Canonical extension of coherent categories

Dion Coumans

Radboud University Nijmegen

TACL, July 2011

Outline

1. Canonical extension of distributive lattices $(\wedge, \vee, \top, \bot)$
2. 'Algebraic' semantics for coherent logic $(\wedge, \vee, \top, \bot, \exists)$:
   - Polyadic distributive lattices (pDL's)
   - Coherent categories
3. Canonical extension of pDL's and coherent categories
4. Relation to other constructions (Makkai's topos of types)
5. Future work

Canonical extension of distributive lattices

$$
\begin{tikzcd}
\mathbf{DL}^+ \arrow[rr, shift left, "\mathcal{J}^\infty"] \arrow[d, "U"] & & \text{posets} \arrow[ll, shift left, "Dwn"] \\
\text{distributive lattices} \arrow[rr, shift left, "PrFl"] & & \text{Priestley spaces} \arrow[ll, shift left, "ClDwn"] \arrow[u, "U"]
\end{tikzcd}
$$

$\mathbf{DL}^+$	$=$	completely distributive algebraic lattices
Priestley spaces	$=$	totally order-disconnected compact Hausdorff spaces

Canonical extension of distributive lattices

$$\begin{tikzcd}
\mathbf{DL}^+ \arrow[rrr, shift left, "\mathcal{J}^\infty"] \arrow[d, shift left, "U"] & & & \text{posets} \arrow[lll, shift left, "Dwn"] \\
\text{distributive lattices} \arrow[u, shift left, "(_)^\delta"] \arrow[rrr, shift left, "PrFl"] & & & \text{Priestley spaces} \arrow[lll, shift left, "ClDwn"] \arrow[u, "U"']
\end{tikzcd}$$

$\mathbf{DL}^+$	$=$	completely distributive algebraic lattices
Priestley spaces	$=$	totally order-disconnected compact Hausdorff spaces

Canonical extension of distributive lattices

$\mathbf{DL}^+$ = completely distributive algebraic lattices.

Canonical extension is left adjoint to $\mathbf{DL}^+ \hookrightarrow \mathbf{DL}$.

Universal characterization of canonical extension:

$$\begin{tikzcd} \mathbf{L} \arrow[r, "e"] \arrow[dr, "f"'] & \mathbf{L}^\delta \arrow[d, dotted, "\tilde{f}"] \\ & \mathbf{K} \end{tikzcd}$$

where $\mathbf{L} \in \mathbf{DL}$ and $\mathbf{K}, \mathbf{L}^\delta \in \mathbf{DL}^+$.

Algebraic semantics for coherent logic

We start from

Signature:	$\Sigma = (f_0, \ldots, f_{k-1}, R_0, \ldots, R_{l-1}, c_0, \ldots, c_{m-1})$
Set of var's / sorts:	$X = \{x_0, x_1, \ldots\}$ / $\{A, B, \ldots\}$
Equality:	$=$
Connectives:	$\wedge, \vee, \top, \bot, \exists$
Derivability notion:	$\vdash$ (given by axioms and rules)

Question:

What properties does the logic over Σ have?

Algebraic semantics for coherent logic

We start from

Signature:	$\Sigma = (f_0, \ldots, f_{k-1}, R_0, \ldots, R_{l-1}, c_0, \ldots, c_{m-1})$
Set of var's / sorts:	$X = \{x_0, x_1, \ldots\}$ / $\{A, B, \ldots\}$
Equality:	$=$
Connectives:	$\wedge, \vee, \top, \bot, \exists$
Derivability notion:	$\vdash$ (given by axioms and rules)

Question:

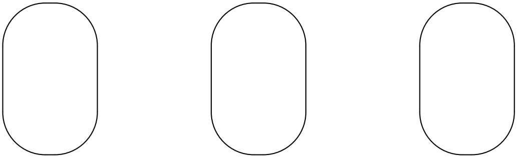
What properties does the logic over Σ have?

