naBL-algebras and quantum structures

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OP Education for Competitiveness
INVESTMENTS IN EDUCATION DEVELOPMENT

- most celebrated fuzzy logic - Hájek's BL-logic: general framework for formalizing statements of fuzzy nature: statements principially true only in a certain degree, no sharp yes-no
- the logic of continuous t-norms and their residua
- algebraic semantics: BL-algebras

- non-associative MV-logic: M.Botur, R.H. (2009), algebraic semantics: commutative basic algebras CBA
- naBL - algebraic semantics for a non-associative BL - M.Botur (2011)

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fuzzy logics vs quantum logics

- quantum logics: formalizing statements in quantum mechanical experiments, statements of a **probabilistic** character
- typically: if some QM yes-no experiment leads to a positive result
- again: statements to which it is not possible to assign a sharp truth value-but this time since the result is **unpredictable**
- formulas of the corresponding logic are interpreted by **effect algebras** EA
- nevertheless, quantum and fuzzy structures have common structural properties:
  lattice EA: pasted by blocks which are MV, Z. Riečanová (1999)
  BL can be viewed as a special class of weak EA, (T.Vetterlein, series of papers)

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Aims

- find a common structure theory for CBA, naBL and EA
- 1st step: characterize CBA and naBL in the languge of EA

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Effect algebras-definition

- Effect algebras: D. Foulis and M. K. Bennett 1994

An **effect algebra** is a system $\mathscr{E} = (E; +, 0, 1)$ where 0 and 1 are two distinguished elements of E, $+$ is a partial binary operation with axioms:

(EA1) $a + b = b + a$ whenever $a + b$ exists;

(EA2) $a + (b + c) = (a + b) + c$ if one of the sides is defined;

(EA3) for every $a \in E$ there exists a unique $a' \in E$ with $a + a' = 1$;

(EA4) if $a + 1$ is defined then $a = 0$.

natural order:

$$a \leq b \quad \text{iff} \quad b = a + c \text{ for some } c \in E$$

$(E; \leq, 0, 1)$...bounded poset
$(E; \leq)$...lattice...$\mathscr{E}$...**lattice effect algebra**
Natural examples: OML's and MV-algebras:

(1) If $(L; \vee, \wedge, ^{\perp}, 0, 1)$ is an orthomodular lattice then defining

$$a + b := a \vee b \quad \text{iff} \quad a \leq b^{\perp},$$

$(L; +, 0, 1)$ is a lattice effect algebra with $a' = a^{\perp}$.

(2) Given an MV-algebra $\mathscr{A} = (A; \oplus, \neg, 0)$, defining

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M.Botur (FSS 2011)

Definition

An algebra $\mathbf{A} = (A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $\langle 2,2,2,2,0,0 \rangle$ is a (non-associative) residuated lattice (RL) if

(A1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,

(A2) $(A, \odot, 1)$ is a commutative groupoid with 1,

(A3) for any $x, y, z \in A$, $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$ (adjointness property).

- BL-algebras: associative RL satisfying divisibility and prelinearity:

$$x \odot (x \rightarrow y) = x \wedge y, (x \rightarrow y) \vee (y \rightarrow x) = 1$$

- α, β-terms:

$$\alpha_b^a(x) := (a \odot b) \rightarrow (a \odot (b \odot x)),$$

$$\beta_b^a(x) := b \rightarrow (a \rightarrow ((a \odot b) \odot x))$$

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naBL-algebras

- **representable** RL: subdirect products of linearly ordered members
- naBL (M. Botur): def.: representable RL satisfying divisibility
- α, β-prelinearities

$$(x \to y) \vee \alpha_b^a(y \to x) = 1 \qquad (\alpha\text{-prelinearity})$$

$$(x \to y) \vee \beta_b^a(y \to x) = 1 \qquad (\beta\text{-prelinearity}),$$

- naBL: RL satisfying divisibility and both α, β-prelinearities
- α, β-prelinearities can be substituted by

$$x \perp y \ \Rightarrow \ x \perp \alpha_b^a(y) \text{ and } x \perp \beta_b^a(y) = 1$$

where $x \perp y$:=$x \vee y = 1$.

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Definition

A binary operation $*$ on the interval $[0,1]$ of reals is said to be a (non-associative) t-norm (nat-norm briefly) if

(t1) $([0,1], *, 1)$ is a commutative groupoid with the neutral element 1,

(t2) $*$ is continuous,

(t3) $*$ is monotone

Theorem

(M.Botur)
$na\mathscr{BL} = \mathrm{IP_S SP_U}(na\mathscr{T})$

Definition

A structure $(L, \leq, \oplus, 0)$ is called a **naturally ordered abelian groupoid** (NAG, briefly) if

- (NAG1) $(L, \leq, 0)$ is a poset with a least element 0
- (NAG2) $(L, \oplus, 0)$ is an abelian groupoid with 0
- (NAG3) $a \leq b$ iff $a \oplus x = b$ for some $x \in L$.

$\mathscr{L}$ is called **bounded** whenever $(L, \leq)$ has a top element 1.

