A GOAL-ORIENTED GRAPH CALCULUS FOR RELATIONS

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Graph calculus, algebras of relations, complement, refutation soundness, completeness

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OUTLINE

1. Introduction — graphical notation
2. Slices — graphical representation
3. Graphs — alternative slices
4. Conclusion — comments

1 Introduction

◁ Diagrams & figures — useful in science & everyday life

Graphs & diagrams — visualization

- Computing — automata, Petri nets, flowcharts
- Foundations of Mathematics — categories, allegories
- Engineering/Architecture — wiring diagrams, blueprints
- Metro journey — diagram of lines

⇝ Venn diagrams

heuristic appeal — not proofs — ∴ — compile

♡ Graph manipulations

precise syntax & semantics — ∴ — proof methods

Formulas traditionally written down on a single line

$\rightsquigarrow$ Notations — economy vs. visualization

1. Polish prefix (parenthesis-free) $\rightarrow \wedge pq \vee rs$

2. usual (with parentheses) $(p \wedge q) \rightarrow (r \vee s)$

3. two-dimensional

$$\begin{pmatrix} p \\ \wedge \\ q \end{pmatrix} \rightarrow \begin{pmatrix} r \\ \vee \\ s \end{pmatrix}$$

♡ Graph calculi — 2-dimensional notation & nodes

◁ Drawings for relations — natural idea

a related to b via relation R — arc $a \overset{\mathrm{R}}{\to} b$

Operations on relations — simple manipulations on arrows

- Bolean intersection $\cap$ — parallel arcs
- Peircean transposal $^{\mathsf{T}}$ — arrow reversal
- Peircean relative product (composition) $|$ — consecutive arcs

∞ Reason about relations — manipulate their representations

∠ visual appeal — 2-dimensional manipulations

Overview

◁ Goal-orientation $P \subseteq Q$ iff $P \cap \overline{Q} \subseteq \emptyset$

♡ Reductions equivalent objects

1. Represent term $P \cap \overline{Q}$ by slice S ∴ goal $S \subseteq \emptyset$
2. Convert slice S to graph G ∴ goal $G \subseteq \emptyset$
3. Expand graph G to H ∴ goal $H \subseteq \emptyset$

∞ (Correctness) Basic graph H: $H \subseteq \emptyset$ iff H inconsistent

▷ Relational terms generated $\begin{cases} \text{from relation names} \\ \text{by relational operations} \end{cases}$

▷ Peircean operations 2-ary relations over set M

(0) Constants

(**I**) Identity (diagonal) $I_M := \{(a,b) \in M^2 \,/\, a = b\}$

(**D**) Diversity $I_M\tilde{} := \{(a,b) \in M^2 \,/\, a \neq b\}$

(1) Unary operation

($\smile$) Transposition $^{\mathsf{T}}$ (reversal) $\mathrm{R}^{\mathsf{T}} := \{(a,b) \in M^2 \,/\, (b,a) \in \mathrm{R}\}$

(2) Binary operations

(;) Relative product $|$ (composition)

$\mathrm{P} \,|\, \mathrm{Q} := \{(a,b) \in M^2 \,/\, \exists c \in M\, [(a,c) \in \mathrm{P} \wedge (c,b) \in \mathrm{Q}]\}$

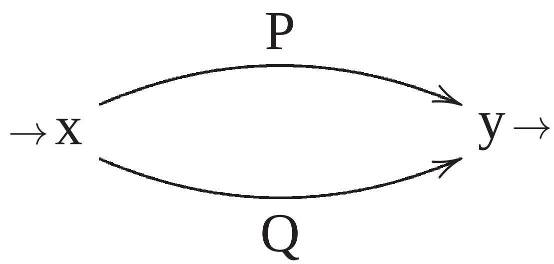
(;) Relative sum $\underline{|}$

$\mathrm{P} \,\underline{|}\, \mathrm{Q} := \{(a,b) \in M^2 \,/\, \forall c \in M\, [(a,c) \in \mathrm{P} \vee (c,b) \in \mathrm{Q}]\}$

2 Slices

♡ Slice

graphical representation



term $\begin{bmatrix} P \\ \sqcap \\ Q \end{bmatrix}$

parallel arcs

intersection ∩

▷ Slice S

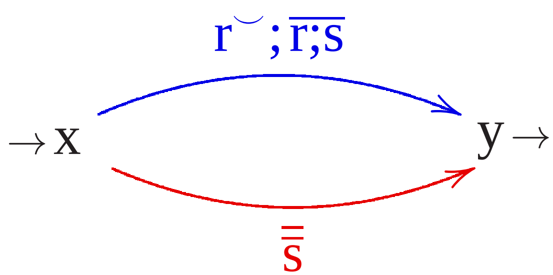
- finite sets of
  - nodes
  - labeled arcs
- 2 distinguished nodes: I/O →

Slice example

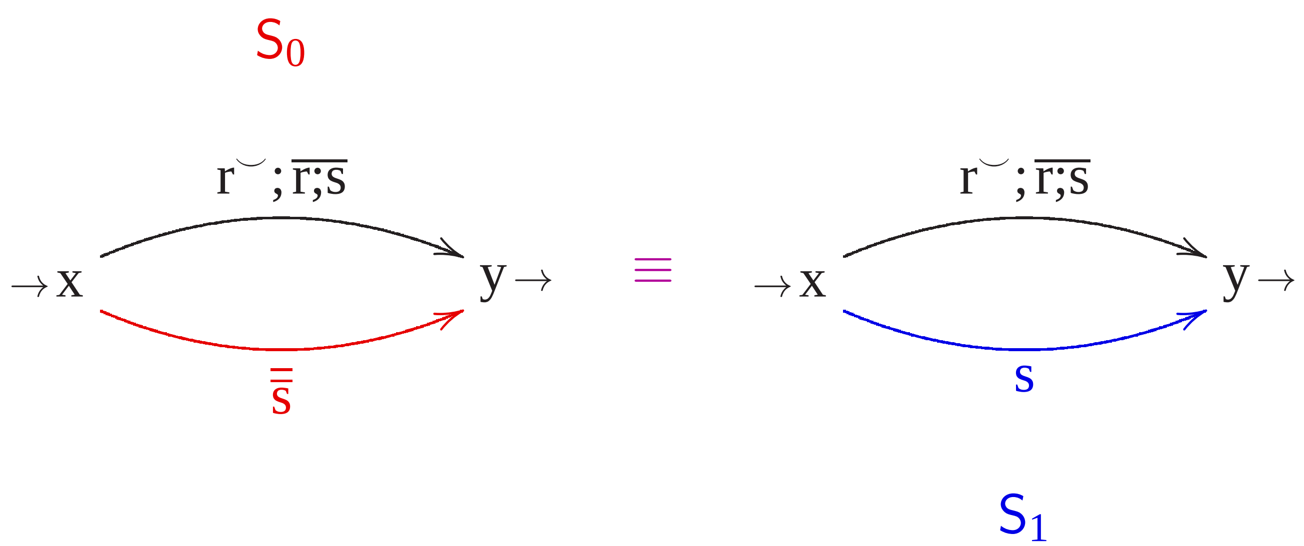
$\rightsquigarrow$ Establish $r^{\smile};\overline{r;s} \sqsubseteq \overline{s}$

$\lhd$ Reduce $r^{\smile};\overline{r;s} \sqsubseteq \overline{s}$ to $\begin{bmatrix} r^{\smile};\overline{r;s} \\ \sqcap \\ \overline{\overline{s}} \end{bmatrix} \sqsubseteq \perp\!\!\!\perp$

(∩) Slice S_0 parallel arcs ∩



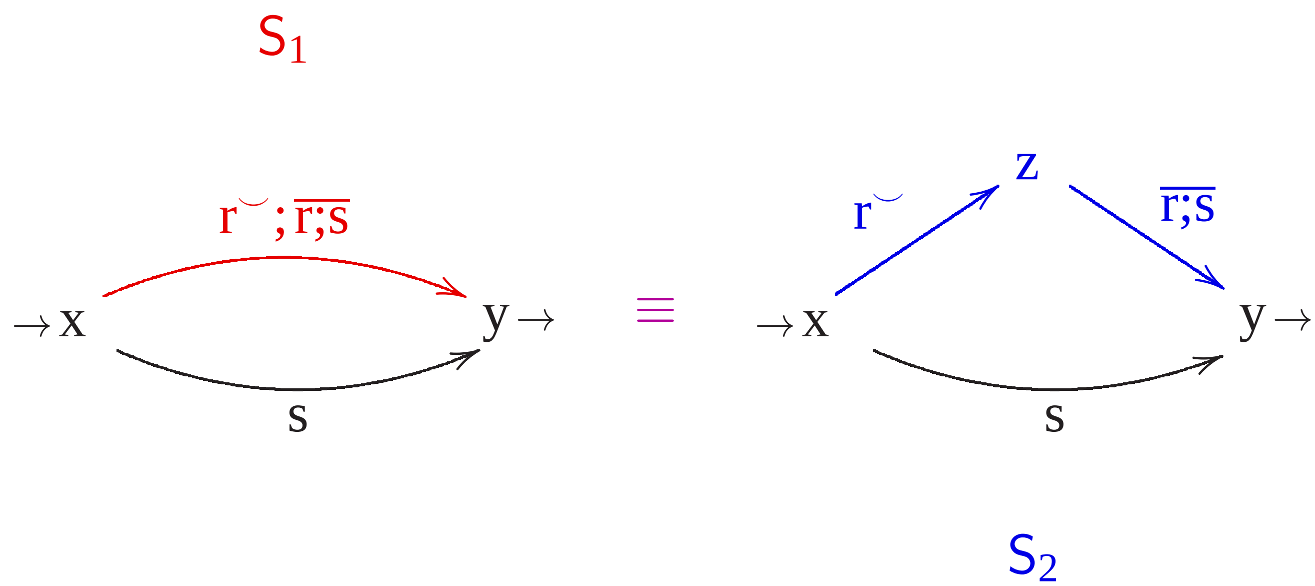
(=) Eliminate double complement $\overline{\overline{s}} \equiv s$