First observation:

For each $n \in \mathbb{N}$,

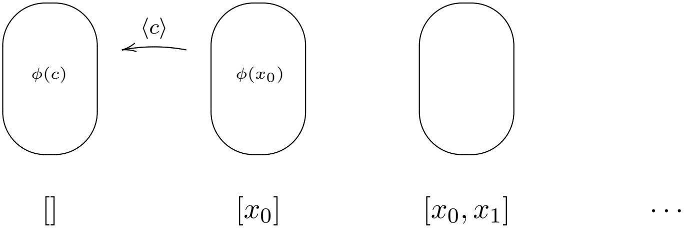
$(Fm(x_0, \ldots, x_{n-1})/_{\vdash \cap \dashv}, \vdash)$ is a distributive lattice.

Algebraic semantics for coherent logic



$[]$ $\quad$ $[x_0]$ $\quad$ $[x_0, x_1]$ $\quad$ $\cdots$

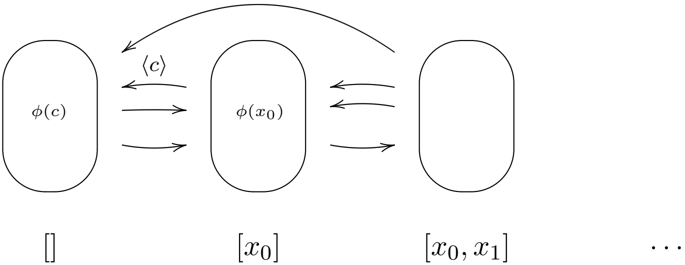
Algebraic semantics for coherent logic



Substitutions:

$$\begin{array}{ccc} x_0 & \mapsto & c \\ \phi(x_0) & \mapsto & \phi(c) \end{array}$$

Algebraic semantics for coherent logic



Substitutions:

$$\begin{array}{ccc} x_0 & \mapsto & c \\ \phi(x_0) & \mapsto & \phi(c) \end{array}$$

Algebraic semantics for coherent logic

Contexts and substitutions form a category $\mathbf{B}$:

Objects:	natural numbers (contexts) / sorts
Morphism $n \to m$:	m-tuple $\langle t_0, \ldots, t_{m-1} \rangle$
	s.t. $FV(t_i) \subseteq \{x_0, \ldots, x_{n-1}\}$

$$
\begin{tikzcd}
{[]} \arrow[r, bend left, "\langle c \rangle"] \arrow[rr, bend right, "\langle c, f(c) \rangle"'] & {[x_0]} \arrow[loop above, "\langle x_0 \rangle"] \arrow[r, bend left, "\langle x_0, f(x_0) \rangle"] & {[x_0, x_1]} \arrow[l, bend left] & \ldots
\end{tikzcd}
$$

Algebraic semantics for coherent logic

Contexts and substitutions form a category $\mathbf{B}$:

Objects:	natural numbers (contexts) / sorts
Morphism $n \to m$:	m-tuple $\langle t_0, \ldots, t_{m-1} \rangle$ s.t. $FV(t_i) \subseteq \{x_0, \ldots, x_{n-1}\}$

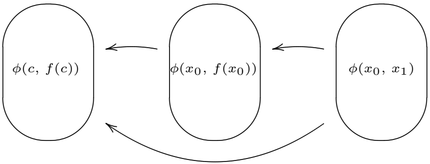
This category has **finite products**:

$$\begin{tikzcd}
[x_0,\, x_1] & [x_0,\, x_1,\, x_2] \arrow[l, "\langle x_0, x_1 \rangle"'] \arrow[r, "\langle x_2 \rangle"] & [x_0] \\
& [\ldots] \arrow[ul, "\langle t_0, t_1 \rangle"] \arrow[u, dotted, "\langle t_0, t_1, s_0 \rangle"'] \arrow[ur, "\langle s_0 \rangle"']
\end{tikzcd}$$

Algebraic semantics for coherent logic

Formulas and substitutions: functor $\mathbf{B}^{op} \to \mathbf{DL}$

$$
\begin{array}{ccccc}
n & \mapsto & Fm(x_0, \ldots, x_{n-1}) & & \\
n \xrightarrow{\langle t_0,\ldots,t_{m-1}\rangle} m & \mapsto & Fm(x_0, \ldots, x_{m-1}) & \to & Fm(x_0, \ldots, x_{n-1}) \\
 & & \phi(x_0, \ldots, x_{m-1}) & \mapsto & \phi(t_0, \ldots, t_{m-1})
\end{array}
$$



$$
\begin{tikzcd}
{[]} \arrow[r, bend left, "\langle c \rangle"] \arrow[rr, bend right, "\langle c, f(c) \rangle"'] & {[x_0]} \arrow[r, "\langle x_0, f(x_0) \rangle"] & {[x_0, x_1]} & \cdots
\end{tikzcd}
$$

