On Augmented Posets And $(\mathcal{Z}_1, \mathcal{Z}_1)$-Complete Posets

Mustafa Demirci

Akdeniz University, Faculty of Sciences,
Department of Mathematics,
07058-Antalya, Turkey,
e-mail: demirci@akdeniz.edu.tr

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1. Banaschewski and Bruns's approach (BB-approach)

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Banaschewski and Bruns's approach (BB-approach)

Primary concept in their approach is augmented poset that is a triple $U = (|U|, \mathfrak{J}U, \mathfrak{M}U)$, consisting of a poset $|U|$, a subset $\mathfrak{J}U$ of $\mathcal{P}(U)$ in which each member has the join in $|U|$ and a subset $\mathfrak{M}U$ of $\mathcal{P}(U)$ in which each member has the meet in $|U|$. Augmented posets together with structure preserving maps constitute a category **P**. A structure preserving map $h : U \to V$ here means a monotone map $h : |U| \to |V|$ with the properties

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(a) For all $S \in \mathfrak{J}U$, $h(S) \in \mathfrak{J}V$ and $h(\bigvee S) = \bigvee h(S)$,

(b) For all $R \in \mathfrak{M}U$, $h(R) \in \mathfrak{M}V$ and $h(\bigwedge R) = \bigwedge h(R)$.

The category of spaces, denoted by **S**, is another central concept in BB-approach. The objects of **S** (the so-called spaces) and its morphisms generalize the notions of topological spaces and continuous functions. A space is defined to be a quadruple $W = (|W|, \mathfrak{D}(W), \Sigma(W), \Delta(W))$ fulfilling the properties

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(S4) $\Delta(W)$ consists of all $\mathfrak{B} \subseteq \mathfrak{D}(W)$ with $\cap\mathfrak{B} \in \mathfrak{D}(W)$.

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(S7) $(f^{\leftarrow})^{\to}(\mathfrak{B}) \in \Delta(W_1)$ and for all $\mathfrak{B} \in \Delta(W_2)$.

Theorem

[3] $\mathbf{P}$ is dually adjoint to $\mathbf{S}$, i.e. there are functors $\Psi : \mathbf{P}^{op} \to \mathbf{S}$ and $T : \mathbf{S} \to \mathbf{P}^{op}$ such that $T \dashv \Psi : \mathbf{P}^{op} \to \mathbf{S}$.

Corollary

[3] The full subcategory of $\mathbf{P}$ of all spatial objects ($\mathbf{SpaP}$) and the full subcategory of $\mathbf{S}$ of all sober objects ($\mathbf{SobS}$) are dually equivalent.

Subset selection-based approach (Z-approach)

Z-approach uses the notion of subset selection: A subset selection $\mathcal{Z}$ is a rule assigning to each poset P a subset $\mathcal{Z}(P)$ of its power set $\mathcal{P}(P)$.

Subset selection $\mathcal{Z}$	elements of $\mathcal{Z}(P)$
$\mathcal{V}$	no subset of P
$\mathcal{V}_\perp$	the empty set $\emptyset$
$\mathcal{P}_n$	nonempty subsets of P with cardinality less than or equal to n
$\mathcal{F}$	finite subsets of P
$\mathcal{CN}$	countable subsets of P
$\mathcal{D}$	directed subsets of P
$\mathcal{BN}$	bounded subsets of P

$\mathcal{C}$	nonempty linearly ordered subsets (chains) of P
$\mathcal{C}_{\perp}$	linearly ordered subsets (including $\emptyset$) of P
$\mathcal{P}$	subsets of P
$\mathcal{A}$	downsets of P
$\mathcal{W}$	well-ordered chains of P

A subset selection $\mathcal{Z}$ is called a subset system [2, 6, 14] iff for each order-preserving function $f : P \to Q$, the implication $M \in \mathcal{Z}(P) \Rightarrow f(M) \in \mathcal{Z}(Q)$ holds.
During this talk, $\mathcal{Z}$ will be assumed as a subset selection unless further assumptions are made.

Definition

[8, 18] A poset P with the property that each $M \in \mathcal{Z}(P)$ has the join (meet) in P is called a $\mathcal{Z}$-join(meet)-complete poset.