(;) Eliminate relative product

consecutive arcs

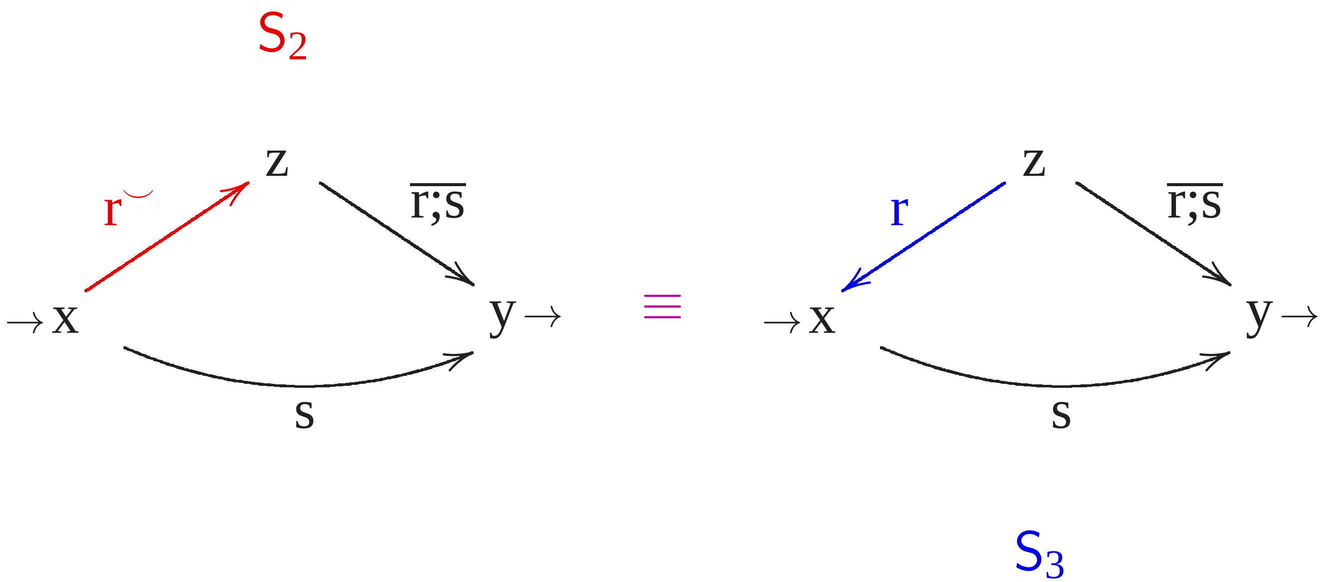
<u>new</u> intermediate node



$$(a,b) \in P;Q \quad \Leftrightarrow \quad \exists c \begin{bmatrix} (a,c) \in P \\ \wedge \\ (c,b) \in Q \end{bmatrix}$$

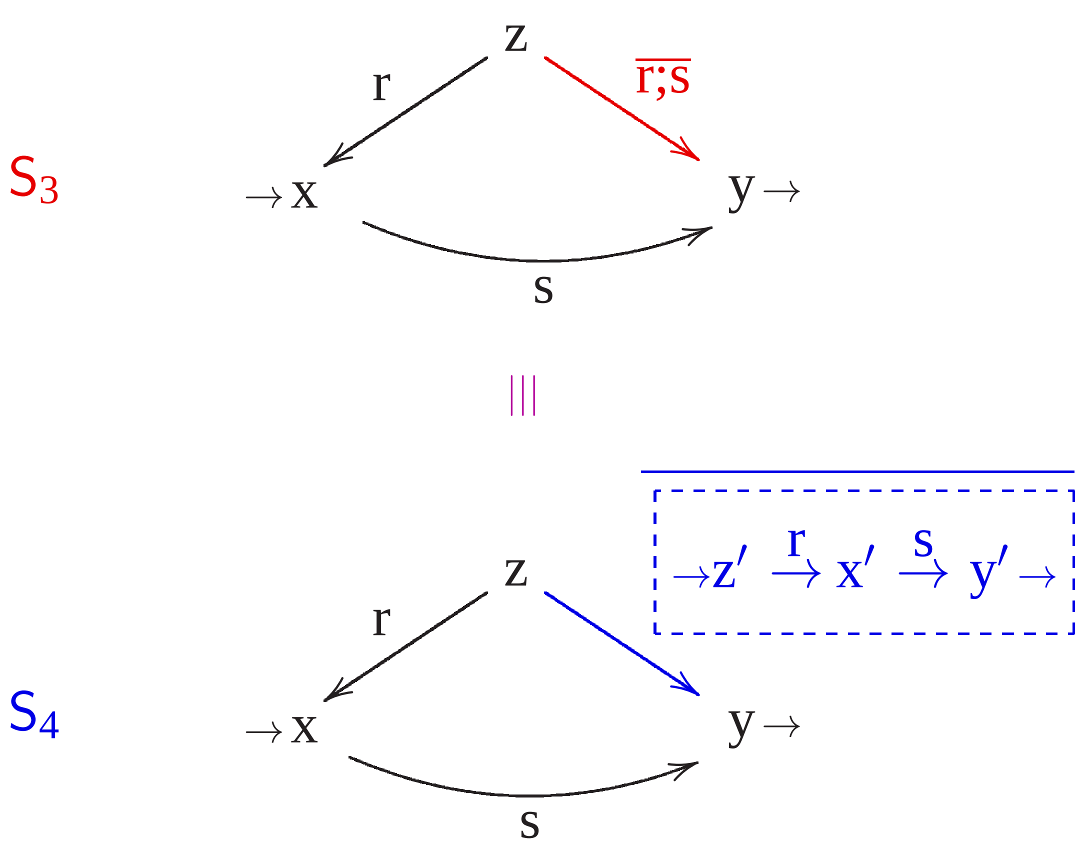
$(^\smile)$ Eliminate converse

invert arrow: $x \xrightarrow{r^\smile} z \equiv x \xleftarrow{r} z$



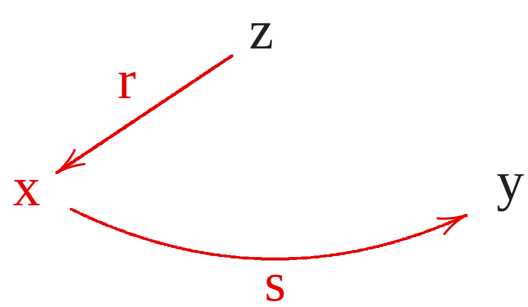
$(a,c) \in R^\smile \quad \Leftrightarrow \quad (c,a) \in R$

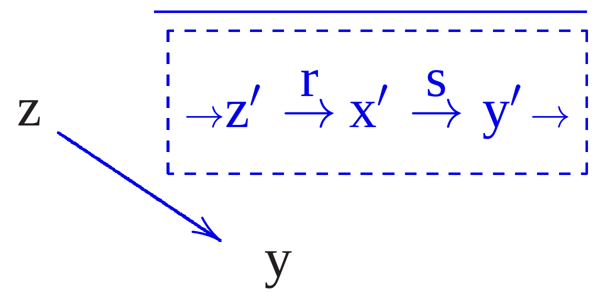
($\bar{;}$) Eliminate complemented relative product label: complemented slice



(⊥) Slice S_4 inconsistent

Parallel paths: from z to y





term r;s

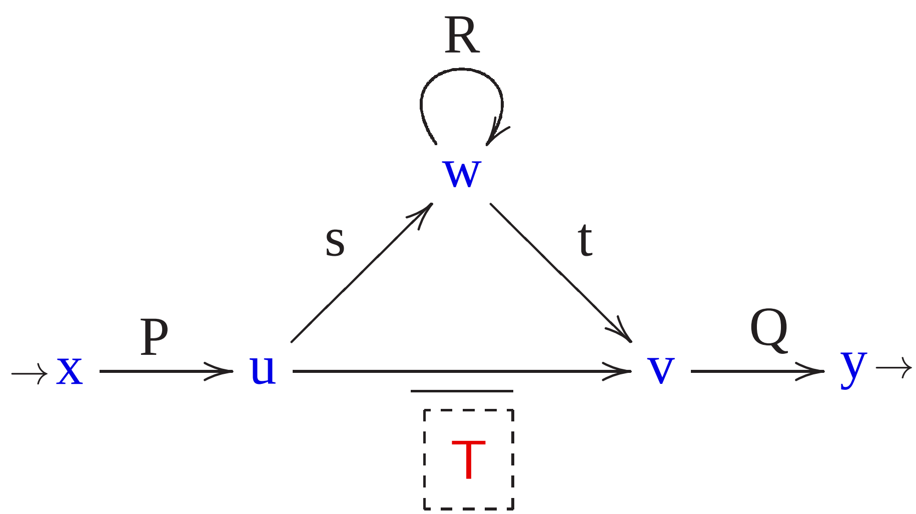
term $\overline{r;s}$

$$r;s \cap \overline{r;s} = \emptyset$$

▷ Slice concepts

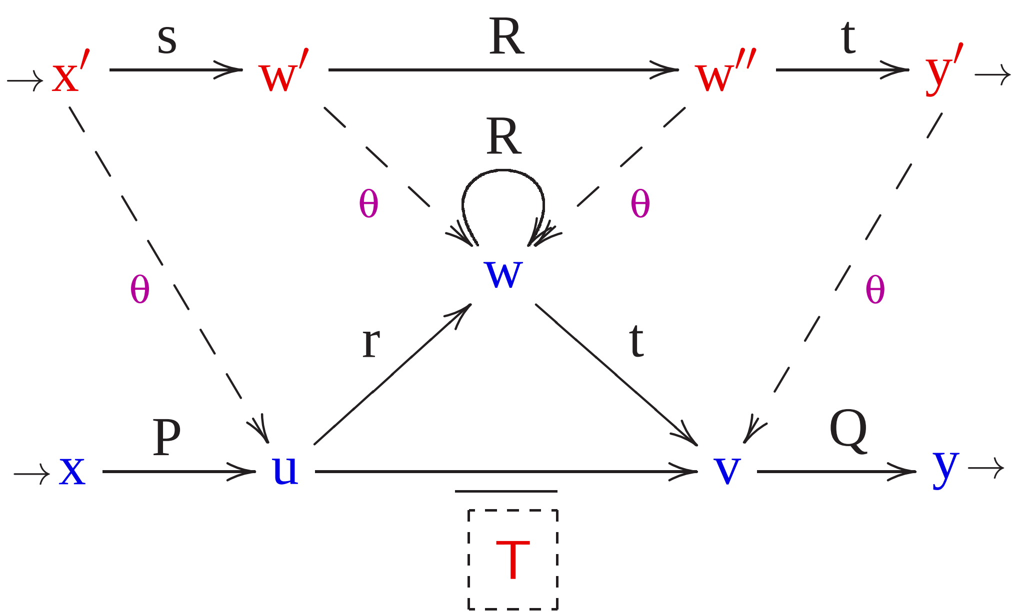
1. Morphism — node mapping preserving arcs
2. Zero slice — parallel incompatible paths