Algebraic semantics for coherent logic

Existential quantification: related to the inclusion map

$$\begin{tikzcd} \phi(x_0) \arrow[r, bend left, "i"] & \psi(x_0, x_1) \arrow[l, bend left, "\exists_{x_1}"] \end{tikzcd}$$

$$\begin{tikzcd} {[x_0]} & {[x_0, x_1]} \arrow[l, "\langle x_0 \rangle"'] \end{tikzcd}$$

$$\begin{array}{ccc} \exists_{x_1}(\psi(x_0, x_1)) & \vdash & \phi(x_0) \\ \hline\hline \psi(x_0, x_1) & \vdash & \phi(x_0) \end{array}$$

Algebraic semantics for coherent logic

Existential quantification: related to the inclusion map

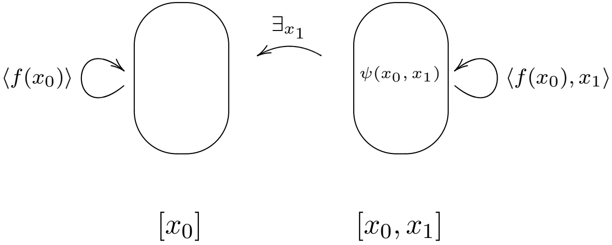
$$
\begin{tikzcd}
\phi(x_0) \arrow[r, bend left, "i"] & \psi(x_0, x_1) \arrow[l, bend left, "\exists_{x_1}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
{[x_0]} & {[x_0, x_1]} \arrow[l, "\langle x_0 \rangle"']
\end{tikzcd}
$$

$$
\frac{\exists_{x_1}(\psi(x_0, x_1)) \quad \vdash_{x_0} \quad \phi(x_0)}{\psi(x_0, x_1) \quad \vdash_{x_0, x_1} \quad i(\phi(x_0))}
$$

Algebraic semantics for coherent logic

Existential quantification: interaction with substitutions



$$\exists_{x_1}(\psi(x_0, x_1))[f(x_0)/x_0] \;=\; \exists_{x_1}(\psi(f(x_0), x_1))$$

(Beck-Chevalley)

Algebraic semantics for coherent logic

Existential quantification: interaction with substitutions

$$\begin{tikzcd}
\phi(x_0) \arrow[rr, bend left, "i"] & & \psi(x_0, x_1) \arrow[ll, bend left, "\exists_{x_1}"] \\
{[x_0]} & & {[x_0, x_1]} \arrow[ll, "\langle x_0 \rangle"']
\end{tikzcd}$$

$$\exists_{x_1}[i(\phi(x_0) \wedge \psi(x_0, x_1)] = \phi(x_0) \wedge \exists_{x_1}[\psi(x_0, x_1)]$$

(Frobenius)

Algebraic semantics for coherent logic

A **polyadic distributive lattice** is a functor $P\colon \mathbf{B}^{\mathbf{op}} \to \mathbf{DL}$ s.t.

1. (Contexts & substitutions)

   $\mathbf{B}$ is a category with finite products;

2. (Existential quantification)

   for all $I, J \in \mathbf{B}$, $P(\pi)\colon P(I) \to P(I \times J)$ has a left adjoint $\exists_\pi$ satisfying Beck-Chevalley and Frobenius;

3. (Equality)

   for all $I, J \in \mathbf{B}$, $P(\delta)\colon P(I \times I \times J) \to P(I \times J)$ has a left adjoint $\exists_\delta$ satisfying Beck-Chevalley and Frobenius,

   (where $\delta = \langle \pi_1, \pi_1, \pi_2 \rangle\colon I \times J \to I \times I \times J$).

