Admissible Multiple-Conclusion Rules

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Three Examples

- Intuitionistic logic has the *disjunction property*, which may be expressed as the admissible multiple-conclusion rule:

$$p \vee q \;/\; p, q.$$

- Similarly, the following multiple-conclusion rule is admissible in infinite-valued Łukasiewicz logic:

$$p \vee \neg p \;/\; p, \neg p.$$

- *Whitman's condition* may be written as a universal formula that holds in all free lattices:

$$p \wedge q \leq r \vee s \;\;\Rightarrow\;\; p \leq r \vee s,\; q \leq r \vee s,\; p \wedge q \leq q,\; p \wedge q \leq s.$$

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This Talk

We consider:

- (admissible) (multiple-conclusion) rules
- characterizations of these rules
- a case study (Kleene and De Morgan algebras).

Some Terminology

To talk about logics and algebras, we need

- **propositional languages** $\mathcal{L}$ consisting of connectives such as $\wedge, \vee, \to, \neg, \bot, \top$ with specified finite arities
- sets $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ of **$\mathcal{L}$-formulas** $\psi, \varphi, \chi, \ldots$ built from a countably infinite set of variables $p, q, r, \ldots$
- endomorphisms on $\mathbf{Fm}_{\mathcal{L}}$ called **$\mathcal{L}$-substitutions**.

Definition

An **$\mathcal{L}$-rule** is an ordered pair (Γ, Δ) with $\Gamma \cup \Delta \subseteq \mathrm{Fm}_{\mathcal{L}}$ *finite*, written $\Gamma \,/\, \Delta$ (**multiple-conclusion** in general, **single-conclusion** if $|\Delta| = 1$).

Logics and Consequence

Definition

A **logic** L on $\mathbf{Fm}_{\mathcal{L}}$ is a set of single-conclusion $\mathcal{L}$-rules satisfying (writing $\Gamma \vdash_{\mathrm{L}} \varphi$ for $(\Gamma, \{\varphi\}) \in \mathrm{L}$):

- $\{\varphi\} \vdash_{\mathrm{L}} \varphi$ (reflexivity)
- if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Gamma \cup \Gamma' \vdash_{\mathrm{L}} \varphi$ (monotonicity)
- if $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \cup \{\varphi\} \vdash_{\mathrm{L}} \psi$, then $\Gamma \vdash_{\mathrm{L}} \psi$ (transitivity)
- if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma\Gamma \vdash_{\mathrm{L}} \sigma\varphi$ for any $\mathcal{L}$-substitution σ (structurality).

An **L-theorem** is a formula φ such that $\emptyset \vdash_{\mathrm{L}} \varphi$ (abbreviated as $\vdash_{\mathrm{L}} \varphi$).

Multiple-Conclusion Consequence

Definition

An **m-logic** L on $\mathbf{Fm}_{\mathcal{L}}$ is a set of (multiple-conclusion) $\mathcal{L}$-rules (writing $\Gamma \vdash_{\mathsf{L}} \Delta$ for $(\Gamma, \Delta) \in \mathsf{L}$) satisfying:

- $\{\varphi\} \vdash_{\mathsf{L}} \varphi$ (reflexivity)
- if $\Gamma \vdash_{\mathsf{L}} \Delta$, then $\Gamma \cup \Gamma' \vdash_{\mathsf{L}} \Delta' \cup \Delta$ (monotonicity)
- if $\Gamma \vdash_{\mathsf{L}} \{\varphi\} \cup \Delta$ and $\Gamma \cup \{\varphi\} \vdash_{\mathsf{L}} \Delta$, then $\Gamma \vdash_{\mathsf{L}} \Delta$ (transitivity)
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Derivable and Admissible Rules

Definition

For a logic L on $\mathbf{Fm}_{\mathcal{L}}$, an $\mathcal{L}$-rule $\Gamma \ / \ \Delta$ is

- **L-derivable**, written $\Gamma \vdash_{\mathsf{L}} \Delta$, if $\Gamma \vdash_{\mathsf{L}} \varphi$ for some $\varphi \in \Delta$.
- **L-admissible**, written $\Gamma \mathrel{|\!\sim}_{\mathsf{L}} \Delta$, if for every $\mathcal{L}$-substitution σ:

$$\vdash_{\mathsf{L}} \sigma\varphi \quad \text{for all } \varphi \in \Gamma \qquad \Rightarrow \qquad \vdash_{\mathsf{L}} \sigma\psi \quad \text{for some } \psi \in \Delta.$$

(Note: $\vdash_{\mathsf{L}}$ and $\mathrel{|\!\sim}_{\mathsf{L}}$ are m-logics.)

