On interpolation in $NEXT(\mathbf{KTB})$

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TOPOLOGY, ALGEBRA AND CATEGORIES IN LOGIC, MARSEILLES

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Brouwerian logic $\mathbf{KTB}$

Axioms CL and

$$K := \Box(p \to q) \to (\Box p \to \Box q)$$
$$T := \Box p \to p$$
$$B := p \to \Box\Diamond p$$

and rules: (MP), (Sub) i (RG).

Extensions of the Brouwer logic $\mathbf{KTB}$ [I.Thomas 1964]

$\mathbf{T_n} = \mathbf{KTB} \oplus (4_n)$, where

$$(4_n) \qquad \Box^n p \to \Box^{n+1} p$$

$$\mathbf{KTB} \subset \ldots \subset \mathbf{T_{n+1}} \subset \mathbf{T_n} \subset \ldots \subset \mathbf{T_2} \subset \mathbf{T_1} = \mathbf{S5}.$$

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Relational semantics - Kripke frames for $\mathbf{T_n}$

$$(tran_n) \quad \forall_{x,y}(\text{if } \; xR^{n+1}y \; \text{ then } \; xR^n y)$$

where the relation of n-step accessibility is defined inductively as follows:

$$xR^0 y \quad \text{iff} \quad x = y$$
$$xR^{n+1} y \quad \text{iff} \quad \exists_z \, (xR^n z \, \wedge \, zRy)$$

In the case of the logic $\mathbf{T_n}$, R is reflexive, symmetric and n-transitive. [Thomas 1964]

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Definitions

- A logic L has the Craig interpolation property (CIP) if for every implication $\alpha \to \beta$ in L, there exists a formula γ (interpolant for $\alpha \to \beta$ in L) such that $\alpha \to \gamma \in L$ and $\gamma \to \beta \in L$ and $Var(\gamma) \subseteq Var(\alpha) \cap Var(\beta)$.
- A logic L is Halldén complete if

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The connection between logics with CIP and Halldén-complete ones

Theorem

If L has only one Post-complete extension and is Halldén-incomplete, then interpolation fails in L. [Schumm, 1986]

G. F. Schumm, *Some failures of interpolation in modal logic*, Notre Dame Journal of Formal Logic, Vol. 27 (1), (1986), 108–110.

Observation

All logics from $NEXT(\mathbf{KTB})$ have only one Post-complete extension, namely the logic $Triv$.

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The logic $\mathbf{KTB}$ *is Halldén complete.[Kripke, 1957]*

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The logic $\mathbf{KTB}$ *has CIP.*

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The logics $\mathbf{T_n}$, $n \geq 1$ *have CIP.*

Proof. A very general method of construction of inseparable tableaux is applicable here (see i.e. Chagrov, Zakharyaschev).

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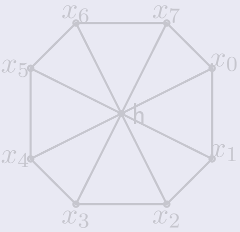
Halldén-completeness and CIP in $NEXT(\mathbf{T_2})$

Logics determined by wheel-frames



We consider infinite families of wheel-frames:

$\{\mathfrak{W}_i : \ i \in A$ and i is prime$\}$, where $A \subseteq \mathbb{N}$,

and logics determined by them:

$$L_A \ := \ L(\{\mathfrak{W}_i : \ i \in A \text{ and } i \text{ is prime}\}) = \\ = \bigcap\{L(\mathfrak{W}_i) : \ i \in A \text{ and } i \text{ is prime}\}.$$

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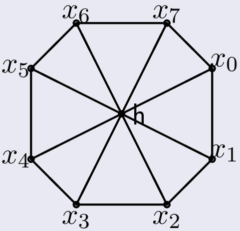

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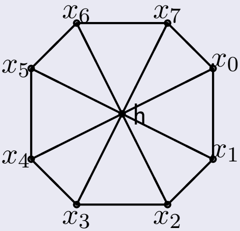

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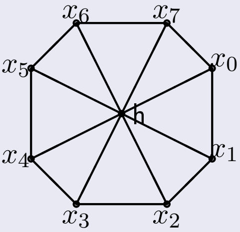

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The family of logics $\{L_A\}_{A\subseteq\mathbb{N}}$ *is uncountably infinite. [Miyazaki, 2005]*

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Nonequivalent formulas in $\mathbf{T_2}$

Let $\alpha := p \wedge \neg\Diamond\Box p$.