Algebraic semantics for coherent logic

Examples of polyadic distributive lattices (pDL's):

- **Syntactic pDL**

  $\mathbf{B}$ = contexts and substitutions

$$\begin{array}{rccc} \mathcal{F}\colon & \mathbf{B}^{op} & \to & \mathbf{DL} \\ & n & \mapsto & Fm(x_0, \ldots, x_{n-1})/_{\vdash\cap\dashv} \end{array}$$

- **Powerset pDL**

  $\mathbf{B} = \mathbf{Set}$

$$\begin{array}{rccc} \mathcal{P}\colon & \mathbf{B}^{op} & \to & \mathbf{BA} \\ & A & \mapsto & \mathcal{P}(A) \\ & A \xrightarrow{f} B & \mapsto & \mathcal{P}(B) \xrightarrow{f^{-1}} \mathcal{P}(A). \end{array}$$

Polyadic distr. lattices and coherent categories

Polyadic distr. lattices

Functor $P\colon \mathbf{B}^{op} \to \mathbf{DL}$ s.t.

- $\mathbf{B}$ has finite products;
- $P(\pi)$ and $P(\delta)$ have left adjoints satisfying BC and Frobenius.

Coherent categories

Category $\mathbf{C}$ s.t.

- $\mathbf{C}$ has finite limits;
- $\mathbf{C}$ has stable finite unions;
- $\mathbf{C}$ has stable images.

Polyadic distr. lattices and coherent categories

Polyadic distr. lattices

Functor $P\colon \mathbf{B}^{op} \to \mathbf{DL}$ s.t.

- $\mathbf{B}$ has finite products;
- $P(\pi)$ and $P(\delta)$ have left adjoints satisfying BC and Frobenius.

Coherent categories

Category $\mathbf{C}$ s.t.

- $\mathbf{C}$ has finite limits;
- $\mathbf{C}$ has stable finite unions;
- $\mathbf{C}$ has stable images.

Proposition

There is an adjunction $\mathcal{A}\colon \mathbf{pDL} \leftrightarrows \mathbf{Coh}\colon \mathcal{S}$, $\mathcal{A} \dashv \mathcal{S}$.

For $\mathbf{C} \in \mathbf{Coh}$, $\mathcal{S}(\mathbf{C}) = \mathcal{S}_{\mathbf{C}}\colon \begin{array}{rcl} \mathbf{C}^{op} & \to & \mathbf{DL} \\ A & \mapsto & Sub_{\mathbf{C}}(A) \end{array}$

and $\mathcal{A}(\mathcal{S}(\mathbf{C})) \simeq \mathbf{C}$.

Canonical extension of pDL's

Recall: canonical extension for DL's is a functor $\mathbf{DL} \xrightarrow{(\_)^\delta} \mathbf{DL}^+$.

Definition

For a pDL $P\colon \mathbf{B} \to \mathbf{DL}$ we define:

$$P^\delta\colon \mathbf{B} \xrightarrow{P} \mathbf{DL} \xrightarrow{(\_)^\delta} \mathbf{DL}.$$

Proposition

For a pDL P, P^δ is again a pDL.

Proof: check that $P^\delta(\pi)$ and $P^\delta(\delta)$ have left adjoints satisfying BC and Frobenius.

Canonical extension of coherent categories

We have:

- adjunction $\mathcal{A}\colon \mathbf{pDL} \leftrightarrows \mathbf{Coh}\colon \mathcal{S}$, $\mathbf{C} \simeq \mathcal{A}(\mathcal{S}_{\mathbf{C}})$
- for a pDL P, $P^\delta\colon \mathbf{B} \xrightarrow{P} \mathbf{DL} \xrightarrow{(\_)^\delta} \mathbf{DL}$

Definition

For a coherent category $\mathbf{C}$ we define:

$$\mathbf{C}^\delta = \mathcal{A}(\mathcal{S}^\delta_{\mathbf{C}})$$

Proposition

For a distributive lattice $\mathbf{L}$, $\mathcal{A}(\mathcal{S}^\delta_{\mathbf{L}}) \simeq \mathbf{L}^\delta$.

Canonical extension of coherent categories

Properties of $\mathbf{C}^\delta = \mathcal{A}(\mathcal{S}^\delta_{\mathbf{C}})$:

1. subobject lattices are in $\mathbf{DL}^+$
2. pullback morphisms are complete lattice homomorphisms

$\mathbf{Coh}^+ =$ coherent categories satisfying (1) and (2).