⇝ Slice S with embedded slice T



$$\mathsf{T} := \;\rightarrow x' \xrightarrow{s} w' \xrightarrow{R} w'' \xrightarrow{t} y' \rightarrow$$

Node mapping $\theta : N_T \dashrightarrow N_S$ preserves arcs



Parallel paths from u to v incompatible

Slice S zero not satisfiable

3 Graphs

♡ Alternative slices — for $\cup$, $\emptyset$

▷ Graph — finite set of alternative slices

$$\rightarrow x \xrightarrow{P \sqcup Q} y \rightarrow \quad \equiv \quad \left\{ \begin{array}{c} \rightarrow x \xrightarrow{P} y \rightarrow, \\ \rightarrow x \xrightarrow{Q} y \rightarrow \end{array} \right\}$$

2 alternative slices

$$\rightarrow x \xrightarrow{\perp\!\!\!\perp} y \rightarrow \quad \equiv \quad \{ \quad \}$$

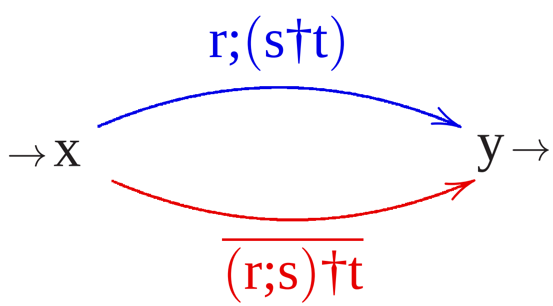
0 alternative slices

Graph example

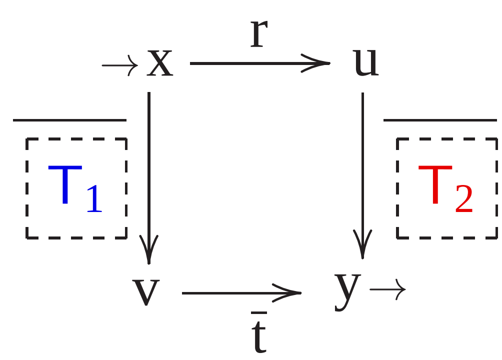
⇝ Establish $r;(s\dagger t) \sqsubseteq (r;s)\dagger t$

◁ Reduce $r;(s\dagger t) \sqsubseteq (r;s)\dagger t$ to $\left[\begin{array}{c} r;(s\dagger t) \\ \sqcap \\ \overline{(r;s)\dagger t} \end{array}\right] \sqsubseteq \perp\!\!\!\perp$

($\cap$) Difference slice $\mathsf{DS}(r;(s\dagger t) \backslash (r;s)\dagger t)$ parallel arcs $\cap$



($\triangleright^*$) Eliminate operations but complement

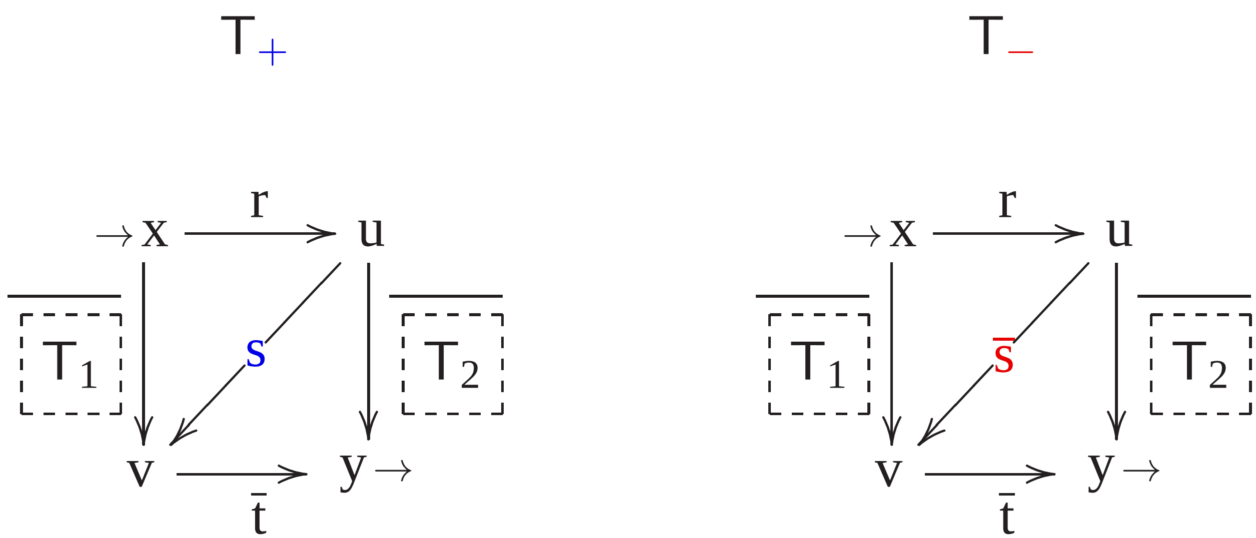


Arc labels complemented slices

$$T_1 \quad := \quad \to x_1 \xrightarrow{r} u_1 \xrightarrow{s} v_1 \to$$

$$T_2 \quad := \quad \to u_2 \xrightarrow{\bar{s}} v_2 \xrightarrow{\bar{t}} y_2 \to$$

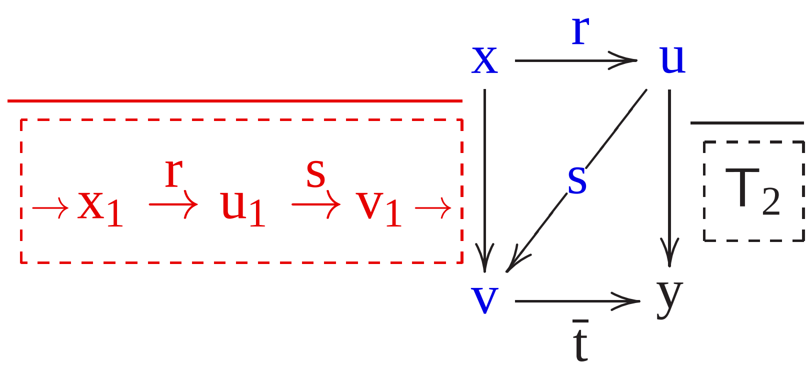
(∪) Expand graph G: 2 alternative slices T_+ & T_-



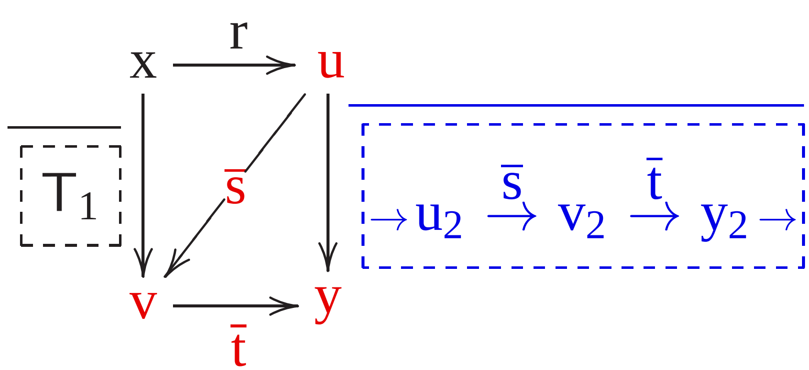
Equivalent alternative paths from u to v $s \cup \bar{s} = M^2$

(⊥) Zero graph inconsistent slices T_+ & T_-

T_+ Parallel paths from x to v: $\overline{r;s}$ $r;s$

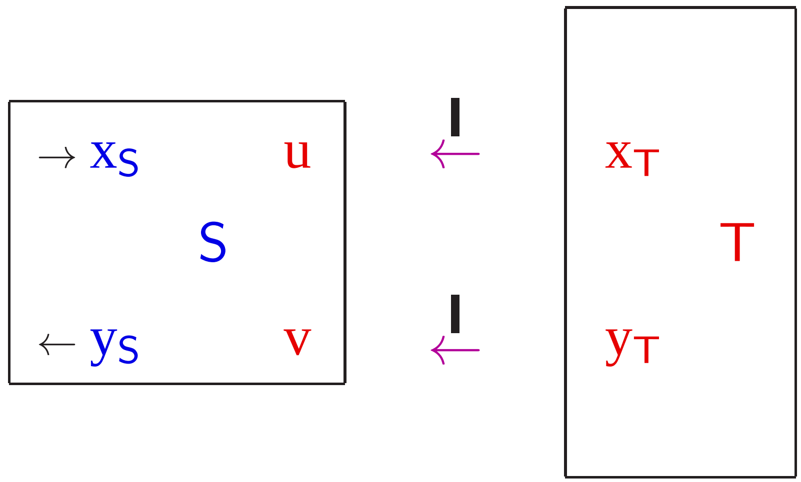


T_- Parallel paths from u to y: $\bar{s};\bar{t}$ $\overline{\bar{s};\bar{t}}$



Gluing onto slice S slice & graph

♡ Gluing of slices addition of slice-label arc $S + u \xrightarrow{T} v$



eliminate I arcs

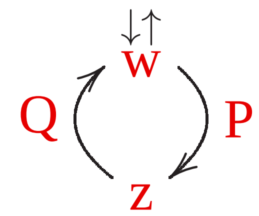
◁ Eliminate arc w I z from slice S rename w to z w (or z to w) in S

