# Properties of relatively pseudocomplemented directoids

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# Outline

1. Introduction
2. Relative pseudocomplement as a residuum
3. Congruence properties
4. References

# Introduction

Relatively pseudocomplemented lattices and semilattices play an important role in the investigation of intuitionistic logics and their reducts. They were intensively studied by G.T. Jones. The operation of relative pseudocomplementation serves as an algebraic counterpart of the intuitionistic connective implication.

To investigate some more general algebraic systems connected with non-classical logic (as e.g. BCK-algebras, BCI-algebras, etc.), we often study ordered sets which are not necessarily semilattices. However, a bit weaker structure was introduced by J. Ježek and R. Quackenbush as follows.

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# Directoids

## Definition

By a directoid is meant a groupoid $\mathcal{D} = (D; \sqcap)$ satisfying identities

(D1) $x \sqcap x = x$,

(D2) $x \sqcap y = y \sqcap x$,

(D3) $x \sqcap ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z$.

# Directoids are partially ordered sets

Every directoid $\mathcal{D} = (D; \sqcap)$ can be converted into an ordered set $(D; \leq)$ via

$$x \leq y \iff x \sqcap y = x.$$

Every downward directed ordered set $(D; \leq)$ can be organized into a directoid taking

$$x \sqcap y = y \sqcap x \in L(x, y) \iff x \parallel y,$$
$$x \sqcap y = y \sqcap x = x \iff x \leq y.$$

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$$\begin{aligned} x \sqcap y = y \sqcap x \in L(x, y) \quad &\Longleftrightarrow \quad x \parallel y, \\ x \sqcap y = y \sqcap x = x \quad &\Longleftrightarrow \quad x \leq y. \end{aligned}$$

# Relative pseudocomplementation on semilattices

Let $(S; \wedge)$ be a $\wedge$-semilattice, $a, b \in S$. By a relative pseudocomplement of $a$ with respect to $b$ in $S$ we mean the greatest element among $x \in S$ satisfying

$$a \wedge x \leq b.$$

Or equivalently (But only if it exists!), $a * b$ is the greatest element among $x \in S$ satisfying

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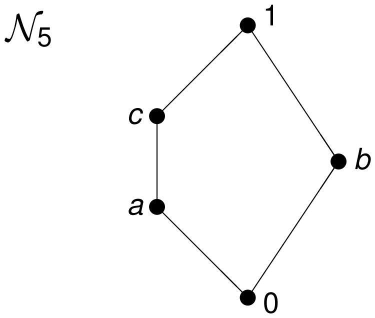


There is a greatest $x \in N_5$ with $c \wedge x = c \wedge a$.

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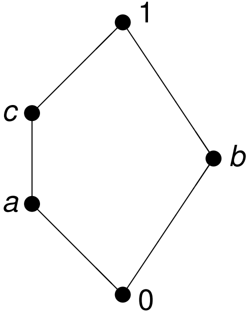


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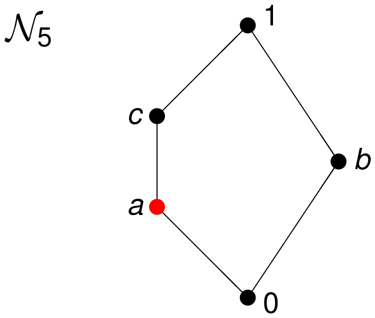


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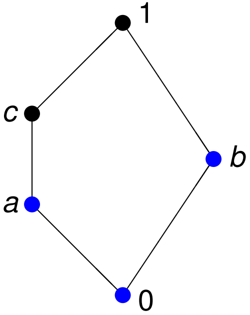


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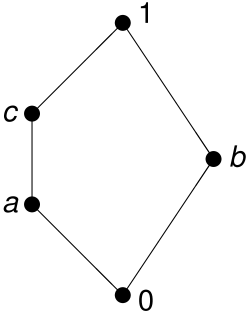


There is a greatest $x \in N_5$ with $c \wedge x = c \wedge a$.
$c * a$ does not exist.

