\times$	Mikulas, 15.40
$\{\smile, ;\}$	$\times$	Bredikhin,1977
$\{+, ;\}$	$\times$	Andreka, 1988
$\{\leq, 1', ;\}$	$\times$	Hirsch
$[\{\mathsf{dom}, ;\}, \{0, 1, \mathsf{dom}, \mathsf{rng}, \mathsf{AntiDom}, 1', ;\}]$	$\times$	Hirsch Mikulas
$[\{\mathsf{dom}, \smile, ;\}, \{\leq, 0, 1, 1', \smile, \mathsf{dom}, \mathsf{rng}, ;\}]$	$\surd$	Bredikhin, 1977

An Oasis of Finite Axiomatisability



$\Longleftarrow \quad \{\leq, 1', \smile, ;, \mathsf{dom}, \mathsf{rng}\}$

FMP

Theorem 1 *Let* $S \subseteq \{0, 1, -, +, \cdot, \leq, 1', ^\smile, ;, \mathsf{dom}, \mathsf{rng}\}$.

1. *If composition is not in* S *then* S *has fmp.*

2. *If* $S \supseteq \{\cdot, ;\}$ *then* S *does not have the fmp.*

3. *If* $\{^\smile, ;, \mathsf{dom}, \mathsf{rng}\} \subseteq S \subseteq \{0, 1, \leq, 1', ^\smile, ;, \mathsf{dom}, \mathsf{rng}\}$ *then* S *is finitely axiomatisable and has fmp.*

Axioms for $\mathsf{R}\{0, 1, \leq, 1', \smile, ;, \mathsf{dom}, \mathsf{rng}\}$

$\leq$ is a partial order, bounds 0, 1,

$\smile$, ;, dom, rng are monotonic and normal.

$(1', \smile, ;)$ is involuted monoid.

Domain/range axioms

$$\mathrm{dom}(a) = (\mathrm{dom}(a))^{\smile} \leq 1' = \mathrm{dom}(1')$$

$$\mathrm{dom}(a) \leq a \,;\, a^{\smile}$$

$$\mathrm{dom}(a^{\smile}) = \mathrm{rng}(a)$$

$$\mathrm{dom}(\mathrm{dom}(a)) = \mathrm{dom}(a) = \mathrm{rng}(\mathrm{dom}(a))$$

$$\mathrm{dom}(a) \,;\, a = a$$

$$\mathrm{dom}(a \,;\, b) = \mathrm{dom}(a \,;\, \mathrm{dom}(b))$$

$$\mathrm{dom}(\mathrm{dom}(a) \,;\, \mathrm{dom}(b)) = \mathrm{dom}(a) \,;\, \mathrm{dom}(b) = \mathrm{dom}(b) \,;\, \mathrm{dom}(a)$$

$$\mathrm{dom}(\mathrm{dom}(a) \,;\, b) = \mathrm{dom}(a) \,;\, \mathrm{dom}(b)$$

Closed Sets

$\mathcal{A} = (A, 0, 1, \leq, 1', {}^{\smile}, ;, \mathsf{dom}, \mathsf{rng}) \models Ax$, $X \subseteq \mathcal{A}$ is closed if

- $X^{\uparrow} = X$,
- $\mathsf{dom}(X); X; \mathsf{rng}(X) \subseteq X$.

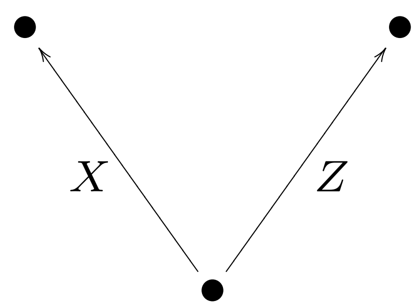
For $X \subseteq A$ let $\gamma(X)$ be the *closed set generated by* X.

- $a^{\uparrow}$ is closed, for $a \in \mathcal{A}$.
- If X, Y are closed, $\mathsf{dom}(X) = \mathsf{dom}(Y)$ and $\mathsf{rng}(X) = \mathsf{rng}(Y)$ then $X \cup Y$ is closed and $Z; (X \cup Y) = Z; X \cup Z; Y$.