∠ Gluing $S \frac{u}{v} T \equiv S + u \xrightarrow{T} v$ pushout

$\rightsquigarrow$ Slices S & T

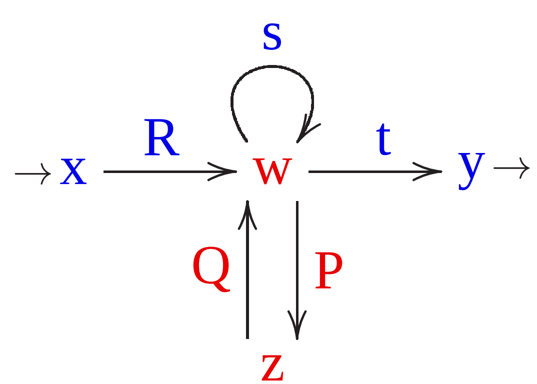
$$S \;:=\; \to x \xrightarrow{R} u \xrightarrow{s} v \xrightarrow{t} y \to$$

$T :=$



Glued slice $S \frac{u}{v} T$ — identify u, v to w

$S \frac{u}{v} T \;=$



▷ Glued graph — glued slices — $S \frac{u}{v} H := \{ S \frac{u}{v} T \,/\, T \in H \}$

4 Graph Calculus

▷ *Labels*

Labels generated:
- from:
  - relation names
  - slices & graphs
- by relational operations

▷ *Basic* objects

Basic: labels:
- relation name <u>or</u>
- complement of basic slice

▷ Label derivations

Rules $\begin{cases} \text{Conversion} & \begin{cases} \text{Operational} & \text{relational operations} \\ \text{Structural} & \text{arc labels} \end{cases} \\ \text{Expansion} & \text{alternatives} \end{cases}$

∞ Valid label inclusion $L \sqsubseteq \mathbb{L}$ iff $L \vdash H$ H: zero graph

▷ *Normal linear derivation* L `Cnv`* G `Exp`* H

1. Convert label L to graph G — e.g. basic form

2. Expand graph G to zero graph H — binary expansion

♡ Operational rules label $\triangleright$ graph any context

- Meaning of operation but complement given by graph
- Double complement $\overline{\overline{L}} \equiv L$

∠ Operational rules

- eliminate relational operations but complement
- may introduce slices or graphs within labels

♡ Structural rules arc labels

- Addition of graph-label arc glued graph
- Label vs. slice $L \equiv \rightarrow x \xrightarrow{L} y \rightarrow$
- de Morgan laws complement of $\cup$, complement of $\cap$

⋈ Conversion $L \triangleright^* L^{\mathtt{bs}}$

Every label L convertible to an equivalent basic graph $L^{\mathtt{bs}}$

▷ Expansion rule replace S by copies $S \frac{u}{v} T$ & $S + u \overline{\overline{T}} v$

$$\texttt{(Exp)} \quad \frac{\{\,S\,\}}{\{\,S \frac{u}{v} T\,,\, S + u \overline{\overline{T}} v\,\}} \qquad (u,v) \in {N_S}^2$$

♢ Label inclusion $L \sqsubseteq \mathbb{L}$ valid iff normal linear derivation

$$L \quad \texttt{Cnv}^* \quad G \quad \texttt{Exp}^* \quad H \quad \text{zero graph}$$

◁

1. Convert label L to (basic) graph G — finite process
2. Expand graph G to zero graph H — unbounded search

5 Conclusion

1. Complement (-) difficult to handle (+) goal-orientation
2. Goal-orientation derive zero graph
3. Graph language more expressive: labels (embedding)
   - easy one single object
   - simple concepts morphism
   - Extension to hypotheses erasing
4. Generalize: labels to *n*-labels first-order predicate logic

A Details

1. Language
   (a) Syntax — objects
   (b) Semantics — meaning
   (c) Concepts — morhism, zero, basic
   (d) Constructions — gluing, transformations (graph $\leftrightarrow$ slice)
2. Calculus
   (a) Conversion — label to basic graph
   (b) Expansion — (basic) slice to (basic) graph
   (c) Correctness — sound & complete
3. Hypotheses
   (a) Semantics — consequence
   (b) Rule — erase slice

◁ Constants & operations — square relational interpretation

Arity	Symbol	Interpretation	
0	r	arbitrary relation	over set M
	$\mathbb{⊥}$	empty relation	$\emptyset$
	$\mathbb{⊤}$	universal relation	square $M^2 := M \times M$
	I	identity relation	diagonal I_M
	D	diversity relation	$I_M{}^{\sim}$
1	$\overline{}$	Boolean complementation $^{\sim}$	$\mathrm{R}^{\sim} := M^2 \setminus \mathrm{R}$
	$\smile$	Peircean transposition $^{\mathsf{T}}$	
2	$\sqcap$	Boolean intersection $\cap$	
	$\sqcup$	Boolean union $\cup$	
	;	Peircean relative product $\vert$	
	†	Peircean relative sum $\underline{\vert}$	

∠ Constants & operations — language interpretation (regular expressions)

Arity	Symbol	Interpretation
0	r	arbitrary language over alphabet A
	$\mathbb{L}$	empty language $\emptyset$
	$\mathbb{T}$	universal language A^*
	I	null-word language $\Lambda := \{\lambda\}$
	D	non-null language $A^+ := \{w \in A^* \,/\, w \neq \lambda\}$
1	$\overline{}$	Language complementation $^\sim$ $L^\sim := A^* \setminus L$
	$\smile$	Language reversal $^{\mathtt{rv}}$ $L^{\mathtt{rv}} := \{w^{\mathtt{rv}} \in A^* \,/\, w \in L\}$
2	$\sqcap$	Language intersection $\cap$
	$\sqcup$	Language union $\cup$
	;	Lang. concatenation $\cdot$ $P \cdot Q := \{u \cdot v \in A^* \,/\, u \in P \wedge v \in Q\}$
	†	Lang. co-concatenation $\underline{\cdot}$ $P \,\underline{\cdot}\, Q := (P^\sim \cdot Q^\sim)^\sim$

A.1 Language

Denumerably infinite sets $\begin{cases} \text{R}n\text{: } \textit{relation names} \\ \mathbb{N}\text{d: } \textit{nodes} \text{ (alphabetical order: x, y, z, ...)} \end{cases}$

▷ Syntax — mutual recursion

(L) *Labels*: generated from (relation names; slices, graphs) by relational operations

(a) *Arc*: triple u L v — u, v: nodes, L: label

(Σ) *Sketch* $\Sigma = \langle N, A \rangle$ — sets N: nodes & A: arcs

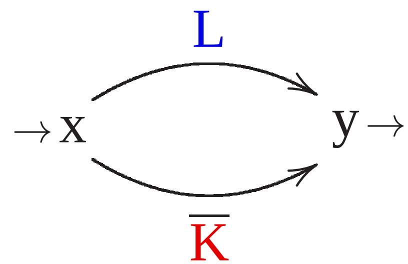
(D) *Draft* $\mathsf{D} = \langle N, A \rangle$ — finite sketch

(S) *Slice* $\mathsf{S} = \langle N, A : \mathrm{x}_{\mathsf{S}}, \mathrm{y}_{\mathsf{S}} \rangle$ $\begin{cases} \textit{underlying } \text{draft } \underline{\mathsf{S}} = \langle N, A \rangle \\ \textit{input, output } \mathrm{x}_{\mathsf{S}}, \mathrm{y}_{\mathsf{S}}\text{: nodes} \end{cases}$

(G) *Graph* G: finite set of slices

(⊑) *Label inclusion* $\mathsf{L} \sqsubseteq \mathsf{K}$ pair of labels

(\) *Difference slice* $\mathsf{DS}(\mathsf{L} \setminus \mathsf{K})$ parallel arcs



▷ *Morphism* $\theta : \Sigma' \dashrightarrow \Sigma''$ node mapping preserving arcs

Nodes	Arcs
N'	$\mathsf{u}\,\mathsf{L}\,\mathsf{v} \in A'$
$\downarrow \theta$	$\Downarrow$
N''	$\mathsf{u}^{\theta}\,\mathsf{L}\,\mathsf{v}^{\theta} \in A''$

◁ Set of morphisms $\mathsf{Mor}[\Sigma', \Sigma'']$

♡ Meaning

labels,slices & graphs	denote	2-ary relations
arcs, sketches & drafts	represent	restrictions

⇝ Pair (a,b) satisfies arc u r v — pair (a,b) in relation of r (of model)

∠ Semantics

- *Model*: relation name $\mapsto$ 2-ary relation — $\mathfrak{M} = \langle M, (\mathrm{r}^{\mathfrak{M}})_{\mathrm{r} \in \mathrm{R}n} \rangle$
- *Assignment* $\mathrm{g} : N \to M$ — $\mathrm{w} \in N \mapsto \mathrm{w}^{\mathrm{g}} \in M$

◁ Model $\mathfrak{M}$ — relational term r $\mapsto$ 2-ary relation $\mathrm{r}^{\mathfrak{M}} \subseteq M^2$

▷ Behavior — mutual recursion

(L) *Relation* of label $[\mathrm{L}]_{\mathfrak{M}} \subseteq M^2$ concrete versions of operations

e.g. $[\mathrm{r}]_{\mathfrak{M}} := \mathrm{r}^{\mathfrak{M}}$, $[\mathrm{L}^{\smile}]_{\mathfrak{M}} := [\mathrm{L}]_{\mathfrak{M}}{}^{\mathsf{T}}$, $[\mathrm{L;K}]_{\mathfrak{M}} := [\mathrm{L}]_{\mathfrak{M}} \mid [\mathrm{K}]_{\mathfrak{M}}$

(a) *Satisfaction* of arc $\mathrm{u} \xrightarrow{\mathrm{L}} \mathrm{v}$ — $(\mathrm{u}^{\mathrm{g}}, \mathrm{v}^{\mathrm{g}}) \in [\mathrm{L}]_{\mathfrak{M}}$ (with $\mathrm{u}, \mathrm{v} \in N$)