# Analogy for directoids?

In general,

$$x \leq y \quad \not\Rightarrow \quad x \sqcap z \leq y \sqcap z.$$

$\mathcal{D}$ :




There is the greatest $x \in D$ such that $a \sqcap x \leq b$.

But $c$ is not the greatest $x \in D$ such that $a \sqcap x = a \sqcap b$.

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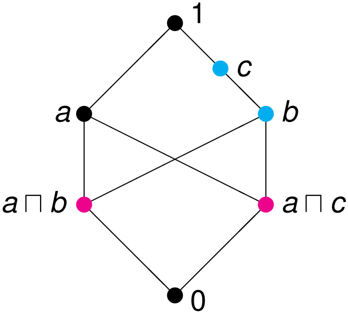


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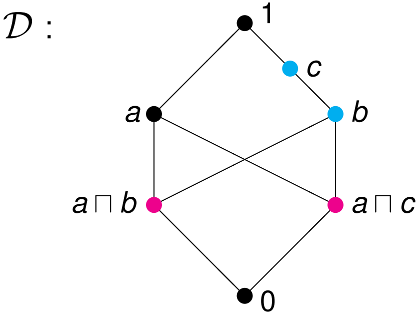


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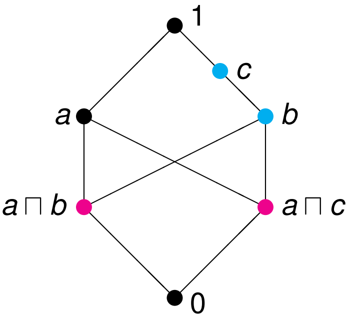


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**Definition (by I. Chajda)**

Let $\mathcal{D} = (D; \sqcap)$ be a directoid and $a, b \in D$. An element $x$ is called a relative pseudocomplement of $a$ with respect to $b$ if it is a greatest element of $D$ such that

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It is denoted by $a * b$.

A directoid $\mathcal{D}$ is relatively pseudocomplemented if there exists $a * b$ for every $a, b \in D$.

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# Identities characterizing relative pseudocomplementation on directoids

## Theorem

*Let* $(D; \sqcap)$ *be a directoid and* $*$ *a binary operation on D. Then* $\mathcal{D} = (D; \sqcap, *)$ *is a relatively pseudocomplemented directoid if and only if it satisfies the following identities*

(S1) $x \sqcap (x * y) = x \sqcap y$

(S2) $(x * y) \sqcap y = y$

(S3) $x * y = x * (x \sqcap y)$

(S4) $x * x = y * y.$

# Axiom system of RPCD

**Theorem**

*Let $\mathcal{D} = (D; \sqcap, *)$ be an algebra with two binary operations. Then $\mathcal{D}$ is a relatively pseudocomplemented directoid if and only if it satisfies the identities (D2), (D3), (S1), (S2) and (S3), i.e.*

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# Adjointness property for RPCD

In relatively pseudocomplemented $\wedge$-semilattices,

$$a \wedge x \leq b \iff x \leq a * b. \qquad (APS)$$

In relatively pseudocomplemented directoids,

$$a \sqcap x = a \sqcap b \implies x \leq a * b, \qquad (I)$$

but not converselly in general.

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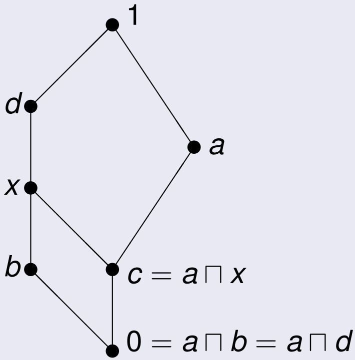


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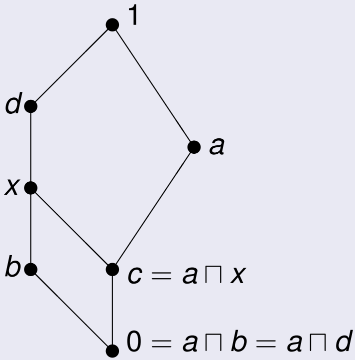


