The Equational Theory of Kleene Lattices

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Kleene algebras

The class KA of Kleene algebras is the collection of algebras of the similarity type $(+, ;, ^*, 0, 1)$ satisfying a certain finite set of quasi-equations (Kozen).

Standard interpretations of KA are

- language algebras, LKA,
  connection with regular expressions and regular languages.
- relation algebras, RKA,
  connection with program semantics and propositional dynamic logic PDL.

Language Kleene algebras

Let Σ be a set (alphabet) and Σ^* denote the free monoid of all finite words over Σ, including the empty word λ. The class LKA of *language Kleene algebras* is defined as the class of subalgebras of algebras of the form

$$(\wp(\Sigma^*), +, ;, ^*, 0, 1)$$

- $+$ is set union,
- ; is complex concatenation (of words)

$$X ; Y = \{xy : x \in X, y \in Y\}$$

- * is the Kleene star operation

$$X^* = \{x_0 x_1 \ldots x_{n-1} : n \in \omega, x_i \in X \text{ for each } i < n\}$$

- 0 is the empty language and
- 1 is the singleton language consisting of the empty word λ.

Relational Kleene algebras

The class RKA of *relational Kleene algebras* is defined as the class of subalgebras of algebras of the form

$$(\wp(W), +, ;, ^*, 0, 1)$$

where $W = U \times U$ for some set U,

- $+$ is set union,
- ; is relation composition

$$x ; y = \{(u, v) \in W : (u, w) \in x \text{ and } (w, v) \in y \text{ for some } w\}$$

- * is reflexive-transitive closure,
- 0 is the emptyset and
- 1 is the identity relation restricted to W

$$1 = \{(u, v) \in W : u = v\}$$

Equational theories of Kleene algebras

LKA $\subseteq$ RKA, whence Eq(RKA) $\subseteq$ Eq(LKA).

Cayley representation f assigns a binary relation to a language X over an alphabet Σ:

$$f(X) = \{(w, wx) : w \in \Sigma^* \text{ and } x \in X\}$$

The Cayley representation respects the Kleene algebra operations: $+, ;, ^*, 0, 1$.

But

RKA $\not\subseteq$ LKA.

The identity $1 = \{\lambda\}$ is an atom (minimal, non-zero element) in language algebras.

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Equational theories of Kleene algebras (ctd.)

Same equational theory:

Eq(RKA) = Eq(LKA).

The free algebras of RKA and LKA coincide — it is the algebra of regular expressions, hence a language Kleene algebra (Németi).

Furthermore,

Kozen:

Eq(KA) = Eq(LKA)(= Eq(RKA)).

Thus the equational theory of RKA and LKA is finitely *quasi-axiomatizable*.

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Redko:

The equational theory of language (relational) Kleene algebras is not finitely axiomatizable.

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Kleene lattices

Note:

- regular languages are closed under intersection,
- intersection in relational interpretation — PDL with intersection

Kleene lattices:

LKL and RKL are defined as expansions of LKA and RKA, respectively, with meet $\cdot$ interpreted as intersection.

Main topic of this talk:

What can we say about the equational theories of LKL and RKL?

Free Kleene lattices

Unlike in the meet-free case free algebras are not language algebras.

Fact:

No free algebra of LKL or RKL with at least one free generator is representable as a language algebra.

Proof: In the free algebra, the terms 0, $x \cdot 1$ and 1 are below 1, and all three of 0, $x \cdot 1$ and 1 are different. (For example, $x \cdot 1 \neq 1$ in the free algebra, because if $x = 0$, then $x \cdot 1 = 0 \neq 1$.) However, in a language representation 1 is the one-element set $\{\lambda\}$ which has only two subsets. □

Fact:

The free algebra of RKL is a relation algebra, it is in RKL.

More language- than relational validities

The Cayley representation f preserves also meet:

LKL $\subseteq$ RKL, whence Eq(RKL) $\subseteq$ Eq(LKL).