(Σ) *Satisfaction* of sketch Σ — $\mathrm{g} : \Sigma \to \mathfrak{M}$ $\Leftrightarrow$ g satisfies all arcs of Σ

(S) *Extension* of slice — $[\![\mathsf{S}]\!]_{\mathfrak{M}}$

values of I/O for assignments satisfying underlying draft

$$[\![\mathsf{S}]\!]_{\mathfrak{M}} := \{(\mathrm{x}_{\mathsf{S}}{}^{\mathrm{g}}, \mathrm{y}_{\mathsf{S}}{}^{\mathrm{g}}) \in M^2 \,/\, \mathrm{g} : \underline{\mathsf{S}} \to \mathfrak{M}\}$$

(G) *Extension* of graph — $[\![\mathsf{G}]\!]_{\mathfrak{M}} := \bigcup_{\mathsf{S} \in \mathsf{G}} [\![\mathsf{S}]\!]_{\mathfrak{M}}$

▷ Label inclusion & equivalence

($\mathfrak{M}$) *holds* $\mathfrak{M} \models \mathrm{L} \sqsubseteq \mathrm{K}$ $\Leftrightarrow$ $[\mathrm{L}]_{\mathfrak{M}} \subseteq [\mathrm{K}]_{\mathfrak{M}}$

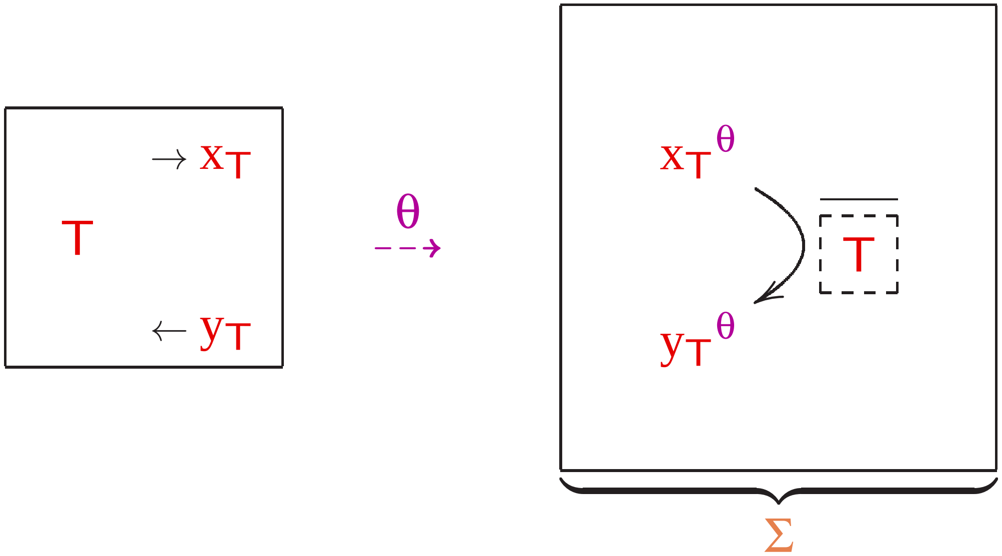
($\models$) *valid* $\models \mathrm{L} \sqsubseteq \mathrm{K}$ $\Leftrightarrow$ $\mathfrak{M} \models \mathrm{L} \sqsubseteq \mathrm{K}$ $(\forall \mathfrak{M})$

($\perp$) *Null* label L $\Leftrightarrow$ $\mathrm{L} \sqsubseteq \perp\!\!\!\perp$ valid

($\equiv$) *Equivalent* labels $\mathrm{L} \equiv \mathrm{K}$ $\Leftrightarrow$ $\mathrm{L} \sqsubseteq \mathrm{K}$ & $\mathrm{K} \sqsubseteq \mathrm{L}$ valid

Concepts

▷ *Zero* sketch — <u>not</u> satisfiable



▷ *Zero* slice & graph — <u>empty</u> extension

(S) Slice T is zero $\Leftrightarrow$ underlying draft $\underline{\mathsf{T}}$ is zero sketch

(G) Graph H is zero $\Leftrightarrow$ all its slices $\mathsf{T} \in \mathsf{H}$ are zero slices

▷ *Basic* labels, arcs, sketches, slices & graphs mutual recursion

(L) Label L is basic $\Leftrightarrow$ L is $\begin{cases} \text{relation name } \underline{\text{or}} \\ \text{complement of basic slice (cf. below)} \end{cases}$

(a) Arc $u \xrightarrow{L} v$ is basic $\Leftrightarrow$ its label L is basic label

Basic sketches, slices & graphs only basic arcs

(Σ) Sketch $\Sigma = \langle N, A \rangle$ is basic $\Leftrightarrow$ all its arcs $a \in A$ are basic arcs

(S) Slice $S = \langle \underline{S} : x_S, y_S \rangle$ is basic $\Leftrightarrow$ underlying draft $\underline{S}$ is basic sketch

(G) Graph G is basic $\Leftrightarrow$ all its slices $S \in G$ are basic slices

Constructions

▷ Gluing slice onto draft: $D \frac{u}{v} T$ — pushout $D \frac{u}{v} T$

$$
\begin{array}{ccccc}
 & & D + \{u, v\} & & \\
 & \overset{\alpha}{\nearrow} & & \overset{\sigma}{\searrow} & \\
\to x \quad y \leftarrow & & \text{PO} & & D \frac{u}{v} T \\
 & \overset{\beta}{\searrow} & & \overset{\tau}{\nearrow} & \\
 & & \underline{T} & &
\end{array}
$$

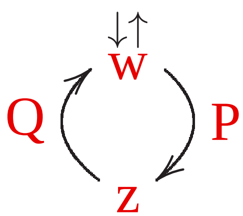
▷ Gluing onto slice S — slice & graph

1. slice T — transfer I/O — $S \frac{u}{v} T := \langle \underline{S} \frac{u}{v} T : x_T{}^{\sigma}, y_T{}^{\sigma} \rangle$
2. graph H — glued slices — $S \frac{u}{v} H := \{ S \frac{u}{v} T \,/\, T \in H \}$

$\rightsquigarrow$ Slices S & T', T''

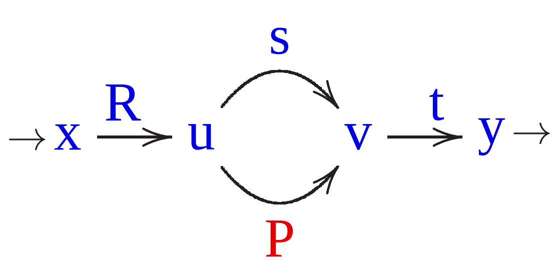
$$\mathsf{S} \quad := \quad \to x \xrightarrow{R} u \xrightarrow{s} v \xrightarrow{t} y \to$$

$$\mathsf{T}' \quad := \quad \to w \xrightarrow{P} z \to$$

$\mathsf{T}'' \quad :=$ 

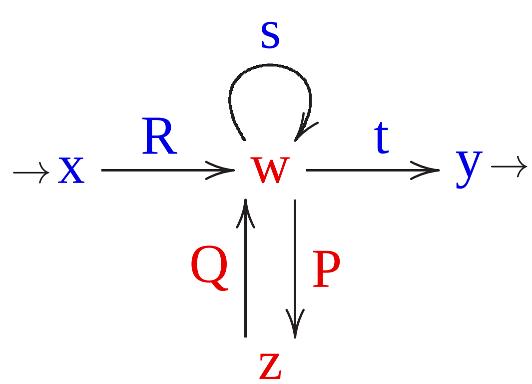
Glued slice $\mathsf{S}\frac{u}{v}\mathsf{T}'$

identify w to u, z to v

$\mathsf{S}\frac{u}{v}\mathsf{T}' \quad =$ 

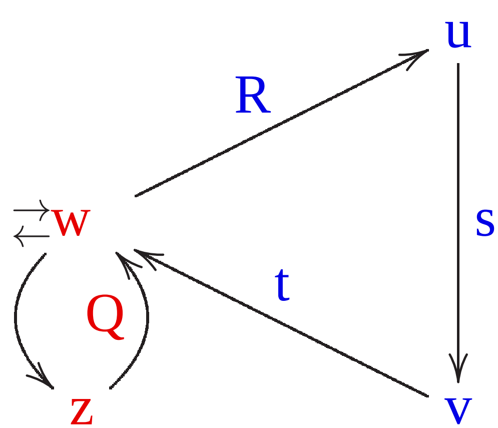
Glued slice $S\frac{u}{v}T''$ identify u, v to w

$S\frac{u}{v}T'' \quad =$



Glued slice $S\frac{x}{y}T''$ identify I/O

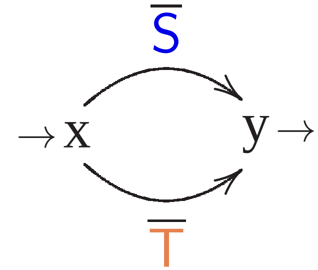
$S\frac{x}{y}T'' \quad =$



▷ Transformations graph $\leftrightarrow$ slice

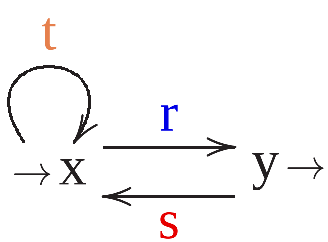
- Slice of graph $\mathsf{Sl}[\mathsf{G}] := \langle \{x, y\}, \{x \xrightarrow{\overline{\mathsf{S}}} y \,/\, \mathsf{S} \in \mathsf{G}\} : x, y \rangle$ parallel arcs

$\rightsquigarrow$ Graph $\{\mathsf{S}, \mathsf{T}\}$ $\mapsto$ slice



- Graph of slice $\mathsf{Gr}(\mathsf{S})$ 1-arc complemented label slice, for each arc

$\rightsquigarrow$ Slice



$\mapsto$ graph $\left\{ \begin{array}{c} \to x \xrightarrow{\bar{r}} y\to, \\ \to x \xleftarrow{\bar{s}} y\to, \\ \to \overset{\bar{t}}{\circlearrowleft x}\ y\to \end{array} \right\}$