Canonical extension of coherent categories

Properties of $\mathbf{C}^\delta = \mathcal{A}(\mathcal{S}^\delta_{\mathbf{C}})$:

1. subobject lattices are in $\mathbf{DL}^+$
2. pullback morphisms are complete lattice homomorphisms

$\mathbf{Coh}^+$ = coherent categories satisfying (1) and (2).

Universal characterization:

$$\begin{tikzcd} \mathbf{C} \arrow[r, "M_0"] \arrow[dr, "M"'] & \mathbf{C}^\delta \arrow[d, dotted, "\tilde{M}"] \\ & \mathbf{E} \end{tikzcd}$$

where $\mathbf{C} \in \mathbf{Coh}$, $\mathbf{E}, \mathbf{C}^\delta \in \mathbf{Coh}^+$, M a coherent functor satisfying:

Canonical extension of coherent categories

Properties of $\mathbf{C}^\delta = \mathcal{A}(\mathcal{S}^\delta_{\mathbf{C}})$:

1. subobject lattices are in $\mathbf{DL}^+$
2. pullback morphisms are complete lattice homomorphisms

$\mathbf{Coh}^+$ = coherent categories satisfying (1) and (2).

Universal characterization:

$$\begin{tikzcd} \mathbf{C} \arrow[r, "M_0"] \arrow[dr, "M"'] & \mathbf{C}^\delta \arrow[d, dotted, "\tilde{M}"] \\ & \mathbf{E} \end{tikzcd}$$

where $\mathbf{C} \in \mathbf{Coh}$, $\mathbf{E}, \mathbf{C}^\delta \in \mathbf{Coh}^+$, M a coherent functor satisfying:

for all $A \xrightarrow{\alpha} B$ in $\mathbf{C}$, ρ (prime) filter in $\mathcal{S}_C(A)$,
$\exists_{M(\alpha)}(\bigwedge\{M(U) \mid U \in \rho\}) = \bigwedge\{\exists_{M(\alpha)}(M(U)) \mid U \in \rho\}$.

Topos of types

Note: $\mathcal{S}^{\delta}_{\mathbf{C}} \colon \mathbf{C}^{op} \to \mathbf{DL}^{+}$ is an internal frame in $\mathbf{Set}^{\mathbf{C}^{op}} = \widehat{\mathbf{C}}$.

Then $Sh_{\widehat{C}}(\mathcal{S}^{\delta}_{\mathbf{C}}) \simeq T(\mathbf{C}) =$ **topos of types** of $\mathbf{C}$.

Topos of types

Note: $\mathcal{S}^{\delta}_{\mathbf{C}} \colon \mathbf{C}^{op} \to \mathbf{DL}^{+}$ is an internal frame in $\mathbf{Set}^{\mathbf{C}^{op}} = \widehat{\mathbf{C}}$.

Then $Sh_{\widehat{C}}(\mathcal{S}^{\delta}_{\mathbf{C}}) \simeq T(\mathbf{C})$ = **topos of types** of $\mathbf{C}$.

Topos of types was introduced by Makkai in 1979 as:

- 'a reasonable codification of the 'discrete' (non topological) syntactical structure of types of the theory'
- a tool to prove representation theorems
- 'conceptual tool meant to enable us to formulate precisely certain natural intuitive questions'

Some later work by: Magnan & Reyes and Butz.

Topos of types and the class of models

For a distributive lattice $\mathbf{L}$,

$$\begin{array}{rcl} \text{prime filters of } \mathbf{L} & = & \text{lattice homomorphisms } \mathbf{L} \to \mathbf{2} \\ & = & \text{`models of } \mathbf{L}\text{'}. \end{array}$$

$\mathbf{L}^\delta = \mathcal{D}(Mod(\mathbf{L}))$

Categorical analogue:

$Mod(\mathbf{C})$ = coherent functors $M \colon \mathbf{C} \to \mathbf{Set}$.

Study: $\mathbf{Set}^{Mod(\mathbf{C})}$.

We have to restrict to an appropriate subcategory $\mathcal{K}$ of $Mod(\mathbf{C})$.

Question: How does $\mathbf{Set}^{\mathcal{K}}$ relate to $T(\mathbf{C}) = Sh_{\hat{\mathbf{C}}}(\mathcal{S}^\delta_{\mathbf{C}})$?

Topos of types and the class of models

$\mathcal{K}$ appropriate subcategory of $Mod(\mathbf{C})$.