However, strict inclusion and not equality holds in this case:

$$(x\,;y)\cdot 1 = (x\cdot 1)\,;(y\cdot 1) \tag{1}$$

$$(x\cdot 1)\,;y = y\,;(x\cdot 1) \tag{2}$$

$$(z+(x\cdot 1)\,;y)^* = z^* + (x\cdot 1)\,;(z+y)^* \tag{3}$$

E.g. equation (1) expresses that λ cannot be written as a concatenation of words distinct from λ.

Main result 1:

Equations (1), (2) and (3) axiomatize Eq(LKL) over Eq(RKL), i.e.

$$\mathrm{Eq(RKL)} \cup \{(1),(2),(3)\} \vdash \mathrm{Eq(LKL)}$$

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$$(x ; y) \cdot 1 = (x \cdot 1) ; (y \cdot 1) \quad (1)$$

$$(x \cdot 1) ; y = y ; (x \cdot 1) \quad (2)$$

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Without identity

Note: all three "distinguishing" equations (1), (2) and (3) use the identity 1.
Recall: $0^* = 1$ and $x^+ := x \,;\, x^*$.

Identity-free Kleene lattices:

Let RKL^- and LKL^- denote the $(+, ;, ^+, 0)$-subreducts of RKL and LKL, respectively.

Main result 2:

The equational theories of LKL^- and RKL^- coincide, i.e.

$$\mathrm{Eq}(\mathsf{LKL}^-) = \mathrm{Eq}(\mathsf{RKL}^-)$$

Also, like in the Kleene algebra case,

Representing free algebras:

The free algebras of LKL^- are representable as language algebras.

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Continuity and ground terms

For a variable x we let $\Gamma(x) = \{x\}$, $\Gamma(0) = \emptyset$, $\Gamma(1) = \{1\}$,

$$\Gamma(\tau + \sigma) = \Gamma(\tau) \cup \Gamma(\sigma)$$
$$\Gamma(\tau \cdot \sigma) = \Gamma(\tau) \cdot \Gamma(\sigma)$$
$$\Gamma(\tau \,;\, \sigma) = \Gamma(\tau) \,;\, \Gamma(\sigma)$$
$$\Gamma(\tau^*) = \bigcup\{\Gamma(\tau^n) : n \in \omega\}$$
$$\Gamma(\tau^+) = \bigcup\{\Gamma(\tau^n) : 0 < n \in \omega\}$$

and we let $GT = \bigcup_\tau \Gamma(\tau)$ denote the set of *ground terms*.

Continuity:

For every term τ, language or relation algebra $\mathfrak{A}$ and valuation k,

$$\tau^{\mathfrak{A}}[k] = \bigcup\{\sigma^{\mathfrak{A}}[k] : \sigma \in \Gamma(\tau)\}$$

E.g. instead of RKL $\models \tau \leq \sigma$ prove that for every $\tau' \in \Gamma(\tau)$, there is $\sigma' \in \Gamma(\sigma)$ with RKL $\models \tau' \leq \sigma'$.

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Term graphs

Term graphs as special 2-pointed, labelled graphs defined by induction on the complexity of ground terms. Let $G(0) = \emptyset$, for variable x, we let