▷ Slice is *small* $\Leftrightarrow$ nodes: input, output

⋈ Small slice S equivalence $\overline{\{\mathsf{S}\}} \equiv \mathsf{Gr}(\mathsf{S})$

A.2 Calculus

▷ Operational rules label ▷ graph any context

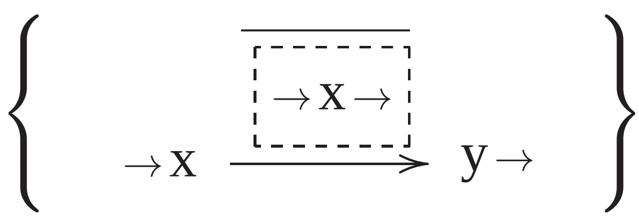
Constants Boolean $\perp\!\!\perp, \top\!\!\top$ & Peircean I, D

($\perp\!\!\perp$) $\perp\!\!\perp \; \triangleright \; \{ \quad \}$ $\perp\!\!\perp \equiv$ empty graph

($\top\!\!\top$) $\top\!\!\top \; \triangleright \; \{ \; \to x \quad y \to \; \}$ $\top\!\!\top \equiv$ 2-node arcless slice $\langle \{x, y\}, \emptyset : x, y \rangle$

(I) $\mathsf{I} \; \triangleright \; \{ \; \to x \to \; \}$ $\mathsf{I} \equiv$ 1-node arcless slice $\to x \to$

(D) $\mathsf{D} \; \triangleright$



$\mathsf{D} \equiv \overline{\mathsf{I}}$ (2-node 1-arc slice)

Boolean operations

unary $^{-}$ & binary $\sqcap, \sqcup$

$(=)\ \overline{\overline{L}} \rhd L$ $\qquad$ $\overline{\overline{L}} \equiv L$

$(\sqcap)\ L \sqcap K \rhd \left\{ \to x \overset{L}{\underset{K}{\rightrightarrows}} y \to \right\}$ $\qquad$ parallel arcs: L & K

$(\sqcup)\ L \sqcup K \rhd \left\{ \begin{array}{l} \to x \xrightarrow{L} y \to , \\ \to x \xrightarrow{K} y \to \end{array} \right\}$ $\qquad$ alternative slices: L & K

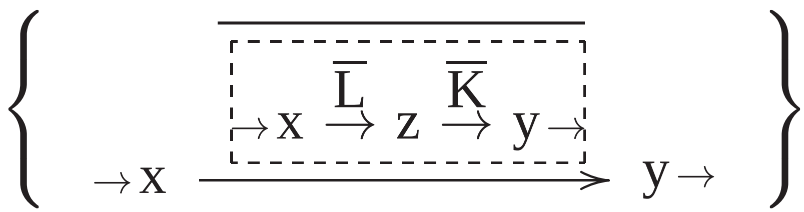
Peircean operations — unary $^\smile$ & binary $;$, $\dagger$

$(\smile)\;\; L^\smile \;\triangleright\; \left\{ \;\to x \xleftarrow{L} y \to \;\right\}$ — reversed arrow

$(;)\;\; L;K \;\triangleright\; \left\{ \;\to x \xrightarrow{L} z \xrightarrow{K} y \to \;\right\}$ — consecutive arcs

L then K

$(\dagger)\;\; L\dagger K \;\triangleright$



$L\dagger K \equiv \overline{\overline{L};\overline{K}}$

Summary operations arity: 0, 1

($\bot$) $\bot \triangleright \{ \quad \}$ empty graph

($\top$) $\top \triangleright \{\langle\{x,y\},\emptyset : x,y\rangle\}$

2-node arcless slice

$\rightarrow x \quad y \rightarrow$

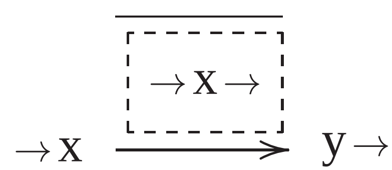
($\mathbf{I}$) $\mathbf{I} \triangleright \{\langle\{x\},\emptyset : x,x\rangle\}$

1-node arcless slice

$\rightarrow x \rightarrow$

($\mathbf{D}$) $\mathbf{D} \triangleright \{\langle\{x,y\},\{\overline{\langle\{x\},\emptyset : x,x\rangle}\} : x,y\rangle\}$

2-node 1-arc slice



($=$) $\overline{\overline{L}} \triangleright L$ replace $\overline{\overline{L}}$ by L

($\smile$) $L^{\smile} \triangleright \{\langle\{x,y\},\{y\,L\,x\} : x,y\rangle\}$

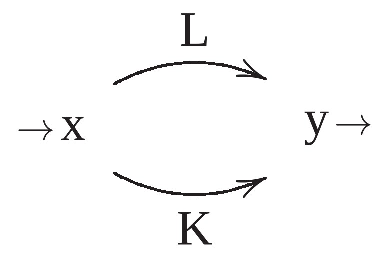
reversed-arc slice

$\rightarrow x \overset{L}{\leftarrow} y \rightarrow$

Summary operations arity: 2

$(\sqcap)\ \ L \sqcap K \ \triangleright\ \{\langle\{x,y\},\{x\,L\,y,\ x\,K\,y\} : x,y\rangle\}$

parallel-arc slice:



$$(\sqcup)\ \ L \sqcup K \ \triangleright\ \left\{ \begin{array}{c} \langle\{x,y\},\{x\,L\,y\} : x,y\rangle\,, \\ \langle\{x,y\},\{x\,K\,y\} : x,y\rangle \end{array} \right\}$$

alternative slices:

$\rightarrow x \xrightarrow{L} y \rightarrow$

$\rightarrow x \xrightarrow{K} y \rightarrow$

$(;)\ \ L;K \ \triangleright\ \{\mathsf{Sl}(L \rightarrow K)\}$

consecutive-arc slice:

$\rightarrow x \xrightarrow{L} z \xrightarrow{K} y \rightarrow$

$(\dagger)\ \ L \dagger K \ \triangleright\ \{\langle\{x,y\},\{x\,\overline{\mathsf{Sl}(\overline{L} \rightarrow \overline{K})}\,y\} : x,y\rangle\}$

complemented label:

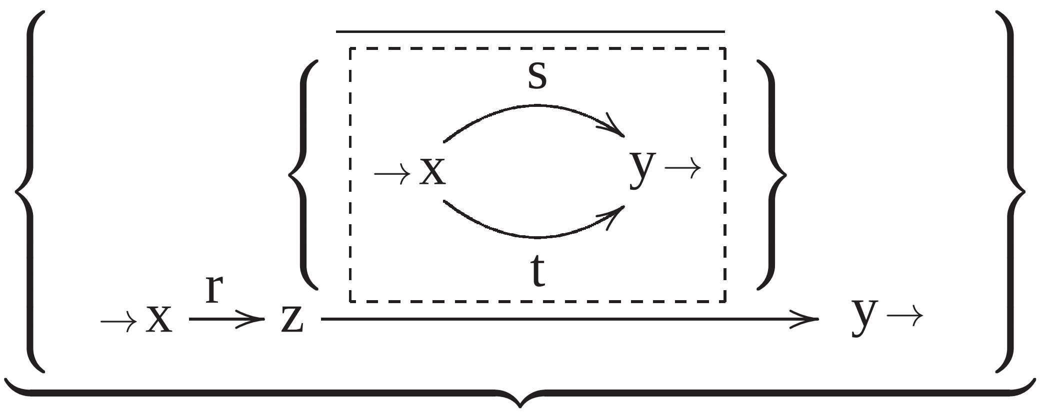
$\overline{\rightarrow x \xrightarrow{\overline{L}} z \xrightarrow{\overline{K}} y \rightarrow}$

∠ Operational rules   composite labels $\rhd^*$ graphs

⇝ Term $r\,;\overline{s \sqcap t}$ graph G

$$r\,;\overline{s \sqcap t} \;\overset{(;)}{\rhd}\; \left\{ \;\rightarrow x \xrightarrow{r} z \xrightarrow{\overline{s \sqcap t}} y \rightarrow \;\right\}$$

$\overset{(\sqcap)}{\rhd}$



G

▷ Structural rules — arc labels

$(\overset{\cup}{\rightarrow})$ $\{S + u\,H\,v\} \triangleright S \frac{u}{v} H$ — replace graph label by glued slices

$(\overline{G})$ $\overline{G} \triangleright \{SI[G]\}$ — replace complemented graph by slice