For simplicity, we call a $\mathcal{Z}_1$-join-complete and $\mathcal{Z}_2$-meet-complete poset a $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete poset. There are two useful subset selections $\mathcal{Z}_1^{\text{sup}}$ and $\mathcal{Z}_2^{\text{inf}}$ derived from given two original subset selections $\mathcal{Z}_1$ and $\mathcal{Z}_2$ by the formulas

$$\begin{aligned} \mathcal{Z}_1^{\text{sup}}(P) &= \left\{ M \in \mathcal{Z}_1(P) \mid \bigvee M \text{ exists in } P \right\}, \\ \mathcal{Z}_2^{\text{inf}}(P) &= \left\{ N \in \mathcal{Z}_2(P) \mid \bigwedge N \text{ exists in } P \right\}, \end{aligned}$$

With the help of these derived subset selections, every poset can be introduced as a $\left(\mathcal{Z}_1^{\text{sup}}, \mathcal{Z}_2^{\text{inf}}\right)$-complete poset.

Definition

A monotone map $f : P \to Q$ is $(\mathcal{Z}_1, \mathcal{Z}_2)$-continuous iff the following two conditions are satisfied:

(i) For each $M \in \mathcal{Z}_1^{\sup}(P)$, $f(\bigvee M) = \bigvee f(M)$,

(ii) For each $N \in \mathcal{Z}_2^{\inf}(P)$, $f(\bigwedge N) = \bigwedge f(N)$.

Under the assumption "$\mathcal{Z}_3$ and $\mathcal{Z}_4$ are subset systems", $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete posets and $(\mathcal{Z}_3, \mathcal{Z}_4)$-continuous maps constitute a category $\mathcal{Q}$-**CPos**, where $\mathcal{Q}$ stands for $(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4)$. $\mathcal{Q}$-**CPos** provides a practically useful categorical framework for many order-theoretic structures.

Example

(Examples of $\mathcal{Q}$-**CPos**)
$(\mathcal{F}, \mathcal{V}, \mathcal{V}, \mathcal{V})$-**CPos**=Category of join-semilattices with $\perp$ and monotone maps (**SEMI** [12]),

$(\mathcal{D}, \mathcal{V}, \mathcal{D}, \mathcal{V})$-**CPos=**Category of directed-complete posets and Scott-continuous functions (**DCPO** [1]),

$(\mathcal{P}, \mathcal{P}, \mathcal{C}, \mathcal{V})$-**CPos=**Category of complete lattices and maps preserving joins of chains (**LC** [13]),

$(\mathcal{P}, \mathcal{P}, \mathcal{V}, \mathcal{V})$-**CPos=**Category of complete lattices and monotone maps (**LI** [13]).

$\mathcal{Q}$-spaces and their category $\mathcal{Q}$-SPC

Z-approach suggests another generalization of topological space called $\mathcal{Q}$-space: A $\mathcal{Q}$-space is, by definition, a pair (X, τ) consisting of a set X and a subset τ (so-called a $\mathcal{Q}$-system on X) of $\mathcal{P}(X)$ such that the inclusion map $i_\tau : (\tau, \subseteq) \hookrightarrow (\mathcal{P}(X), \subseteq)$ is a $\mathcal{Q}$-**CPos**-morphism.

Example

(Examples of $\mathcal{Q}$-systems)

$(\mathcal{V}, \mathcal{V}, \mathcal{V}, \mathcal{V})$-system=System [7, 8], ,

$(\mathcal{P}, \mathcal{F}, \mathcal{P}, \mathcal{F})$-system=Topology [7, 8],

$(\mathcal{F}, \mathcal{P}, \mathcal{F}, \mathcal{P})$-system=Topological closure system [7, 8],

$(\mathcal{V}, \mathcal{P}, \mathcal{V}, \mathcal{P})$-system=Closure system [7, 8],

$(\mathcal{D}, \mathcal{P}, \mathcal{D}, \mathcal{P})$-system=Algebraic closure system [7, 8],

$(\mathcal{D}, \mathcal{F}, \mathcal{D}, \mathcal{F})$-system=Pretopology [11],

$(\mathcal{D}, \mathcal{F}, \mathcal{P}, \mathcal{F})$-system τ=Pretopology τ such that for each $V \subseteq \tau$, if V is not directed but has the join in $(\tau, \subseteq)$, then $\bigcup V \in \tau$,

$(\mathcal{P}, \mathcal{P}, \mathcal{C}_{\perp}, \mathcal{V})$-system τ=$(\tau, \subseteq)$ is a complete lattice such that joins of chains are exactly unions of chains.