$$G(x) = (\{\iota_x, o_x\}, \{(\iota_x, x, o_x)\}, \iota_x, o_x)$$

where $\iota_x \neq o_x$, and

$$G(1) = (\{\iota_1\}, \emptyset, \iota_1, \iota_1)$$

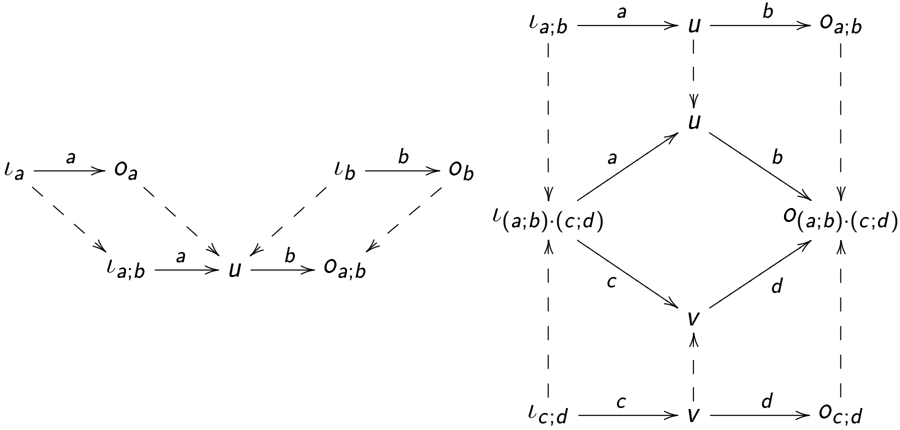
i.e. in this case $\iota_1 = o_1$. For terms σ and τ, we set

$$G(\sigma \,;\, \tau) = G(\sigma) \,;\, G(\tau) \quad \text{concatenation}$$

and

$$G(\sigma \cdot \tau) = G(\sigma) \cdot G(\tau) \quad \text{almost disjoint union}$$

Graph example: $G((a\,;b)\cdot(c\,;d))$



Graphs and maps

Map Lemma 1 (Andréka–Bredikhin):

Let τ be a ground term, U be a set and k be an evaluation of the variables of τ in $\wp(U \times U)$. Then for every $(u, v) \in U \times U$, the following are equivalent.

1. $(u, v) \in \tau[k](= \tau[k]^{\wp(U \times U)})$.
2. There is a map $h_\tau\colon \text{nodes}(G(\tau)) \to U$ such that $h_\tau(\iota_\tau) = u$, $h_\tau(o_\tau) = v$, and for every edge $(i, x, j) \in \text{edges}(G(\tau))$, we have $(h_\tau(i), h_\tau(j)) \in k(x)$.

Corollary:

RKL $\models \tau \leq \sigma$ iff there is a graph homomorphism $g\colon G(\sigma) \to G(\tau)$.

Words and maps

Map Lemma 2:

Let τ be a ground term, Σ be an alphabet and k be an evaluation of the variables of τ in $\wp(\Sigma^*)$. Then for every $w = w_1 \dots w_n \in \Sigma^*$, the following are equivalent.

1. $w \in \tau[k]$.
2. There is an order-preserving map $f_\tau : \text{nodes}(G(\tau)) \to n+1 = \{0, 1, \dots, n\}$ such that $f_\tau(\iota_\tau) = 0$, $f_\tau(o_\tau) = n$, and for every edge $(i, x, j) \in \text{edges}(G(\tau))$, we have $w(f_\tau(i), f_\tau(j)) \in k(x)$ where $w(p, q) = w_p \dots w_q$.

Say, $ab \in x[k]$ and $cd \in y[k]$ so that $abcd \in x\,;y[k]$:

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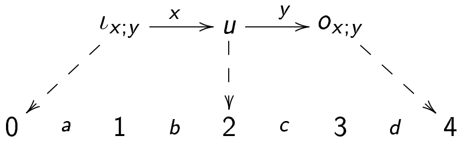

Term words

For every ground term τ, define the *term word* w_τ:

- $w_x = \ell_x$
- $w_1 = \lambda$
- $w_{\tau;\sigma} = w_\tau w_\sigma$
- if $w_\tau = u_1 \dots u_n$ and $w_\sigma = v_1 \dots v_n$ (by adding extra letters to the shorter one), then $w_{\tau\cdot\sigma} = u_1 v_1 u_2 v_2 \dots u_n v_n$.

E.g. $w_a = \ell_a$, $w_b = \ell_b$, $w_c = \ell_c$, $w_d = \ell_d$, whence $w_{a;b} = \ell_a \ell_b$, $w_{c;d} = \ell_c \ell_d$ and

$$w_{(a;b)\cdot(c;d)} = \ell_a \ell_c \ell_b \ell_d$$

Maps and valuations

Map:

There is an order-preserving map

$$f_\tau : \text{nodes}(G(\tau)) \longrightarrow \text{length}(w_\tau) + 1$$

Furthermore, if τ is identity-free, then f_τ is injective.