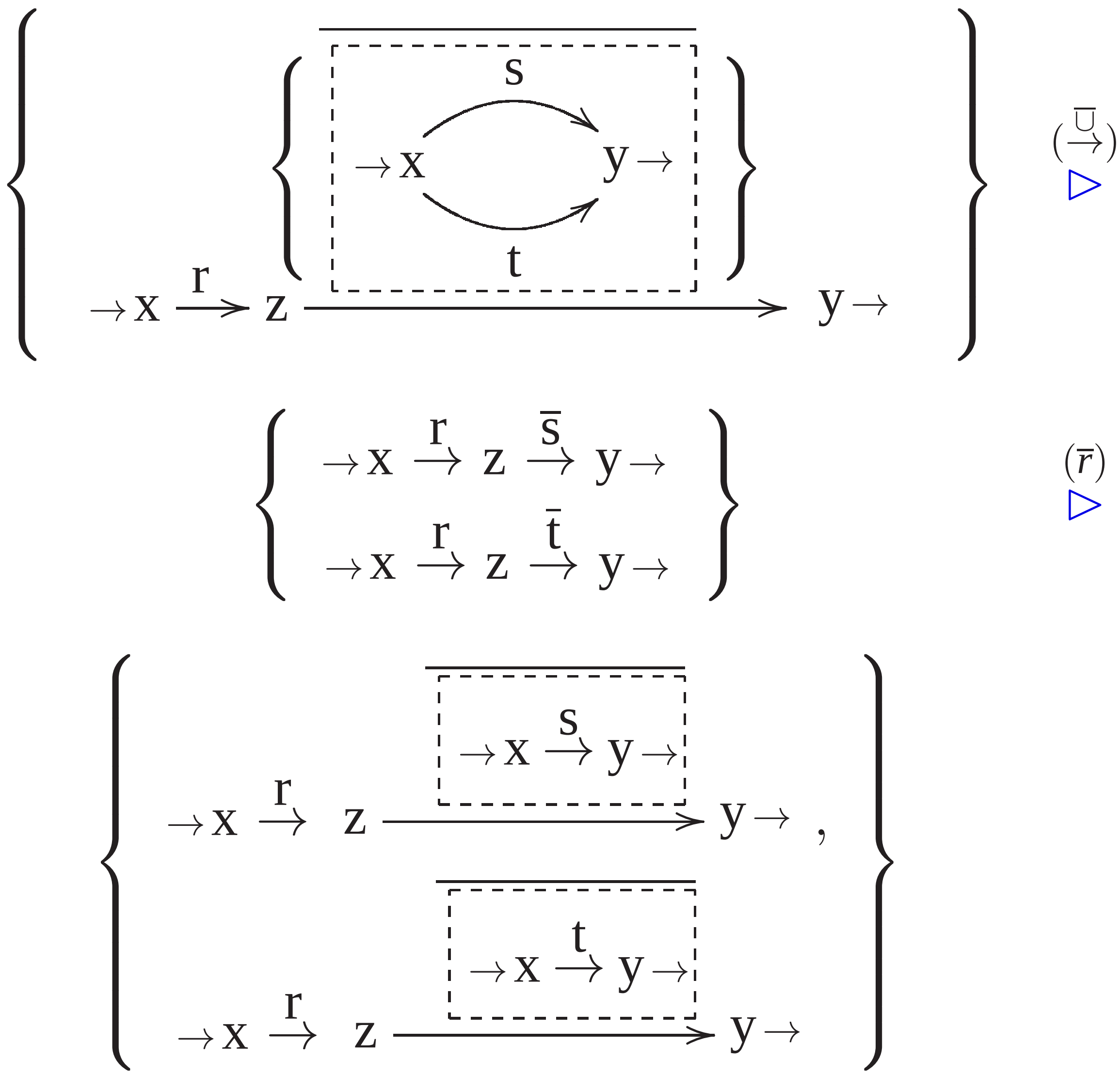
$(\overline{S})$ small S: $\overline{\{S\}} \triangleright Gr(S)$ — move $^{-}$ inside small slice

$(\bar{r})$ $\bar{r} \triangleright \overline{\rightarrow x \overset{r}{\rightarrow} y \rightarrow}$ — replace label $\bar{r}$ by complemented slice

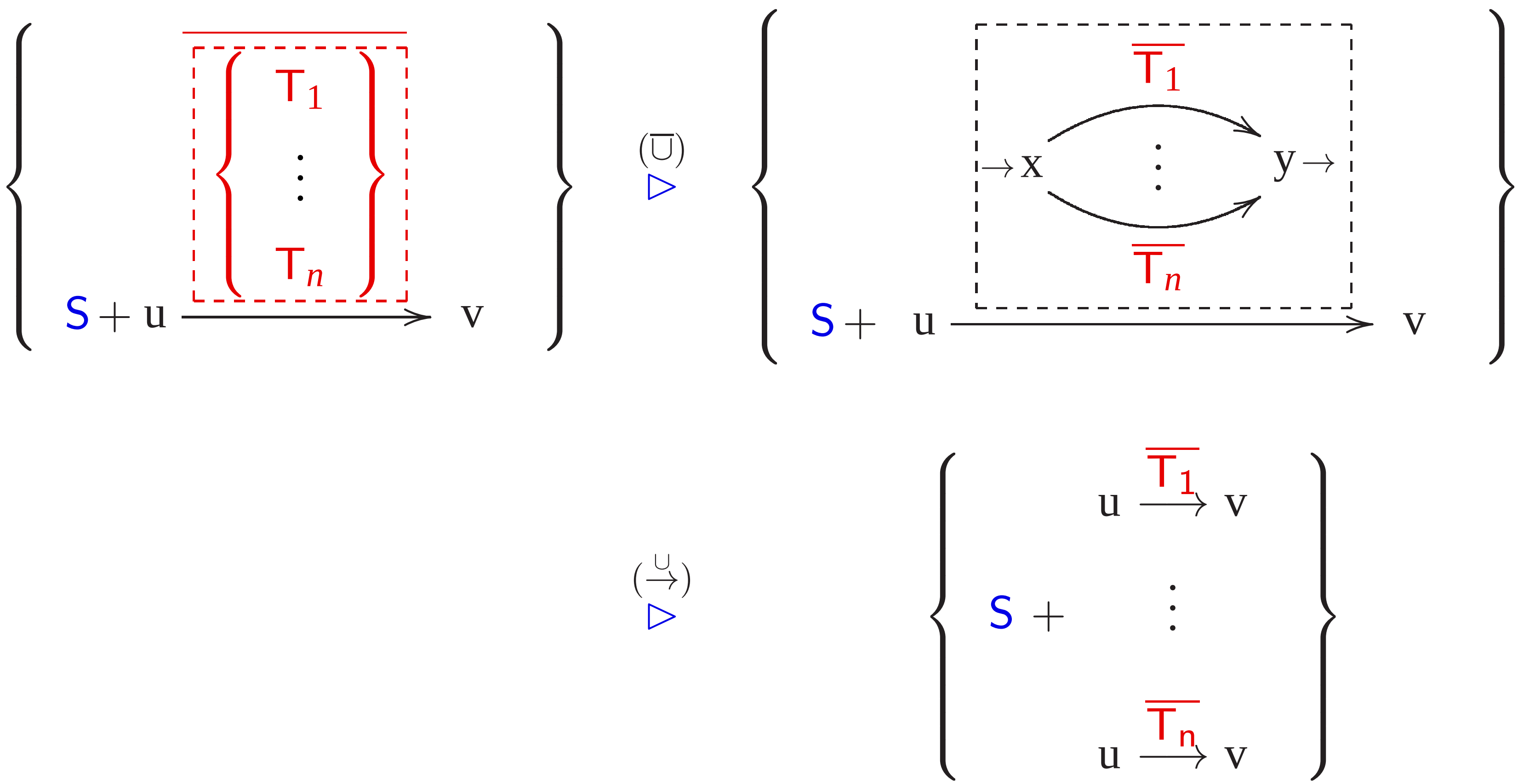
⋈ Derived structural rule — replace compl. graph label by compl. slices

$(\overset{\overline{\cup}}{\rightarrow})$ $\{S + u\,\overline{H}\,v\} \triangleright \{S + \{u\,\overline{T}\,v \,/\, T \in H\}\}$ — replace $u \overset{\overline{H}}{\rightarrow} v$ by $\{u \overset{\overline{T}}{\rightarrow} v \,/\, T \in H\}$

⇝ Graph G (cont'd)



◁ Derived structural rule $(\overset{\cup}{\rightarrow})$ rules $(\overline{\cup})$ & $(\overset{\cup}{\rightarrow})$



♡ Completeness

1. Family of zero slices $\mathcal{Z}_0$
2. Family of eventually zero slices: $\mathsf{S} \in \mathcal{Z}_* \iff \mathsf{S}\ \mathtt{Exp}^*\ \mathsf{H} \subseteq \mathcal{Z}_0$
3. Family of non-eventually zero slices: $\mathsf{S} \in \mathcal{Z}_\infty \iff \mathsf{S} \notin \mathcal{Z}_*$

◁ If $\mathsf{G} \subseteq \mathcal{Z}_*$ then $\mathsf{G} \equiv \mathbb{L}$

⋈ If $\mathsf{S} \in \mathcal{Z}_\infty$ then $\mathsf{S}\frac{\mathrm{u}}{\mathrm{v}}\mathsf{T} \in \mathcal{Z}_\infty$ or $\mathsf{S} + \mathrm{u}\overline{\mathsf{T}}\mathrm{v} \in \mathcal{Z}_\infty$

Chain of slices $\mathsf{S} \in \mathcal{Z}_\infty$ $\mathsf{S}_0 := \mathsf{S}$

$\underline{\mathsf{S}}_0 \overset{\varphi_0}{\rightarrow} \underline{\mathsf{S}}_1 \ \ldots \overset{\varphi_n}{\rightarrow} \underline{\mathsf{S}}_{n+1}$ underlying drafts $\mathsf{S}_n \in \mathcal{Z}_\infty$

Co-limit sketch Σ counter-model $\mathfrak{C}$ $[\![\mathsf{S}]\!]_{\mathfrak{C}} \neq \emptyset$

⋈ If $\mathsf{G} \not\subseteq \mathcal{Z}_*$ then $[\![\mathsf{G}]\!]_{\mathfrak{C}} \neq \emptyset$

▷ Model $\mathfrak{M} = \langle M, (r^{\mathfrak{M}})_{r \in Rn} \rangle$ *natural* for sketch $\Sigma = \langle N_\Sigma, A_\Sigma \rangle$

$$M = N \qquad r^{\mathfrak{M}} = \{(w,z) \in M^2 \,/\, w\,r\,z \in A\}$$

⋈ Discrimination assignments & morphisms natural model $\mathfrak{C}$ for sketch Σ

Basic draft D with $\mathtt{ES}[D] \subseteq \mathtt{ES}[\Sigma]$ $\qquad$ $g : D \to \mathfrak{C}$ iff $g : D \dashrightarrow \Sigma$

◁ Induction on $\mathtt{rk}(D) \in \mathbb{N}$

▷ Basic objects: *rank* and *set of embedded slices* $\qquad$ structural measure

(r) $r \in Rn$: $\quad \mathsf{rk}(r) := 0 \quad \mathsf{ES}[r] := \emptyset$

(-) compl. slice: $\quad \mathsf{rk}(\overline{T}) := \mathtt{rk}(T) + 1 \quad \mathsf{ES}[\overline{T}] := \mathsf{ES}[T] \cup \{T\}$

(a) arc: $\quad \mathsf{rk}(u\,L\,v) := \mathsf{rk}(L) \quad \mathsf{ES}[u\,L\,v] := \mathsf{ES}[L]$

