Enrichable Elements in Heyting Algebras

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Outline

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Outline

Logic KM

Propositional languages $\mathcal{L}$ and $\mathcal{L}^-$:

- an infinite set of propositional variables $p, q, \ldots$;
- connectives: $\wedge$, $\vee$, $\rightarrow$, $\neg$ (assertoric connectives) and $\Box$ (a unary modality);

$\mathcal{L}$ is the full language above, $\mathcal{L}^-$ is the assertoric part of $\mathcal{L}$. Formulas in $\mathcal{L}^-$ are denoted by letters $A, B, \ldots$.

Logic KM

KM is **Int** understood in language $\mathcal{L}$ plus the following formulas as axioms:

- $p \rightarrow \Box p$
- $(\Box p \rightarrow p) \rightarrow p$
- $\Box p \rightarrow (q \vee (q \rightarrow p))$

closed under substitution and detachment (*modus ponens*).

Kuznetsov's Theorem

Theorem (Kuznetsov's Theorem)

For any formulas A and B of $\mathcal{L}^-$,

$$\mathbf{Int} + A \vdash B \Leftrightarrow \mathbf{KM} + A \vdash B.$$

- Why is **KM** interesting?
- Why is Kuznetsov's Theorem interesting?

KM nowadays is mentioned in connection with **Lax** (Fairtlough-Mendler) or **mHC** (Esakia). However, having been defined in the end of the 1970s, it stemmed from a different source.

Kuznetsov's Theorem

The following diagram is commutative:

$$\begin{array}{ccc} Ext\mathbf{GL} & \xrightarrow{\kappa} & Ext\mathbf{KM} \\ \mu \downarrow & & \downarrow \lambda \\ Ext\mathbf{Grz} & \xrightarrow{\sigma} & Ext\mathbf{Int} \end{array}$$

Here κ is a lattice isomorphism (Muravitsky), λ is a meet epimorphism (Kuznetsov's Theorem) and μ is also a meet epimorphism (Keznetsov-Muravitsky).
This in particular implies that any intermediate logic is the superintuitionistic fragment of some **GL**-logic.

KM-algebras

$\mathfrak{A} = (\mathcal{A}, \wedge, \vee, \neg, \mathbf{0}, \mathbf{1}, \Box)$, where $(\mathcal{A}, \wedge, \vee, \neg, \mathbf{0}, \mathbf{1})$ is a Heyting algebra (the Heyting reduct of $\mathfrak{A}$) and $\Box$ is subject to the following conditions (identities):

- $\Box x \leq x$
- $\Box x \rightarrow x = x$
- $\Box x \leq y \vee (y \rightarrow x)$

Theorem (algebraic version of Kuznetsov's Theorem)

Any Heyting algebra can be embedded into a **KM***-algebra such that the Heyting reduct of the latter generates the same variety as the initial algebra.*

Enrichable Heyting Algebras

In the remaining part of the this presentation, $\mathfrak{A}$ will denote a Heyting algebra.

Question: In how many ways can one make a Heyting algebra a **KM**-algebra?

Enriched Elements

Definition

Given algebra $\mathfrak{A}$ and its elements a and a^*, the pair (a, a^*) is called an $\mathcal{E}$-pair if the following (in)equalities hold:

- $a \leq a^*$
- $a^* \to a = a$
- $a^* \leq b \vee (b \to a)$, *for any* $b \in \mathfrak{A}$.

If (a, a^*) is an $\mathcal{E}$-pair, we say that a is enriched by a^* in $\mathfrak{A}$, or a^* enriches a.

Enrichable Heyting Algebras

Observation

If (a, a') and (a, a'') are $\mathcal{E}$-pairs of $\mathfrak{A}$ then $a' = a''$.

Corollary

There may be only one way to make a Heyting algebra a **KM***-algebra, if each element of the former is enrichable.*

Definition

An algebra is called enrichable if each element of it is enrichable.

Theorem (Kuznetsov's Theorem revisited)

Every Heyting algebra is embedded into an enrichable algebra such that the latter and the former generate the same variety.

Question: How can such an embedding be done?

$\mathcal{E}$-completion

Let us fix this notation:

- an (initial) algebra $\mathfrak{A}$,
- $(\mu_{\mathfrak{A}}, \subseteq)$, the poset of the prime filters of $\mathfrak{A}$,
- $\mathcal{H}(\mathfrak{A})$, the Heyting algebra of the upward cones over $(\mu_{\mathfrak{A}}, \subseteq)$,
- $h : \mathfrak{A} \to \mathcal{H}(\mathfrak{A})$, Stone embedding,
- $\triangle X = \{F \in \mu_{\mathfrak{A}} \mid \forall F'(F \subset F' \Rightarrow F' \in X)\}$,
- $\mathcal{B}_{\triangle}(\mathfrak{A})$ is the subalgebra of $\mathcal{H}(\mathfrak{A})$ generated by $\{h(a) \mid a \in \mathfrak{A}\} \cup \{\triangle h(a) \mid a \in \mathfrak{A}\}$.