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\begin{aligned}
A_1 &:= \neg p \wedge \Box\neg\alpha\,, \\
A_2 &:= \neg p \wedge \neg A_1 \wedge \Diamond A_1\,, \\
A_3 &:= \alpha \wedge \Diamond A_2 \wedge \neg\Diamond A_1\,, \\
&\vdots \\
A_{2n} &:= \neg p \wedge \Diamond A_{2n-1} \wedge \neg\Diamond A_{2n-2} \quad \text{for } n \geq 2\,, \\
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Lemma

The formulas $\{A_i\}$, $i \geq 1$ *are non-equivalent in the logic* $\mathbf{T_2}$. *[Z.K. 2006]*

Proof

Let us take the following model $\mathfrak{M} = \langle W, R, V \rangle$:

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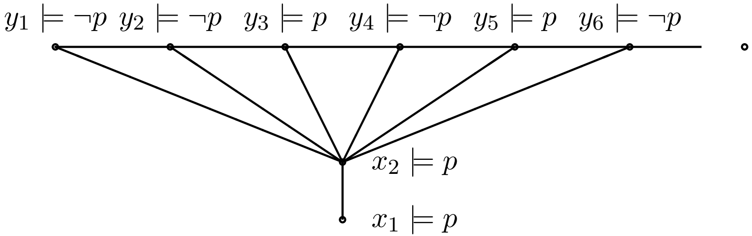
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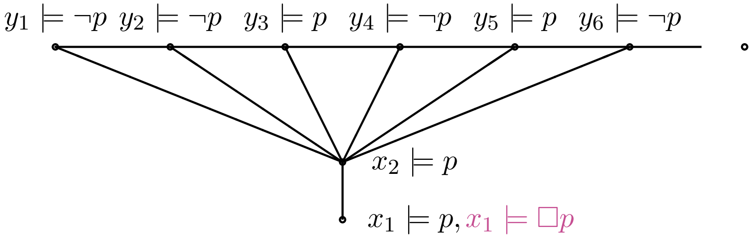
$y_5 \models p$
$y_6 \models \neg p$
$x_2 \models p$
$x_1 \models p$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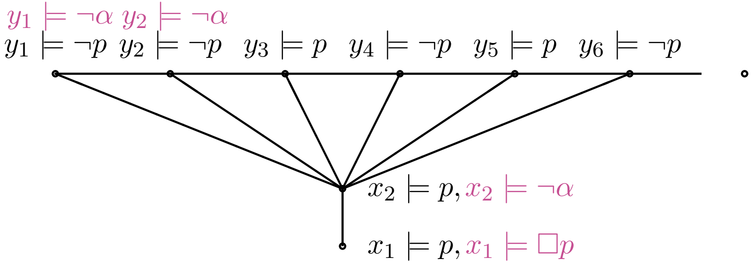

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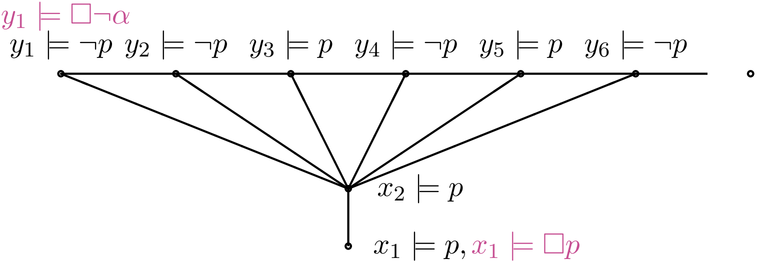