Representation

Base: the set $\Gamma(\mathcal{A})$ of closed subsets of $\mathcal{A}$.
Isomorphism: $\iota(a) = \{(X, Y) : X; a \subseteq Y \wedge Y; a^{\smile} \subseteq X\}$.
Faithful: $a \not\leq b \;\Rightarrow\; (\mathsf{dom}(a)^{\uparrow}, a^{\uparrow}) \in \iota(a) \setminus \iota(b)$.

Composition



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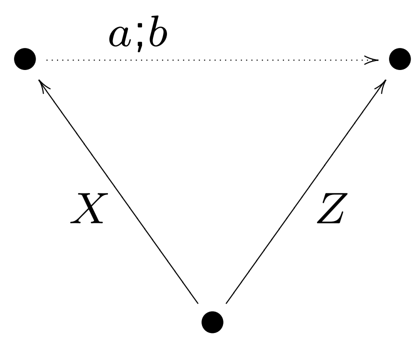
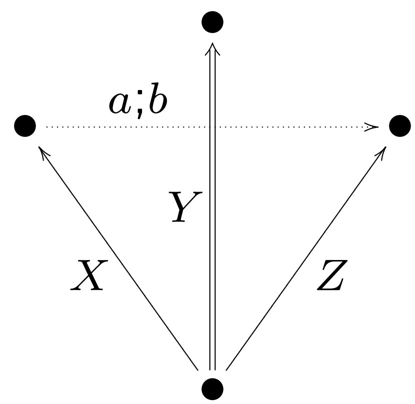
Composition



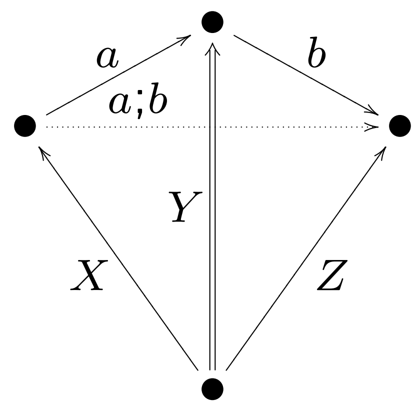
$X; a; b \subseteq Z, \quad Z; b^{\smile}; a^{\smile} \subseteq X$.

Composition



$$Y = X; a^{\uparrow}; \mathsf{rng}(Z; b^{\smile}) \ \cup \ Z; b^{\smile}; \mathsf{rng}(X; a).$$

Composition



$$Y = X; a^{\uparrow}; \mathsf{rng}(Z; b^{\smile}) \; \cup \; Z; b^{\smile}; \mathsf{rng}(X; a).$$

Algebra of closed sets

Domain algebra $\mathcal{A} = (A, \leq, 1', ^\smile, ;\,, \mathsf{dom}, \mathsf{rng})$
$\mathcal{A}^* = (\Gamma(\mathcal{A}), \supseteq, ^\smile, *, \mathsf{dom}, \mathsf{rng})$
where $S * T = \gamma(S; T)$.

1. $(\Gamma(\mathcal{A}), \subseteq, \emptyset, \Gamma(\mathcal{A}))$ is complete distributive lattice, $\mathcal{A}$ embeds $\bigwedge$-densely in $\mathcal{A}^*$ via $a \mapsto a^{\uparrow}$.

2. $\mathcal{A}^*$ satisfies all axioms for domain algebras except $X * X^{\smile} \geq \mathsf{dom}(X)$.

3. The embedding preserves these existing infima: $\mathsf{dom}(a) = 1' \wedge a; a^{\smile}$, $d_1; a \wedge d_2; a = d_1; d_2; a$ (also, order duals), in other respects it is freely generated DL.

4. Suppose $\mathsf{dom}(X_0) \subseteq \mathsf{dom}(X_1) \subseteq \ldots$ and $\mathsf{rng}(X_0) \subseteq \mathsf{rng}(X_1) \subseteq \ldots$. Then

$$(\bigwedge_i X_i) * Z = \bigwedge_i (X_i * Z)$$

Problems

1. $\mathcal{A}^*$ is what kind of completion of $\mathcal{A}$?

2. Does $\{., ;\}$ have FMP?

3. Does $\{\mathsf{dom}, \mathsf{rng}, \mathsf{AntiDom}, \smile, ;\}$ have FMP?

4. What about signatures containing $\{;, +\}$?