$\mathcal{E}$-completion

Definition ($\mathcal{E}$-completion)

We first define the following sequence of algebras:

- $\mathfrak{A}_0 = \mathfrak{A}$,
- $\mathfrak{A}_{i+1} = \mathcal{B}_\vartriangle(\mathfrak{A}_i), \;\; i < \omega$.

Next we observe that $\{\mathfrak{A}_i\}_{i<\omega}$ is a direct family of algebras and define $\overrightarrow{\mathfrak{A}}$ to be the direct limit of $\{\mathfrak{A}_i\}_{i<\omega}$.

Observation

We observe the following:

- $\overrightarrow{\mathfrak{A}}$ *belongs to the variety generated by all* $\mathfrak{A}_i$.
- *If* $\mathfrak{A}$ *is subdirectly irreducible, then all* $\mathfrak{A}_i$ *and* $\overrightarrow{\mathfrak{A}}$ *are subdirectly irreducible as well.*

Simple completion

Notation:

- $\mathfrak{A} \preccurlyeq \mathfrak{B}$ means that $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$ (up to isomorphism).
- $\mathfrak{A} \precsim \mathfrak{B}$ means that $\mathfrak{A}$ is a relative subalgebra (up to isomorphism) of $\mathfrak{B}$, in which case $\mathfrak{A}$, considered by itself, can be regarded a partial algebra.

Definition (simple completion, *a*-completion)

Let $\mathfrak{A} \preccurlyeq \mathfrak{B}$, $a \in \mathfrak{A}$ and (a, a^*) be an $\mathcal{E}$-pair in $\mathfrak{B}$. Then $\mathfrak{B}$ is called a simple completion of $\mathfrak{A}$ if $\mathfrak{B}$ is generated by $\mathfrak{A} \cup \{a^*\}$. A simple completion which depends on a is called an a-completion.

Two Questions:

- Why is a simple completion interesting to investigate?
- Given $\mathfrak{A}$ and $a \in \mathfrak{A}$, do all a-completions form any structure?

$\sim$-negation

Definition ($\sim$-negation)

A unary operation on a Heyting algebra is called a $\sim$-negation if it satisfies the following conditions (identities):

$$
\begin{array}{ll}
(1) & \sim x \wedge \sim\sim x = \sim \mathbf{1}; \\
(2) & x \wedge \sim x \leq \sim \mathbf{1}; \\
(3) & \sim x \vee \sim\sim x = \sim \mathbf{0}; \\
(4) & x \vee \sim x \geq \sim \mathbf{0}; \\
(5) & \sim x \leftrightarrow \sim\sim x = \sim \mathbf{1}; \\
(6) & x \rightarrow y \leq \sim y \rightarrow \sim x; \\
(7) & \sim\sim \mathbf{0} = \sim \mathbf{1}; \\
(8) & \sim\sim \mathbf{1} = \sim \mathbf{0}.
\end{array}
$$

A Heyting algebra $\mathfrak{A}$ with $\sim$-negation will be denoted by $(\mathfrak{A}, \sim)$ and called an expansion.

$\sim$-negation

Observation

In an expansion $(\mathfrak{A}, \sim)$, $[\sim\mathbf{1}, \sim\mathbf{0}], \vee, \wedge, \sim)$ *is a Boolean algebra.*

Proposition

If (a, a^*) *is an* $\mathcal{E}$*-pair in* $\mathfrak{A}$*, then the operation*

$$\sim x = (x \rightarrow a) \wedge a^*$$

is a $\sim$*-negation in* $\mathfrak{A}$*. Conversely, if* $(\mathfrak{A}, \sim)$ *is an expansion then* $(\sim\mathbf{1}, \sim\mathbf{0})$ *is an* $\mathcal{E}$*-pair of* $\mathfrak{A}$*.*

Corollary

Given $a \in \mathfrak{A}$*, let* $\mathfrak{B}$ *be an a-completion of* $\mathfrak{A}$*. Then a* $\sim$*-negation can be defined in* $\mathfrak{B}$ *such that* $a = \sim\mathbf{1}$*, in which case a unique* a^* *that enriches a equals* $\sim\mathbf{0}$*.*

a-expansion, $\triangle a$-expansion

Definition (a-expansion)

Given $a \in \mathfrak{A}$, an expansion $(\mathfrak{A}, \sim)$ is called an a-expansion of $\mathfrak{A}$ if $\sim\mathbf{1} = a$.

Observation

Let $a \in \mathfrak{A}$. Then the algebra $\mathfrak{A}_a^{\triangle}$ which is defined as the subalgebra of $\mathcal{H}(\mathfrak{A})$ generated by $\{h(x) \mid x \in \mathfrak{A}\} \cup \{\triangle h(a)\}$ is (up to isomorphism) an a-expansion of $\mathfrak{A}$.