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  (or, any logic L' extending L has new theorems $\emptyset \vdash_{\mathrm{L}'} \varphi$)

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Exact Sets of Formulas

Definition

$\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ is **L-exact** if for some substitution σ, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}$:

$$\Gamma \vdash_{\mathsf{L}} \varphi \quad \text{iff} \quad \vdash_{\mathsf{L}} \sigma\varphi.$$

Lemma

If Γ is L-exact, then $\Gamma \mathrel{|\!\sim}_{\mathsf{L}} \Delta$ if and only if $\Gamma \vdash_{\mathsf{L}} \Delta$.

Proof.

($\Leftarrow$) Easy. ($\Rightarrow$) Let σ be an "exact" substitution for Γ and suppose that $\Gamma \mathrel{|\!\sim}_{\mathsf{L}} \Delta$. Since $\vdash_{\mathsf{L}} \sigma\varphi$ for all $\varphi \in \Gamma$, we have $\vdash_{\mathsf{L}} \sigma\psi$ for some $\psi \in \Delta$. Hence $\Gamma \vdash_{\mathsf{L}} \psi$ and $\Gamma \vdash_{\mathsf{L}} \Delta$ as required. □

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Fragments of Intuitionistic Logic

Theorem (Prucnal, Minari and Wroński)

The $\{\to\}$, $\{\to, \land\}$, and $\{\to, \land, \neg\}$ fragments of intuitionistic logic (in fact, all intermediate logics) are universally complete.

Proof.

Show that each finite set of formulas in the fragment is exact. E.g., in the $\{\to, \land\}$ fragment, $\sigma(p) = \varphi \to p$ is an exact substitution for $\{\varphi\}$. □

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Fragments of Intuitionistic Logic

Theorem (Cintula and Metcalfe)

The following "Wroński rules" ($n \in \mathbb{N}$):

$$(W_n) \quad (p_1 \to \ldots \to p_n \to \bot) \ / \ (\neg\neg p_1 \to p_1), \ldots, (\neg\neg p_n \to p_n).$$

axiomatize the admissible rules of the $\{\to, \neg\}$ fragment of intuitionistic logic (in fact, all intermediate logics).

P. Cintula and G. Metcalfe. Admissible rules in the implication-negation fragment of intuitionistic logic. *Annals of Pure and Applied Logic* 162(2): 162-171 (2010).

Intuitionistic Logic and the Visser Rules

Iemhoff and Rozière established independently that the “Visser rules”

$$\bigwedge_{i=1}^{n}(p_i \to q_i) \to (p_{n+1} \lor p_{n+2}) \;/\; \bigvee_{j=1}^{n+2}(\bigwedge_{i=1}^{n}(p_i \to q_i) \to p_j)$$

for $n = 2, 3, \ldots$ together with the disjunction property axiomatize the admissible rules of **intuitionistic logic**.

Iemhoff has also shown that the Visser rules axiomatize admissibility in certain **intermediate logics**, and Jeřàbek has given axiomatizations of admissible rules for a wide range of **transitive modal logics** and **Łukasiewicz logics**.

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Let $\mathbf{F}_{\mathcal{Q}}$ denote the **free algebra with countably many generators** of a quasivariety $\mathcal{Q}$.

Definition

$\mathcal{Q}$ is **structurally complete** if $\mathcal{Q} = \mathbb{Q}(\mathbf{F}_{\mathcal{Q}})$. (Or, any proper subquasivariety of $\mathcal{Q}$ generates a proper subvariety of $\mathbb{V}(\mathcal{Q})$.)

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$\mathcal{Q}$ is **universally complete** if $\mathcal{Q} = \mathbb{U}(\mathbf{F}_{\mathcal{Q}})$. (Or, any proper sub universal class of $\mathcal{Q}$ generates a proper sub positive universal class of $\mathbb{U}^{+}(\mathcal{Q})$.)

An *algebraizable logic* L is structurally (universally) complete if and only if its equivalent quasivariety is structurally (universally) complete.

The Algebraic Perspective

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Characterizations

Let $\mathcal{Q}$ be a quasivariety and call each algebra $\mathbf{A} \in \mathbb{IS}(\mathbf{F}_{\mathcal{Q}})$ **exact**.

Lemma

If $\mathcal{Q} = \mathbb{Q}(\mathcal{K})$ and each $\mathbf{A} \in \mathcal{K}$ is exact, then $\mathcal{Q}$ is structurally complete.

Lemma

If each non-trivial finitely presented $\mathbf{A} \in \mathcal{Q}$ is exact, then $\mathcal{Q}$ is universally complete.

Theorem

For any finite algebra $\mathbf{A}$:

(a) $\mathbb{Q}(\mathbf{A})$ *is structurally complete iff* $\mathbf{A} \in \mathbb{ISP}(\mathbf{F}_{\mathbb{Q}(\mathbf{A})})$.

(b) $\mathbb{Q}(\mathbf{A})$ *is universally complete iff each finite non-trivial* $\mathbf{B} \in \mathbb{Q}(\mathbf{A})$ *is exact.*

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A Case Study

Definition

De Morgan algebras are algebras $\langle A, \wedge, \vee, \neg, \bot, \top \rangle$ such that

- $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded distributive lattice
- $\neg\neg x = x$, $\neg(x \wedge y) = \neg x \vee \neg y$, and $\neg(x \vee y) = \neg x \wedge \neg y$.

