Sahlqvist preservation for modal μ-algebras

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Overview

In classical modal logic Sahlqvist's theorem proves that logics axiomatized by 'Sahlqvist formulas' correspond to varieties of modal algebras closed under canonical extensions.

These logics are sound and complete wrt first-order definable classes of frames.

Sambin and Vaccaro (1989) gave a proof of the Sahlqvist theorem using descriptive frames and topology.

Givant and Venema (1999) showed that every Sahlqvist modal formula is preserved under completions of conjugated modal algebras (BAOs).

Our goal is to extend Sahlqvist's theorem and the methods of Sambin and Vaccaro, and Givant and Venema from modal logic to modal fixed point logic.

Modal algebras

A modal algebra is a pair $(A, \Diamond)$, where A is a Boolean algebra and $\Diamond$ a unary operation on A satisfying for each $a, b \in A$,

1. $\Diamond 0 = 0$,
2. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$.

Theorem. Every modal logic is sound and complete wrt modal algebras.

Language of the modal μ-calculus

- countably infinite set of propositional variables, $x, y, z, \ldots$
- constants $\bot$ and $\top$,
- connectives $\wedge$, $\vee$, $\neg$,
- modal operators $\Diamond$ and $\Box$,
- $\mu x \varphi$ for all formulas φ that are positive in x (i.e., x occurs under the scope of an even number of negations).

Modal μ-algebras

Let $(A, \mathcal{F})$ be a pair such that A is a modal algebra and $\mathcal{F} \subseteq A$. A map h from propositional variables to A is called an assignment. We define a (possibly partial) semantics for modal μ-formulas by induction.

- $[\bot]^{\mathcal{F}}_h = 0$
- $[\top]^{\mathcal{F}}_h = 1$
- $[x]^{\mathcal{F}}_h = h(x)$, where x is a propositional variable,
- $[\varphi \wedge \psi]^{\mathcal{F}}_h = [\varphi]^{\mathcal{F}}_h \wedge [\psi]^{\mathcal{F}}_h$,
- $[\varphi \vee \psi]^{\mathcal{F}}_h = [\varphi]^{\mathcal{F}}_h \vee [\psi]^{\mathcal{F}}_h$,
- $[\neg\varphi]^{\mathcal{F}}_h = \neg[\varphi]^{\mathcal{F}}_h$,
- $[\Diamond\varphi]^{\mathcal{F}}_h = \Diamond[\varphi]^{\mathcal{F}}_h$,
- $[\Box\varphi]^{\mathcal{F}}_h = \neg\Diamond\neg[\varphi]^{\mathcal{F}}_h$,

Modal μ-algebras

Let $a \in A$. We denote by $h[x \mapsto a]$ a new algebra assignment such that $h[x \mapsto a](x) = a$ and $h[x \mapsto a](y) = h(y)$ for each propositional variable $y \neq x$.

- If φ is positive in x then

$$[\mu x \varphi]_h^{\mathcal{F}} = \bigwedge \{a \in \mathcal{F} : [\varphi]_{h[x \mapsto a]}^{\mathcal{F}} \leq a\},$$

  if this meet exists; otherwise, the semantics for $\mu x \varphi$ is undefined.

The pair $(A, \mathcal{F})$ is called a modal μ^*-algebra if $[\varphi]_h^{\mathcal{F}}$ is defined for any modal μ-formula φ and any assignment h.

A modal μ^*-algebra of type (A, A) is called a modal μ-algebra.

Axiomatization

The axiomatization of Kozen's system $\mathbf{K}^{\mu}$ consists of the following axioms and rules

propositional tautologies,	
$\vdash \Box(p \to q) \to (\Box p \to \Box q)$	(**K**-axiom),
If $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$	(Modus Ponens),
If $\vdash \varphi$, then $\vdash \varphi[\psi/x]$	(Substitution),
If $\vdash \varphi$, then $\vdash \Box\varphi$	(Necessitation).

$\vdash \varphi[\mu x\varphi/x] \to \mu x\varphi$	(Fixed Point axiom),
If $\vdash \varphi[\psi/x] \to \psi$, then $\vdash \mu x\varphi \to \psi$	(Fixed Point rule),

where no free variable of ψ is bound in φ.

Completeness

Theorem (Ambler et al., 1995). Let Φ be a set of modal μ-formulas and $L = \mathbf{K}^{\mu} + \Phi$ a modal fixed point logic axiomatized by Φ.

Then L is sound and complete with respect to modal μ^*-algebras (A, A) such that $(A, A) \models \Phi$.

Sahlqvist fixed point formulas

A modal μ-formula is positive if it does not contain a negation.

Definition. A formula $\varphi(p_1, \ldots, p_n)$ is called a Sahlqvist fixed point formula if it is obtained from formulas of the form $\neg\Box^m p_i$ $(m \in \omega, i \leq n)$ and positive formulas (in the language with the μ-operator) by applying the operations $\vee$ and $\Box$.

Examples:

$\Box p \to p \equiv \neg\Box p \vee p$,

$\Diamond\Box p \to \Box\Diamond p \equiv \neg\Diamond\Box p \vee \Box\Diamond p \equiv \Box(\neg\Box p) \vee (\Box\Diamond p)$,

$\Box\Diamond p \to \Diamond\Box p$ is the McKinsey formula and is not Sahlqvist!

$\mu x \Box x$ is a Sahlqvist fixed point formula,

$\Box p \to \Diamond^* p \equiv \neg\Box p \vee \mu x(p \vee \Diamond x)$ is a Sahlqvist fixed point formula,

$\Diamond p \to \Box\Diamond^* p \equiv \Box\neg p \vee \Box\mu x(p \vee \Diamond x)$ is a Sahlqvist f. p. formula.

Sahlqvist canonicity for modal μ^*-algebras

Theorem. Let φ be a Sahlqvist fixed point formula and A a modal μ-algebra. Then

$$(A, A) \models \varphi \text{ implies } (A^\sigma, A) \models \varphi,$$

where A^σ is the canonical extension of A.

The proof uses duality and a fixed point version of Esakia's lemma.

There exist a modal μ^*-algebra (A, A) and Sahlqvist fixed point formula ψ such that

$$(A, A) \models \psi \text{ and } (A^\sigma, A^\sigma) \not\models \psi.$$

Sahlqvist theorem for modal fixed point logic

Let LFP denote FO augmented with least fixed point operator.

An LFP-formula ξ is said to be an LFP-frame condition if it does not contain free variables or predicate symbols.

To each Sahlqvist fixed point formula φ corresponds an LFP-frame condition $\chi(\varphi)$.

Theorem. Let Φ be a set of Sahlqvist fixed point formulas and $L = \mathbf{K}^{\mu} + \Phi$ the modal fixed point logic axiomatized by Φ. Then

L is sound and complete wrt μ^*-algebras (A, A) such that

- $(A^{\sigma}, A) \models \Phi$ (canonicity)

and

- $(A_*, A) \models \{\chi(\varphi) : \varphi \in \Phi\}$ (correspondence),

where A_* is the dual space (canonical frame) of A.

Sahlqvist preservation for completions

Theorem. Let $(A, \mathcal{F})$ be a conjugated modal μ^*-algebra and φ a Sahlqvist fixed point formula. Then

$$(A, \mathcal{F}) \models \varphi \text{ implies } (\overline{A}, \mathcal{F}) \models \varphi,$$

where $\overline{A}$ is the Dedekind-MacNeille completion of A.

The proof of this result does not use duality or an analogue of Esakia's lemma.

Corollary (Givant and Venema, 1999). Every Sahlqvist modal formula is preserved under completions of conjugated modal algebras (BAOs).