Continuous functions turn into $\mathcal{Q}$-space-continuous functions in Z-approach:

A map $f : (X, \tau) \to (Y, \nu)$ between $\mathcal{Q}$-spaces (X, τ) and (Y, ν) is $\mathcal{Q}$-space-continuous if the usual requirement of continuity (i.e. $(f^{\leftarrow})^{\to}(\nu) \subseteq \tau$) is satisfied.
$\mathcal{Q}$-spaces and $\mathcal{Q}$-space-continuous maps form a category $\mathcal{Q}$-**SPC** extending the familiar category of topological spaces (**Top**) to the Z-approach.

Relation between BB-approach and Z-approach

We describe this relation via the functors $G_{\mathcal{Q}} : \mathcal{Q}\text{-}\mathbf{CPos}\rightarrow\mathbf{P}$ and $H_{\mathcal{Q}} : \mathcal{Q}\text{-}\mathbf{SPC}\rightarrow\mathbf{S}$, defined by

$$\begin{aligned} G_{\mathcal{Q}}(P) &= \left(P, \mathcal{Z}_3^{\sup}(P), \mathcal{Z}_4^{\inf}(P)\right),\ G_{\mathcal{Q}}(f) = f, \\ H_{\mathcal{Q}}(X,\tau) &= \left(X, \tau, \mathcal{Z}_3^{\sup}(\tau), \mathcal{Z}_4^{\inf}(\tau)\right) \text{ and } H_{\mathcal{Q}}(g) = g \end{aligned}$$

It is easy to check that $G_{\mathcal{Q}}$ and $H_{\mathcal{Q}}$ are full embeddings, and so BB-approach is more general than Z-approach. Using these full embeddings, we may formulate spatiality and sobriety in Z-approach as follows:

Definition

(i) A $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete poset P is $\mathcal{Q}$-spatial iff $G_{\mathcal{Q}}(P)$ is spatial,

(ii) A $\mathcal{Q}$-space (X,τ) is $\mathcal{Q}$-sober iff $H_{\mathcal{Q}}(X,\tau)$ is sober.

Theorem

(Main result) Assume that $\mathcal{Z}_1$, $\mathcal{Z}_2$ are iso-invariant subset selections and $\mathcal{Z}_3$, $\mathcal{Z}_4$ are subset systems. Let $\mathcal{Q}$-$\mathbf{CPos}_s$ and $\mathcal{Q}$-$\mathbf{SPC}_s$ denote the full subcategory of $\mathcal{Q}$-$\mathbf{CPos}$ of all $\mathcal{Q}$-spatial objects and the full subcategory of $\mathcal{Q}$-$\mathbf{SPC}$ of all $\mathcal{Q}$-sober objects.

(i) $\mathcal{Q}$-$\mathbf{CPos}_s$ and $\mathcal{Q}$-$\mathbf{SPC}$ are dually adjoint to each other.

(ii) $\mathcal{Q}$-$\mathbf{CPos}_s$ is dually equivalent to $\mathcal{Q}$-$\mathbf{SPC}_s$.

(iii) If $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are surjectivity-preserving subset systems, then $(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_1, \mathcal{Z}_2)$-$\mathbf{CPos}$ is dually adjoint to $(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_1, \mathcal{Z}_2)$-$\mathbf{SPC}$.

A subset selection $\mathcal{Z}$ is iso-invariant [6] iff for each order-isomorphism $f : P \to Q$, the implication $M \in \mathcal{Z}(P) \Rightarrow f(M) \in \mathcal{Z}(Q)$ holds

A subset system $\mathcal{Z}$ is surjectivity-preserving iff for each surjective monotone map $f : P \to Q$ and for each $M \in \mathcal{Z}(Q)$, there exists at least one $N \in \mathcal{Z}(P)$ such that $f(N) = M$.

Thank you for attending....

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