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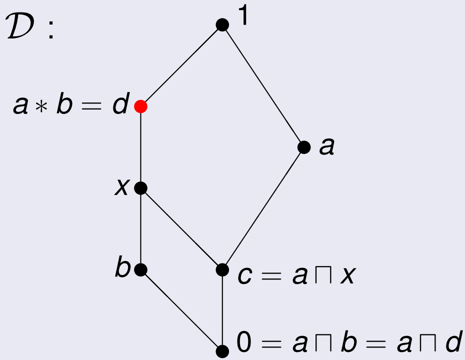


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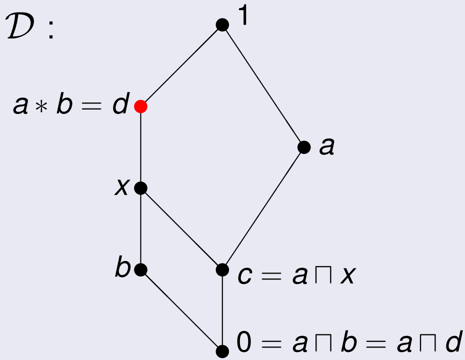


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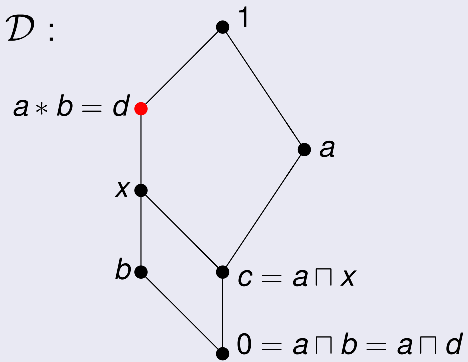


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# Adjointness property for RPCD

So the right hand side of ($I$) must be completed to obtain a condition in the form of an equivalence.

**Theorem**

*Let $(D; \sqcap)$ be a directoid and $*$ be a binary operation on $D$. Then $\mathcal{D} = (D; \sqcap, *)$ is a relatively pseudocomplemented directoid if and only if the following adjointness property holds*

$$a \sqcap x = a \sqcap b \quad \text{iff} \quad x \leq a * b \text{ and } a \sqcap (a * b) = a \sqcap x. \quad (APD)$$

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# Variety of RPCD is CD

**Theorem**

*The terms*

$$\begin{aligned}
t_0(x,y,z) &= x,\\
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# Variety of RPCD is WR and C3-P

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*Variety of RPCD is weakly regular*

$$r_1(x, y) = (x * y) \sqcap (y * x)$$

*and congruence 3-permutable*

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# The question of CP for the variety $\mathcal{V}$ of RPCD

- The problem is open.
- We are able to find proper subvarieties of $\mathcal{V}$ which are CP.
- The variety $\mathcal{R}$ of relatively pseudocomplemented semilattices.
- $\mathcal{W}$ ... relatively pseudocomplemented directoids satisfying the identity

$$((x * y) * y) \sqcap x = x \qquad (T)$$

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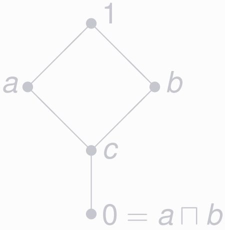
## $\mathcal{R} \subset \mathcal{W}$

Let $(D; \sqcap, *)$ be a relatively pseudocomplemented directoid, where $D = \{0, c, a, b, 1\}$ and $\sqcap, *$ are defined by the following Hasse diagram and the table:




| $*$ | $0$ | $c$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|---|
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| $c$ | $0$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $b$ | $c$ | $1$ | $b$ | $1$ |
| $b$ | $a$ | $c$ | $a$ | $1$ | $1$ |
| $1$ | $0$ | $c$ | $a$ | $b$ | $1$ |

$(D; \sqcap, *) \in \mathcal{W}$, but $(D; \sqcap, *) \notin \mathcal{R}$ since $a \wedge b = c \neq a \sqcap b$.