Question: How does $\mathbf{Set}^{\mathcal{K}}$ relate to $T(\mathbf{C}) = Sh_{\hat{\mathbf{C}}}(\mathcal{S}^{\delta}_{\mathbf{C}})$?

Evaluation functor
$$
\begin{array}{rcl}
ev\colon \mathbf{C} & \to & \mathbf{Set}^{\mathcal{K}} \\
A & \mapsto & \begin{array}{rcl} ev(A)\colon \mathcal{K} & \to & \mathbf{Set} \\ M & \mapsto & M(A) \end{array}
\end{array}
$$

Gives a geometric morphism $\phi_{ev}\colon \mathbf{Set}^{\mathcal{K}} \to \mathbf{Set}^{C^{op}}$

$$
\begin{tikzcd}
 & T(\mathbf{C}) \arrow[d] \\
\mathbf{Set}^{\mathcal{K}} \arrow[ur] \arrow[r, "\phi_{ev}"'] & \mathbf{Set}^{C^{op}}
\end{tikzcd}
$$

$\mathcal{K}$ appropriate subcategory of $Mod(\mathbf{C})$.

Question: How does $\mathbf{Set}^{\mathcal{K}}$ relate to $T(\mathbf{C}) = Sh_{\hat{\mathbf{C}}}(\mathcal{S}^{\delta}_{\mathbf{C}})$?

Evaluation functor

$$\begin{array}{rcl} ev\colon \mathbf{C} & \to & \mathbf{Set}^{\mathcal{K}} \\ A & \mapsto & \begin{array}{rcl} ev(A)\colon \mathcal{K} & \to & \mathbf{Set} \\ M & \mapsto & M(A) \end{array} \end{array}$$

Gives a geometric morphism $\phi_{ev}\colon \mathbf{Set}^{\mathcal{K}} \to \mathbf{Sh}(\mathbf{C}, \mathbf{J_{coh}})$

$$\begin{tikzcd} & T(\mathbf{C}) \arrow[d, two heads] \\ \mathbf{Set}^{\mathcal{K}} \arrow[ur, two heads] \arrow[r, two heads, "\phi_{ev}"'] & \mathbf{Sh}(\mathbf{C}, \mathbf{J_{coh}}) \end{tikzcd}$$

Topos of types and the class of models

$\mathcal{K}$ appropriate subcategory of $Mod(\mathbf{C})$.

Question: How does $\mathbf{Set}^{\mathcal{K}}$ relate to $T(\mathbf{C}) = Sh_{\hat{\mathbf{C}}}(\mathcal{S}^{\delta}_{\mathbf{C}})$?

Evaluation functor
$$\begin{array}{rcl} ev\colon \mathbf{C} & \to & \mathbf{Set}^{\mathcal{K}} \\ A & \mapsto & \begin{array}{rcl} ev(A)\colon \mathcal{K} & \to & \mathbf{Set} \\ M & \mapsto & M(A) \end{array} \end{array}$$

Gives a geometric morphism $\phi_{ev}\colon \mathbf{Set}^{\mathcal{K}} \to \mathbf{Sh(C, J_{coh})}$

$$\begin{tikzcd} & T(\mathbf{C}) \arrow[d, two heads] \\ \mathbf{Set}^{\mathcal{K}} \arrow[ur, two heads] \arrow[r, two heads, "\phi_{ev}"'] & \mathbf{Sh(C, J_{coh})} \end{tikzcd}$$

Makkai: $T(\mathbf{C}) \simeq$ functors in $\mathbf{Set}^{\mathcal{K}}$ with finite support property

Future work

We have: notion of canonical extension for coherent categories

We would like to:

- Study the following diagram (where $\mathcal{K} \subseteq Mod(\mathbf{C})$):

$$\begin{tikzcd} & T(\mathbf{C}) \arrow[d, two heads] \\ \mathbf{Set}^{\mathcal{K}} \arrow[ur, two heads] \arrow[r, two heads, "\phi_{ev}"'] & Sh(\mathbf{C}, J_{coh}) \end{tikzcd}$$

- Generalize to Heyting categories and study addition of axioms
- Apply the developed theory in the study of first order logics
- In particular: study interpolation problems for first order logics, e.g. for IPL + $(\phi \to \psi) \vee (\psi \to \phi)$