Definition ($\triangle a$-expansion)

Given an algebra $\mathfrak{A}$ and $a \in \mathfrak{A}$, let us consider the $h(a)$-expansion $(\mathfrak{A}_a^{\triangle}, \sim)$ of $\mathfrak{A}_a^{\triangle}$ that corresponds to the $\mathcal{E}$-pair $(h(a), \triangle h(a))$ of $\mathfrak{A}_a^{\triangle}$. Restricting the operation $\sim$ to $\mathfrak{A}$, we define $(\mathfrak{A}, \sim) \precsim (\mathfrak{A}_a^{\triangle}, \sim)$ and call the former (possibly) **partial algebra** *a $\triangle a$-expansion of $\mathfrak{A}$. One can observed that, given $a \in \mathfrak{A}$, a $\triangle a$-expansion is unique (up to isomorphism).*

packing, relation $\lhd$

If $(\mathfrak{A}, \sim)$ is a $\triangle a$-expansion, for some $a \in \mathfrak{A}$, then $a = \sim\mathbf{1}$. This observation gives rise to the following.

Definition (packing, relation $\lhd$)

Let $\mathfrak{A} \preccurlyeq \mathfrak{B}$. *If a* $\sim$*-negation can be defined in* $\mathfrak{B}$ *so that* $\sim\mathbf{1} \in \mathfrak{A}$ *and the expansion* $(\mathfrak{B}, \sim)$ *is generated by* $\mathfrak{A}$, *we say that* $\mathfrak{A}$ *is packed in* $\mathfrak{B}$ *w.r.t. this* $\sim$*-negation.*
Accordingly, we write $(\mathfrak{A}, \sim)\lhd(\mathfrak{B}, \sim)$ *if the following conditions are fulfilled:*

$$\begin{array}{ll} (1) & \mathfrak{A} \preccurlyeq \mathfrak{B}; \\ (2) & (\mathfrak{B}, \sim) \textit{ is an expansion}; \\ (3) & (\mathfrak{A}, \sim) \precsim (\mathfrak{B}, \sim); \\ (4) & \sim\mathbf{1} \in \mathfrak{A}; \\ (5) & (\mathfrak{B}, \sim) \textit{ is generated by } \mathfrak{A}. \end{array}$$

Theorem 1

Remark:

Thus $(\mathfrak{A}, \sim) \lhd (\mathfrak{B}, \sim)$, for some $\sim$, iff $\mathfrak{A}$ is packed in $\mathfrak{B}$.

Observation

If an algebra $\mathfrak{A}$ is packed in an algebra $\mathfrak{B}$ w.r.t $\sim$, then $\mathfrak{B}$ is generated as a Heyting algebra by $\mathfrak{A} \cup \{\sim\mathbf{0}\}$.

Theorem

Given a $\triangle$a-expansion $(\mathfrak{A}, \sim)$, there is an expansion $(\mathfrak{A}^*, \sim)$ such that $(\mathfrak{A}, \sim) \lhd (\mathfrak{A}^*, \sim)$. Moreover, for any expansion $(\mathfrak{B}, \sim)$, if $(\mathfrak{A}, \sim) \lhd (\mathfrak{B}, \sim)$ then there is a homomorphism of $(\mathfrak{A}^*, \sim)$ onto $(\mathfrak{B}, \sim)$, which is an isomorphism on $\mathfrak{A}$.

poset $(\mathcal{A}, \leq)$

Definition

Given a $\triangle$a-expansion $(\mathfrak{A}, \sim)$, we define

$$\mathcal{A} = \{(\mathfrak{A}_i, \sim) \mid (\mathfrak{A}, \sim) \triangleleft (\mathfrak{A}_i, \sim), i \in I\}.$$

Also, we define

$$(\mathfrak{A}_i, \sim) \leq (\mathfrak{A}_j, \sim)$$

if there is a homomorphism of the latter onto the former, which is an isomorphism on $\mathfrak{A}$.

Theorem

$(\mathcal{A}, \leq)$ is a poset which is a join semilattice with the top element $(\mathfrak{A}^*, \sim)$ and minimal elements.

minimal elements of $(\mathcal{A}, \leq)$

Definition (relation ◀)

We write $(\mathfrak{A}, \sim) \blacktriangleleft (\mathfrak{B}, \sim)$ *if the following conditions are satisfied:*

- $(\mathfrak{A}, \sim) \lhd (\mathfrak{B}, \sim)$;
- $\mathfrak{A}$ *and* $\mathfrak{B}$ *are subdirectly irreducible*;
- $\mathfrak{A}$ *and* $\mathfrak{B}$ *share their pre-top element.*

Theorem

If an initial algebra $\mathfrak{A}$ *is s.i. then, an expansion* $(\mathfrak{A}_i, \sim)$ *is minimal in* $(\mathcal{A}, \leq)$ *iff* $(\mathfrak{A}, \sim) \blacktriangleleft (\mathfrak{A}_i, \sim)$.

Corollary

If $\mathfrak{A}$ *is s.i. then any its expansion* $(\mathfrak{A}_a^{\triangle}, \sim)$ *is minimal in* $(\mathcal{A}, \leq)$.

Thank you