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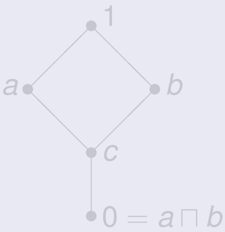


| $*$ | 0 | $c$ | $a$ | $b$ | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| $c$ | 0 | 1 | 1 | 1 | 1 |
| $a$ | $b$ | $c$ | 1 | $b$ | 1 |
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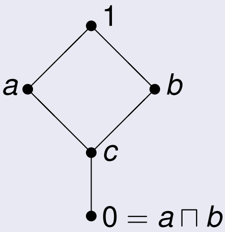


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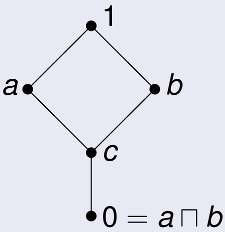


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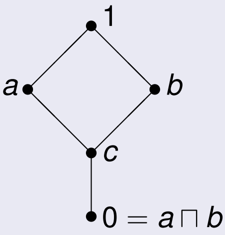


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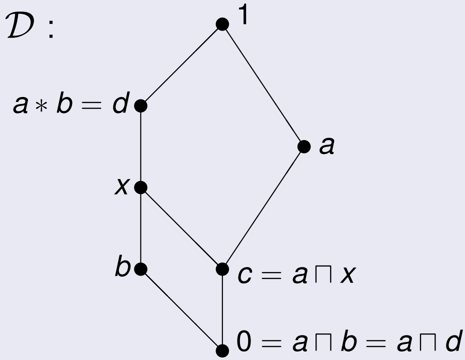


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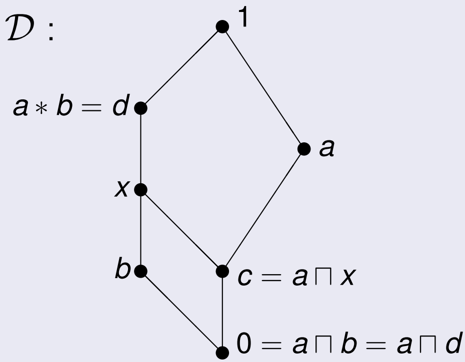


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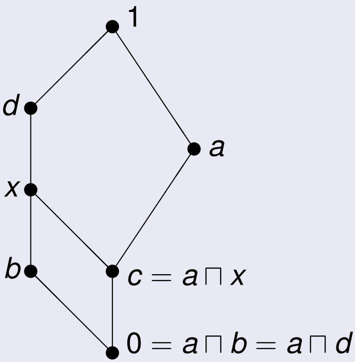


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**Theorem**

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$$p(x,y,z) = ((x*y)*z) \sqcap ((z*y)*x)$$

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# Subdirectly irreducible members of $\mathcal{V}$

- $\mathcal{D} \oplus \mathbf{1}$ ... the directoid constructed from a directoid $\mathcal{D}$ with a greatest element $q$ by adding a new greatest element 1
- If $\mathcal{D}$ is a RPCD then also $\mathcal{D} \oplus \mathbf{1}$ is.

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# Are there any other SI members in $\mathcal{V}$?

$\mathcal{N}_5$ :




Two non-trivial (congruence) partitions:

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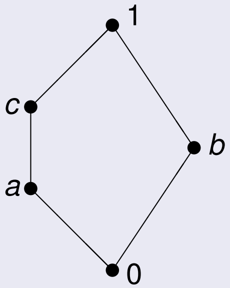


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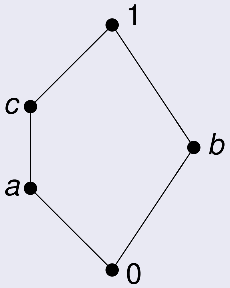


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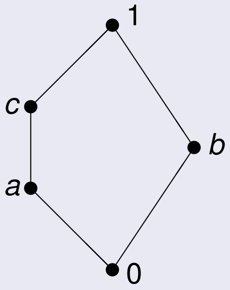


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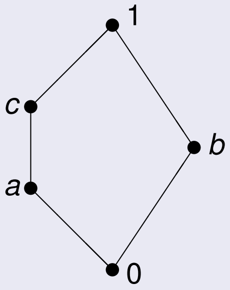


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# Outline

1 Introduction

2 Relative pseudocomplement as a residuum

3 Congruence properties

4 References

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