The class DMA of De Morgan algebras is an equational class generated as a quasivariety by $\mathbf{D_4} = \langle \{\bot, \mathrm{n}, \mathrm{b}, \top\}, \wedge, \vee, \neg, \bot, \top \rangle$.

$\top$

n b

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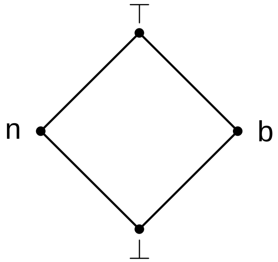

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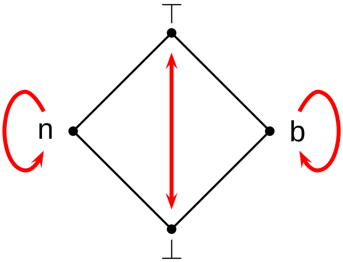

Subvarieties

DMA has only two proper non-trivial subvarieties:

- The class of **Boolean algebras**, BA $= \mathbb{Q}(\mathbf{C}_2)$.
- The class of **Kleene algebras**, KA $= \mathbb{Q}(\mathbf{C}_3)$.

BA is universally complete, but not KA $= \mathbb{Q}(\mathbf{C}_3)$ or DMA $= \mathbb{Q}(\mathbf{D}_4)$; e.g.

$$p \approx \neg p \;\Rightarrow\; p \approx q \qquad (1)$$

holds in $\mathbf{F}_{\mathrm{KA}}$ and $\mathbf{F}_{\mathrm{DMA}}$, but not $\mathbf{C}_3$ or $\mathbf{D}_4$.

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Axiomatizing Admissible Single-Conclusion Rules

$\mathbb{Q}(\mathbf{C_4})$ *is* structurally complete, since $\mathbf{C_4}$ embeds into $\mathbf{F}_{\mathbb{Q}(\mathbf{C_4})}$:

Moreover, $\mathbb{Q}(\mathbf{C_4})$ is axiomatized relative to KA by

$$\neg p \leq p, \quad p \wedge \neg q \leq \neg p \vee q \quad \Rightarrow \quad \neg q \leq q \tag{2}$$

But (2) holds in $\mathbf{F}_{\text{KA}}$, so it axiomatizes $\mathbb{Q}(\mathbf{F}_{\text{KA}})$ relative to KA.

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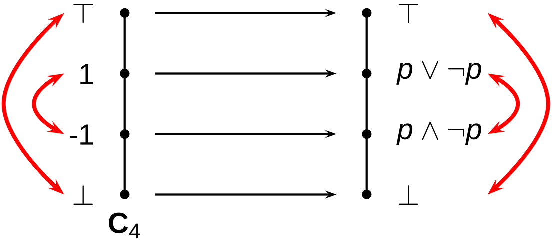

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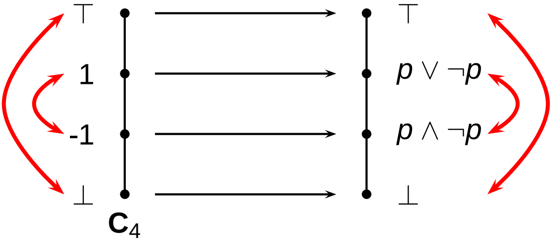

Moreover, $\mathbb{Q}(\mathbf{C_4})$ is axiomatized relative to KA by

$$\neg p \leq p, \;\; p \wedge \neg q \leq \neg p \vee q \quad \Rightarrow \quad \neg q \leq q \tag{2}$$

But (2) holds in $\mathbf{F}_{\mathrm{KA}}$, so it axiomatizes $\mathbb{Q}(\mathbf{F}_{\mathrm{KA}})$ relative to KA.

Axiomatizing Admissible Multiple-Conclusion Rules

Theorem

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$$p \vee q \approx \top \;\Rightarrow\; p \approx \top, q \approx \top. \tag{3}$$

Also, these universal formulas axiomatize $\mathbb{U}(\mathbf{F}_{\mathrm{KA}})$ *relative to* KA.

Theorem

A finite non-trivial De Morgan algebra is exact iff it satisfies (1) and (3). Also, these universal formulas axiomatize $\mathbb{U}(\mathbf{F}_{\mathrm{DMA}})$ *relative to* DMA.

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Concluding Remarks

- (Multiple-conclusion) admissible rules can be used to express properties of logics / classes of algebras.
- Can these rules be useful? E.g., for completeness / generation proofs or for speeding up proof search?
- Do we even have the right notion of admissibility for multiple-conclusion rules? E.g., should $\emptyset \ / \ p, \neg p$ be admissible in classical logic?

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