Two results on compact congruences

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Congruence lattices

Problem. For a given class $\mathcal{K}$ of algebras describe Con $\mathcal{K}$ =all lattices isomorphic to Con A for some $A \in \mathcal{K}$.

- very few relevant classes $\mathcal{K}$ with a satisfactory answer;
- many partial results;
- well documented complexity of the problem.

Number of compact congruences

Let A be an infinite algebra. What can we say about the cardinality of $\operatorname{Con}_c A$? (The $\vee$-semilattice of all compact (finitely generated) congruences)?
Obvious:

$$|\operatorname{Con}_c A| \leq |A|.$$

When the equality holds?

Theorem

Suppose that A is a subdirect product of finite algebras, $A \leq \Pi_{i \in I} A_i$. Then A has at least $|I|$ congruences, and the inequality

$$|\operatorname{Con} A| \leq 2^{|\operatorname{Con}_c A|}$$

gives a lower bound for $|\operatorname{Con}_c A|$. In fact, for $|A| = \aleph_0$ we are done.

Theorem

Let A be an infinite subalgebra of the direct product $\Pi_{i \in I} A_i$. Suppose that there exists a natural number n such that $|A_i| \leq n$ for every $i \in I$. Then $|\operatorname{Con}_c A| = |A|$.

Common bound necessity

The common finite bound on the cardinalities of A_i is necessary.
The ring of p-adic integers

- is a subdirect product of finite rings,
- its cardinality is continuum,
- has only countably many congruences.

Crucial part of the proof

Lemma

Let n be a natural number, X an infinite set and $F \subseteq \{0, \dots, n-1\}^X$. Suppose that $D(x, y) = \{f \in F \mid f(x) \neq f(y)\}$ is nonempty for every $x, y \in X$. Then there are $|X|$ mutually different sets of the form $D(x, y)$.

Application

Theorem

Let A be an infinite algebra in a finitely generated congruence-distributive variety. Then $|\operatorname{Con}_c A| = |A|$.

Compact intersection property

We say that a variety $\mathcal{V}$ has the compact intersection property (CIP) if, for every $A \in \mathcal{V}$, the compact congruences of A are closed under intersection. (That is, $\mathrm{Con}_c A$ is a lattice.)
It seems that we only have a good description of $\mathrm{Con}\,\mathcal{V}$ when

- $\mathcal{V}$ is congruence-distributive, and
- $\mathcal{V}$ has CIP.

A systematic study of this situation - a joint work with my PhD student F. Krajník.

Characterization

Theorem

Let $\mathcal{K}$ be a locally finite congruence-distributive variety. The following conditions are equivalent.

(1) *For every $A \in \mathcal{K}$ the set $\mathrm{Con}_c(A)$ is closed under intersection.*

(2) *Every finite subalgebra of a subdirectly irreducible algebra in $\mathcal{K}$ is itself subdirectly irreducible.*

(3) *For every embedding $f : A \to B$ of algebras in $\mathcal{K}$ with A finite, the mapping $\mathrm{Con}_c(f)$ preserves meets.*

The equivalence of (1) and (2) has been claimed by K. Baker and proved by Blok-Pigozzi (using the concept of equationally definable principal meet).

Examples: Boolean algebras, distributive lattices, pseudocomplemented distributive lattices...

More examples

Take any finite, subdirectly irreducible algebra A, generating a congruence-distributive variety. Enhance the type of A by taking all elements of A as constants (nullary operations). Then the resulting algebra A^* generates a variety satisfying (2).

Sample of applications 1

Suppose that $\mathcal{V}$ is a locally finite congruence-distributive variety with CIP, such that

- every subdirectly irreducible algebra is simple;
- no simple algebra has a one-element subalgebra.

Then the following conditions are equivalent:

(1) $L \in \operatorname{Con}_c \mathcal{V}$;

(2) L is a Boolean algebra.

Example: bounded distributive lattices

Sample of application 2

Suppose that $\mathcal{V}$ is a locally finite congruence-distributive variety with CIP, such that

- every subdirectly irreducible algebra is simple;
- there is a simple algebra with a one-element subalgebra.

Then the following conditions are equivalent:

(1) $L \in \mathrm{Con}_c\,\mathcal{V}$;

(2) L is a generalized Boolean algebra (sectionally complemented distributive lattice).

Example: distributive lattices

Sample of application 3

Suppose that $\mathcal{V}$ is a locally finite congruence-distributive variety with CIP, such that

- every subdirectly irreducible algebra is simple or has the 3-element chain as its congruence lattice;
- every subalgebra of a subdirectly irreducible algebra S has the same congruence lattice as S.

Then the following conditions are equivalent:

(1) $L \in \mathrm{Con}_c \mathcal{V}$;

(2) L is a dually Stone lattice in which codense elements form a Boolean algebra.

Example: principal Stone algebras

Conjecture

For a locally finite congruence-distributive variety $\mathcal{V}$ with CIP, the class $\operatorname{Con}\mathcal{V}$ is determined by all possible diagrams of the form $\operatorname{Con}_c\mathcal{D}$, where $\mathcal{D}$ is a diagram in $\mathcal{V}$ consisting of subdirectly irreducible algebras and proper embeddings between them.