Definition

A NAG $\mathscr{L}$ is called of **type naBL** if it fulfils

- (NAG4) $\forall a, b, c \in L \exists c_1, c_2 \in L :$
  $a \oplus (b \oplus c) = (a \oplus b) \oplus c_1, (a \oplus b) \oplus c = a \oplus (b \oplus c_2)$ and,
  $\forall y \in L : y \perp c \Rightarrow y \perp c_1, y \perp c_2$
- (RP) (**residuation property**):
  $\forall a, b, c \in L$ there is the least $x \in L$ with $a \oplus x \geq b$
- (RDP) (**Riesz decomposition property**)
  $c \leq a \oplus b \Rightarrow c = a_1 \oplus b_1$ for some $a_1 \leq a$, $b_1 \leq b$
- (CP) (**compatibility property**)
  $\forall a, b \in L \exists a_1, b_1, c \in L : a = a_1 \oplus c, b = b_1 \oplus c, a_1 \wedge b_1 = 0.$

- Given an naBL-algebra $(L, \leq_n, \odot, \rightarrow, 0_n, 1_n)$, define its **dual** $(L, \leq, \oplus, \ominus, 0, 1)$:
  put $a \leq b$ iff $b \leq_n a$
  $a \oplus b := a \odot b$
  $1 := 0_n, 0 := 1_n$
  $a \ominus b := b \rightarrow a$

Theorem

Let $(L, \leq, \oplus, \ominus, 0, 1)$ be a dual naBL-algebra. Then $(L, \leq, \oplus, 0, 1)$ is a bounded NAG of type naBL.
Conversely, given a bounded NAG of type naBL $(L, \leq, \oplus, 0, 1)$, then it can be expanded in a unique way to a dual naBL-algebra $(L, \leq, \oplus, \ominus, 0, 1)$.

NAG's (of type naBL) and naBL-algebras

- Given an naBL-algebra $(L, \leq_n, \odot, \rightarrow, 0_n, 1_n)$, define its **dual** $(L, \leq, \oplus, \ominus, 0, 1)$:
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Sketch of proof

- $(L, \leq)$ is a lattice:
  (CP) $\Rightarrow a = a_1 \oplus c, b = b_1 \oplus c, a_1 \wedge b_1 = 0 \Rightarrow$
  $c = a \wedge b, a_1 \oplus b = b_1 \oplus a = a \vee b$
- $b \ominus a := max\{x : a \oplus x = a \vee b\}$
- α-prelinearity:
  $(L, \leq, \oplus, \ominus, 0, 1)$ is a dual of a residuated lattice:
  (NAG4) gives $a \oplus (b \oplus c) = (a \oplus b) \oplus c_1$ for some $c_1 \in L$
  $c_1 \geq \alpha_b^a(c)$
  $\forall y \in L : y \perp c \Rightarrow y \perp c_1 \Rightarrow y \perp \alpha_b^a(c)$
  this by Botur's result is equivalent to α-prelinearity
- β-prelinearity: analogous

Definition

A structure $\mathscr{L} = (L, \leq, \ominus, 0)$ is called a **quasi-BCK-algebra** (QBCK-algebra, briefly) if

- (QBCK1) $(L, \leq, 0)$ is a poset with a least element 0
- (QBCK2) $\ominus$ is a binary operation on L satisfying for all $a, b, c \in L$:
  (a) $a \ominus (a \ominus b) \leq b$
  (b) $a \geq b \Rightarrow c \ominus a \leq c \ominus b, a \ominus c \geq b \ominus c$
  (c) $a \ominus 0 = 0$
- (QBCK3) $a \leq b$ iff $a \ominus b = 0$.
  $\mathscr{L}$ is called **bounded** whenever $(L, \leq)$ has a top element 1.

- BCK: (b) $(a \ominus b) \ominus (a \ominus c) \leq c \ominus b$
  $\{\rightarrow, 1\}$-subreducts of commutative integral RL

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  $\{\to, 1\}$-subreducts of commutative integral RL

Definition

A bounded QBCK-algebra $\mathscr{L}$ is called of **type naBL** if it fulfils

- (OP) $\forall a, b \in L \exists d \in L \forall c \in L : d \geq c \Leftrightarrow a \geq c \ominus b$
- (SC) (**strong cancellability**):
  $\forall a, b, c \in L, c \leq a, b : a \leq b \Leftrightarrow a \ominus c \leq b \ominus c$
- (CP) (**compatibility property**)
  $\forall a, b \in L : (a \ominus b) \wedge (b \ominus a) = 0$
- (α, β)-property:
  $\forall a, b, c \in L$ let $d_1, d_2, d_3 \in L$ be the greatest elements with
  $b \geq d_1 \ominus c$
  $a \geq d_2 \ominus d_1$
  $a \geq d_3 \ominus b$.
  Then $\forall y \in L : y \perp c \Rightarrow y \perp (d_2 \ominus d_3), y \perp (d_2 \ominus b) \ominus a$.

Theorem

Let $(L, \leq, \oplus, \ominus, 0, 1)$ *be a dual naBL-algebra. Then* $(L, \leq, \ominus, 0, 1)$ *is a bounded QBCK-algebra of type naBL.*
Conversely, given a bounded QBCK-algebra of type naBL $(L, \leq, \ominus, 0, 1)$*, then it can be expanded in a unique way to a dual naBL-algebra* $(L, \leq, \oplus, \ominus, 0, 1)$*.*

Basic algebras

- =lattices with SAI
- BA-assigned algebras $\mathscr{A}(\mathbf{L}) = (L, \oplus, \neg, 0)$

(B1) $x \oplus 0 = x$
(B2) $\neg\neg x = x$
(B3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$
(B4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1.$

- **commutative** basic algebras CBA: $\oplus$ is commutative
- used also in another (independent) context: algebraic model of *BPC* (Alizadeh, Ardeshir) (close to Heyting algebras)

Theorem

Let $(L, \leq, \oplus, \ominus, 0, 1)$ *be a dual naBL-algebra. Then L is a CBA iff*

- (CBA1) *If* a *is the least element* x *with* $x \oplus b = 1$*, then* b *is the least* y *with* $a \oplus y = 1$
- (CBA2) *Let* y *be the least element with* $a \oplus y = 1$,
  z *the least element with* $b \oplus z = 1$,
  x *the least element with* $a \oplus x = a \vee b$.
  Then $z \oplus x \geq y$.