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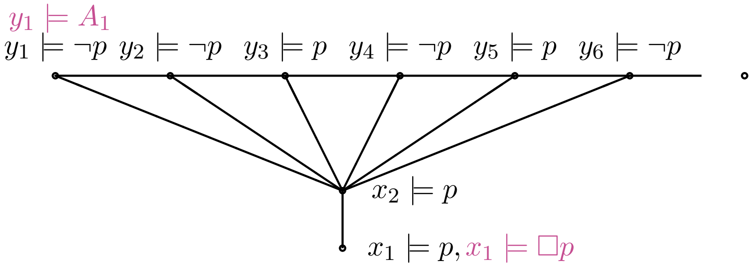
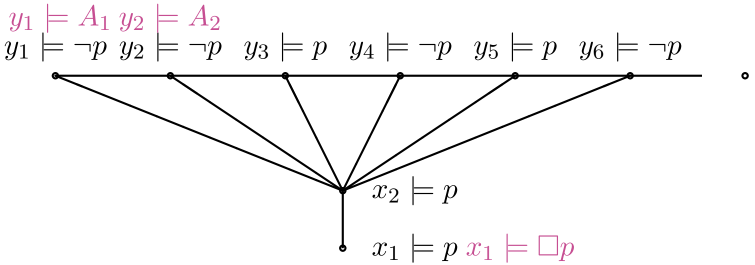
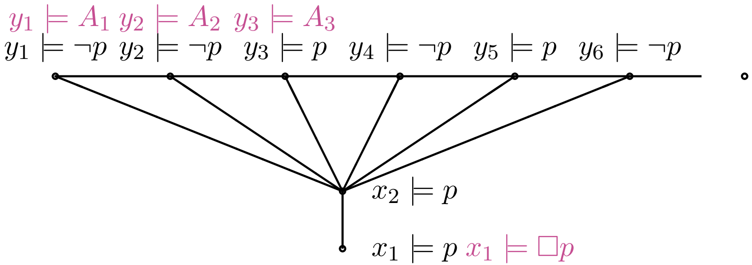


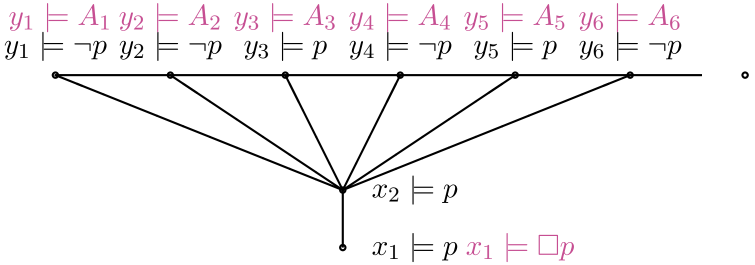
where $A_1 := \neg p \wedge \Box \neg \alpha$.



where $A_2 := \neg p \wedge \neg A_1 \wedge \Diamond A_1$



where $A_3 := \alpha \wedge \Diamond A_2 \wedge \neg \Diamond A_1$



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For any $i \geq 1$ and for any $x \in W$ the following holds:

$$x \models A_i \quad \text{iff} \quad x = y_i$$

Then:

$$y_i \not\models A_i \to A_j \quad \text{and} \quad y_j \not\models A_j \to A_i$$

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Formulas G_k

$G_k := (\Box p \wedge \bigwedge_{i=2}^{k-1} C_i \wedge D_{k-1} \wedge E) \to \Diamond^2 A_k$ and

$$\beta := \neg\Box p \wedge \Diamond\Box p, \quad \gamma := \beta \wedge \Diamond A_1 \wedge \neg\Diamond A_2 \wedge \neg\Diamond A_3,$$

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Lemma

Let $k \geq 5$ *and* k*- odd number.*

$\mathfrak{W}_i \not\models G_k$ *iff* i *is divisible by* $k+2$*.[Z.K. 2007]*

Z. K. *On the existence of a continuum of logics in* $\mathrm{NEXT}(KTB \oplus \Box^2 p \to \Box^3 p)$, BSL, Vol. 36 (1), (2007), 1–7.

Theorem

Each logic L_A (such that card $A \geq 2$) is Halldén incomplete.

Proof. Straightforward. Let $n_1, n_2 \in A$, and n_1, n_2 be prime. Let $G_{n_1-2}(p)$ be the above defined formula and $G_{n_2-2}(q)$ be the appropriate formula whose variable is q.
Then the disjunction $G_{n_1-2}(p) \vee G_{n_2-2}(q) \in L_A$ but none of $G_{n_1-2}(p)$ and $G_{n_2-2}(q)$ belongs to L_A.

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Corollary

There are uncountably many logics in $NEXT(\mathbf{T_2})$ *which are Halldén incomplete and hence - without (CIP).*

Problem

To characterize all logics in $NEXT(\mathbf{KTB})$ (and in $NEXT(\mathbf{T_n})$) having CIP (for example by finding the so-called conservative formulas).

Thank you for your attention.