(D) draft: $\quad \mathtt{rk}(D) := \sum_{a \in A_D} \mathsf{rk}(a)$

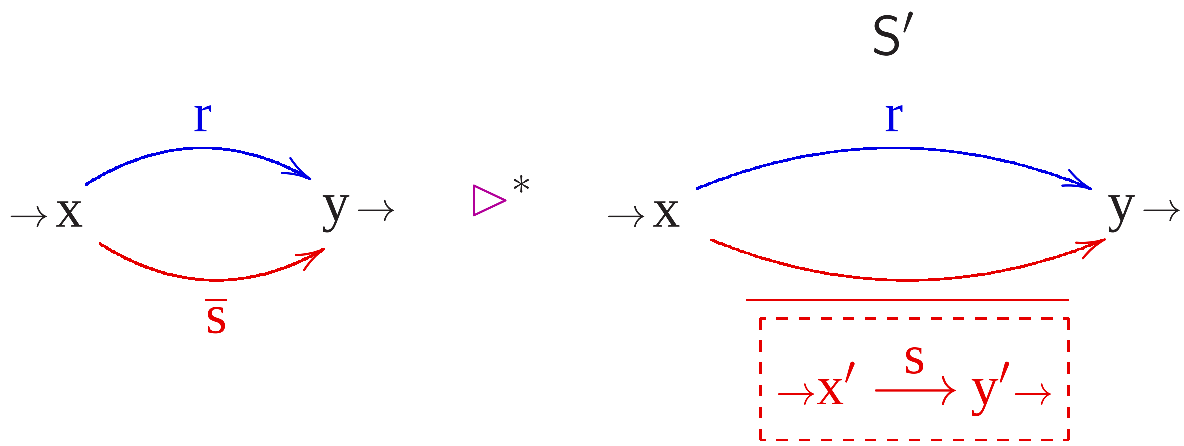
(Σ) sketch: $\quad \mathtt{ES}[\Sigma] := \bigcup_{a \in A_\Sigma} \mathsf{ES}[a]$

(S) slice: $\quad \mathtt{rk}(S) := \mathtt{rk}(\underline{S}) \quad \mathtt{ES}[S] := \mathtt{ES}[\underline{S}]$

A.3 Hypotheses

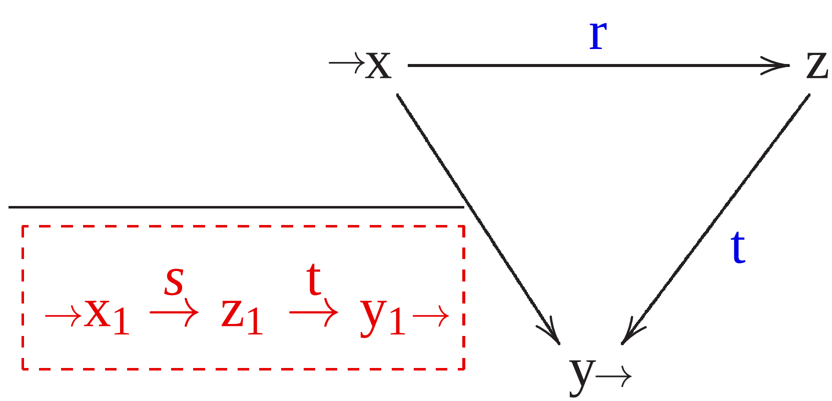
$\rightsquigarrow \quad r \sqsubseteq s \Rightarrow r;t \sqsubseteq s;t \qquad \text{iff} \qquad \begin{pmatrix} r \\ \sqcap \\ \overline{s} \end{pmatrix} \sqsubseteq \mathbb{\bot} \Rightarrow \begin{pmatrix} r;t \\ \sqcap \\ \overline{s;t} \end{pmatrix} \sqsubseteq \mathbb{\bot}$

0. Hypothesis $r \sqsubseteq s \quad \mapsto \quad$ diff. slice $\mathsf{DS}(r \backslash s) \quad \triangleright^* \quad$ basic graph $\{\mathsf{S}'\}$



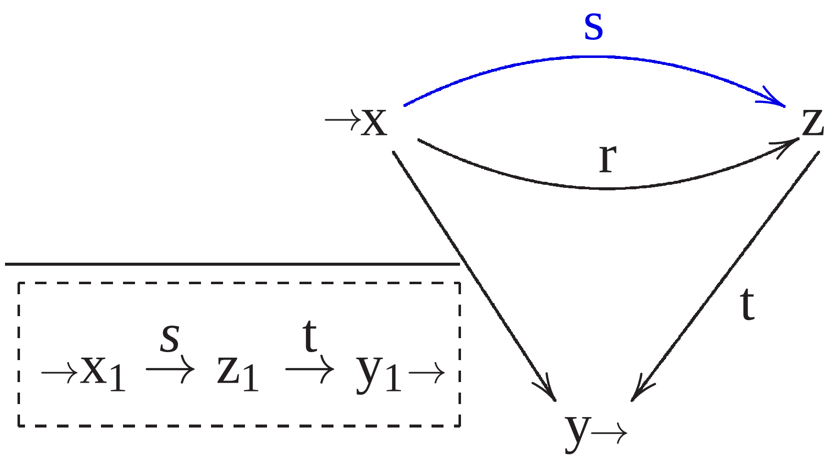
1. Difference slice DS(r;t\s;t) converts to basic slice:

$S_1 :=$

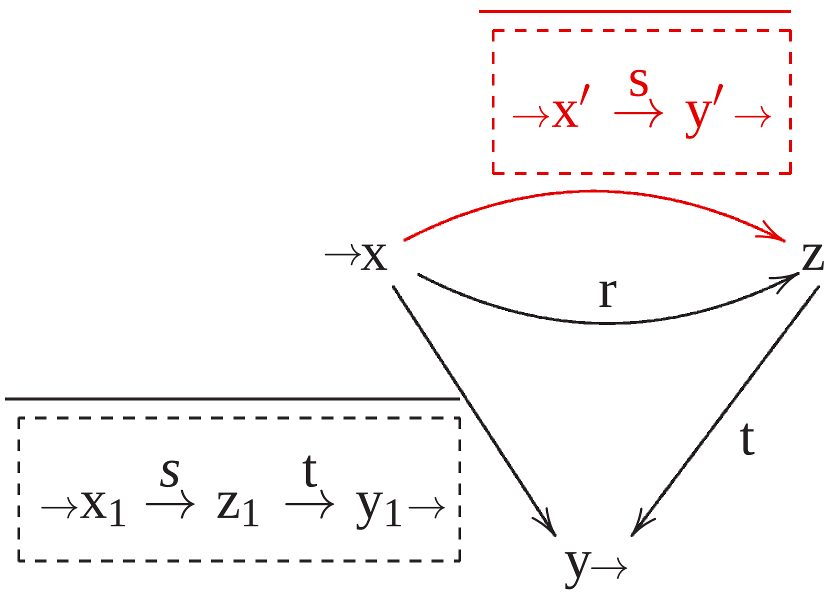


2. Expand $\{S_1\}$ to graph $H = \{S_+, S_-\}$ with $T := \to x \xrightarrow{s} y \to$

$S_+ :=$



$S_- :=$



$\angle$ Graph $H = \{S_+, S_-\}$ $\{S'\}$-erasable

- Slice S_+ parallel paths from x to y: terms s;t & $\overline{s;t}$ $\therefore$ zero
- Slice S_- morphism $\theta : \underline{S'} \dashrightarrow \underline{S_-}$ with $x \mapsto x, y \mapsto z$ $\therefore$ erasable

$\therefore$

Graph H has empty extension in any model
where $r \sqcap \overline{s}$ has empty extension

i. e.

$$\underbrace{\text{hypothesis } r \sqsubseteq s \text{ holds}}_{[r]_{\mathfrak{M}} \subseteq [s]_{\mathfrak{M}}} \Rightarrow \underbrace{\text{inclusion } r;t \sqsubseteq s;t \text{ holds}}_{[r;t]_{\mathfrak{M}} \subseteq [s;t]_{\mathfrak{M}}}$$

▷ Models & consequences

1. Inclusion $L \sqsubseteq K$ — $\mathsf{Mod}(L \sqsubseteq K)$ — models where $[L]_{\mathfrak{M}} \subseteq [K]_{\mathfrak{M}}$
2. Set Λ of inclusions — $\mathsf{Mod}(\Lambda) := \bigcap_{L' \sqsubseteq K' \in \Lambda} \mathsf{Mod}(L' \sqsubseteq K')$
3. $L \sqsubseteq K$ *follows from* Λ — $\Lambda \models L \sqsubseteq K \quad \Leftrightarrow \quad \mathsf{Mod}(\Lambda) \subseteq \mathsf{Mod}(L \sqsubseteq K)$

▷ Slice S is Γ-*erasable* $\Leftrightarrow$ $\mathsf{Mor}[\underline{\mathsf{S}'}, \underline{\mathsf{S}}] \neq \emptyset$ for some $\mathsf{S}' \in \Gamma$

▷ Rule for hypothesis — can erase Γ-erasable slice

$$(\mathtt{Hyp}[\Gamma]) \quad \frac{\{\mathsf{S}\}}{\{\,\}} \qquad \text{if slice } \mathsf{S} \text{ is } \Gamma\text{-erasable}$$

◇ Basic graph G, set Γ of basic slices — $\Lambda[\Gamma] := \{\mathsf{S}' \sqsubseteq \mathbb{L} \,/\, \mathsf{S}' \in \Gamma\}$

$\Lambda[\Gamma] \models \mathsf{G} \sqsubseteq \mathbb{L}$	$\mathsf{G} \sqsubseteq \mathbb{L}$ follows from $\Lambda[\Gamma]$	iff
$\mathsf{G} \ \vdash^{(\mathtt{Exp} \cup \mathtt{Hyp}[\Gamma])} \ \mathsf{H}$	H : zero graph	iff
$\mathsf{G} \ \vdash^{(\mathtt{Exp})} \ \mathsf{H}'$	H' : zero or Γ-erasable graph	