Valuation:

There is a valuation k_τ into the language algebra $\mathfrak{A}_\tau$ over the alphabet consisting of the letters of w_τ such that

$$w_\tau \in \tau^{\mathfrak{A}_\tau}[k_\tau]$$

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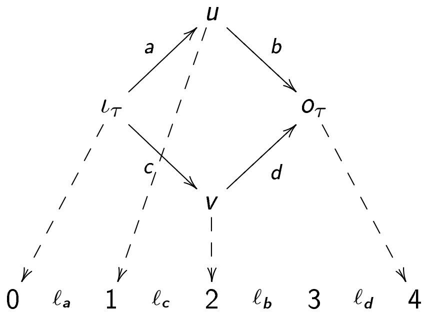
Example for $\tau = (a\,;b)\cdot(c\,;d)$

The map f_τ:



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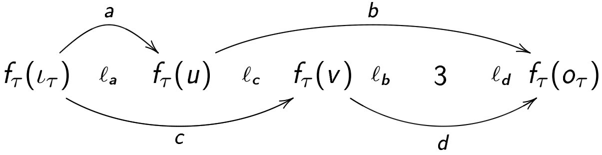

Proof of $\mathrm{Eq}(\mathsf{LKL}^-) \subseteq \mathrm{Eq}(\mathsf{RKL}^-)$

Let $\mathsf{LKL}^- \models \tau \leq \sigma$. By continuity we can assume that τ is a ground term. By the construction of w_τ, k_τ and $\mathfrak{A}_\tau$, we have $w_\tau \in \tau^{\mathfrak{A}_\tau}[k_\tau]$. Hence $w_\tau \in \sigma^{\mathfrak{A}_\tau}[k_\tau]$ by $\mathfrak{A}_\tau \in \mathsf{LKL}^-$.
By Map Lemma 2 there is an order-preserving map $f_\sigma \colon G(\sigma) \to \mathrm{length}(w_\tau) + 1$.
Also there is an order-preserving, injective map $f_\tau \colon G(\tau) \to \mathrm{length}(w_\tau) + 1$.
Then $h = f_\sigma \circ f_\tau^{-1} \colon G(\sigma) \to G(\tau)$ is the desired homomorphism that witnesses $\mathsf{RKL}^- \models \tau \leq \sigma$.

The proof of the finite axiomatizability of Eq(LKL) over Eq(RKL) goes along similar lines, but it is more technical.

Finite (quasi-)axiomatizability?

Question:

Are the equational theories Eq(LKL) and Eq(RKL) finitely axiomatizable?

Task:

Find a finitely axiomatizable quasi-variety that generates the same variety as RKL.

Let RKL^* and LKL^* denote the $(\cdot, +, ;, ^*)$-reducts of RKL and LKL, respectively.

Question:

Is $Eq(LKL^*)$ finitely axiomatizable over $Eq(RKL^*)$?

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Question:

Is Eq(LKL*) finitely axiomatizable over Eq(RKL*)?

Converse and Domain

Let $\mathsf{RKA}^{\smile}$ denote RKA expanded with inverse (interpreted as relation converse).

Ésik et al.:

$\mathsf{Eq}(\mathsf{RKA}^{\smile})$ is not finitely axiomatizable, but it is finitely quasi-axiomatizable.

Do these extend to $\mathsf{Eq}(\mathsf{RKL}^{\smile})$?

Using inverse and meet the operations domain and range are definable:

$$\mathsf{Dom}(x) := 1 \cdot (x \,;\, x^{\smile}) \text{ and } \mathsf{Ran}(x) := 1 \cdot (x^{\smile} \,;\, x)$$

Let $\mathsf{RKA}^{\mathsf{D},\mathsf{R}}$ denote the expansion of RKA with domain and range.

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