On interpolation in $\text{NEXT}(\text{KTB})$

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TOPOLOGY, ALGEBRA AND CATEGORIES IN LOGIC, MARSEILLES

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Brouwerian logic $\textbf{KTB}$

Axioms CL and

\[
\begin{align*}
K & := \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) \\
T & := \Box p \rightarrow p \\
B & := p \rightarrow \Box \Diamond p
\end{align*}
\]

and rules: (MP), (Sub) i (RG).
Extensions of the Brouwer logic $\text{KTB}$ [l. Thomas 1964]

$$T_n = \text{KTB} \oplus (4_n), \text{ where}$$

$$(4_n) \quad \Box^n p \rightarrow \Box^{n+1} p$$

$\text{KTB} \subset \ldots \subset T_{n+1} \subset T_n \subset \ldots \subset T_2 \subset T_1 = \text{S5}.$
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Relational semantics - Kripke frames for $T_n$

\[(\text{tran}_n) \quad \forall x, y (\text{if } xR^{n+1}y \text{ then } xR^ny)\]

where the relation of $n$-step accessibility is defined inductively as follows:

\[xR^0y \quad \text{iff} \quad x = y\]

\[xR^{n+1}y \quad \text{iff} \quad \exists z (xR^nz \land zRy)\]

In the case of the logic $T_n$, $R$ is reflexive, symmetric and $n$-transitive. [Thomas 1964]
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**Definitions**

- A logic $L$ has the **Craig interpolation property (CIP)** if for every implication $\alpha \rightarrow \beta$ in $L$, there exists a formula $\gamma$ (interpolant for $\alpha \rightarrow \beta$ in $L$) such that $\alpha \rightarrow \gamma \in L$ and $\gamma \rightarrow \beta \in L$ and $\text{Var}(\gamma) \subseteq \text{Var}(\alpha) \cap \text{Var}(\beta)$.

- A logic $L$ is **Halldén complete** if

  $$\varphi \lor \psi \in L \text{ implies } \varphi \in L \text{ or } \psi \in L$$

  for all $\varphi$ and $\psi$ containing no common variables.
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The connection between logics with CIP and Halldén-complete ones

Theorem

If $L$ has only one Post-complete extension and is Halldén-incomplete, then interpolation fails in $L$. [Schumm, 1986]


Observation

All logics from $\text{NEXT(}K\text{TB)}$ have only one Post-complete extension, namely the logic $\text{Triv}$. 
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Halldén-completeness for \textbf{KTB}

\begin{quote}
\textbf{Theorem}

The logic \textbf{KTB} is Halldén complete. [Kripke, 1957]

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**Theorem**

The logic \textbf{KTB} is Halldén complete.\cite{Kripke, 1957}

The logic KTB has CIP.

The logics $T_n$, $n \geq 1$ have CIP.

Proof. A very general method of construction of inseparable tableaux is applicable here (see i.e. Chagrov, Zakharyaschev).

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CIP for KTB and $T_n$

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**Corollary**

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The logics \( \textbf{T}_n, n \geq 1 \) are Halldén-complete.
Halldén-completeness and CIP in \( \text{NEXT}(T_2) \)
We consider infinite families of wheel-frames:
\( \{ W_i : i \in A \text{ and } i \text{ is prime} \} \), where \( A \subseteq \mathbb{N} \), and logics determined by them:

\[
L_A := L(\{ W_i : i \in A \text{ and } i \text{ is prime} \}) = \\
= \bigcap \{ L(W_i) : i \in A \text{ and } i \text{ is prime} \}.
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The family of logics $\{L_A\}_{ACN}$ is uncountably infinite. [Miyazaki, 2005]

The family of logics $\{L_A\}_{A \subseteq N}$ is uncountably infinite. [Miyazaki, 2005]

Nonequivalent formulas in \( T_2 \)

Let \( \alpha := p \land \neg \Box \Diamond p. \)

\[
A_1 := \neg p \land \Box \neg \alpha,
\]
\[
A_2 := \neg p \land \neg A_1 \land \Diamond A_1,
\]
\[
A_3 := \alpha \land \Diamond A_2 \land \neg \Diamond A_1,
\]
\[
\vdots
\]
\[
A_{2n} := \neg p \land \Diamond A_{2n-1} \land \neg \Diamond A_{2n-2} \text{ for } n \geq 2,
\]
\[
A_{2n+1} := \alpha \land \Diamond A_{2n} \land \neg \Diamond A_{2n-1} \text{ for } n \geq 2
\]
\[
\vdots
\]
Nonequivalent formulas in $T_2$

Let $\alpha := p \land \neg \Box \neg p$.

1. $A_1 := \neg p \land \Box \neg \alpha$,
2. $A_2 := \neg p \land \neg A_1 \land \Diamond A_1$,
3. $A_3 := \alpha \land \Diamond A_2 \land \neg \Diamond A_1$,
   ... 
4. $A_{2n} := \neg p \land \Diamond A_{2n-1} \land \neg \Diamond A_{2n-2}$ for $n \geq 2$,
5. $A_{2n+1} := \alpha \land \Diamond A_{2n} \land \neg \Diamond A_{2n-1}$ for $n \geq 2$
   ...
Lemma

The formulas \( \{A_i\}, \ i \geq 1 \) are non-equivalent in the logic \( T_2 \). [Z.K. 2006]

Proof

Let us take the following model \( M = \langle W, R, V \rangle \):
The formulas \( \{A_i\}, i \geq 1 \) are non-equivalent in the logic \( T_2 \). [Z.K. 2006]

Proof

Let us take the following model \( \mathcal{M} = \langle W, R, V \rangle \):
$y_1 \models \neg p \quad y_2 \models \neg p \quad y_3 \models p \quad y_4 \models \neg p \quad y_5 \models p \quad y_6 \models \neg p$

$x_1 \models p$

$x_2 \models p$
\[ y_1 \models \neg p \quad y_2 \models \neg p \quad y_3 \models p \quad y_4 \models \neg p \quad y_5 \models p \quad y_6 \models \neg p \]

\[ x_2 \models p \]

\[ x_1 \models p, \quad x_1 \models \Box p \]
where $\alpha := p \land \neg \lozenge \Box p$. 

\[ y_1 \models \neg \alpha \quad y_2 \models \neg \alpha \]
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\[ y_1 \models \square \neg \alpha \]

\[ y_1 \models \neg p \quad y_2 \models \neg p \quad y_3 \models p \quad y_4 \models \neg p \quad y_5 \models p \quad y_6 \models \neg p \]

\[ x_2 \models p \]

\[ x_1 \models p, \quad x_1 \models \square p \]

where \( \alpha := p \land \neg \Diamond \square p \).
where $A_1 := \neg p \land \Box \neg \alpha$. 

$y_1 \models A_1$

$y_1 \models \neg p$  $y_2 \models \neg p$  $y_3 \models p$  $y_4 \models \neg p$  $y_5 \models p$  $y_6 \models \neg p$

$x_2 \models p$

$x_1 \models p$, $x_1 \models \Box p$
$y_1 \models A_1 \ y_2 \models A_2$

$y_1 \models \neg p \ y_2 \models \neg p \ y_3 \models p \ y_4 \models \neg p \ y_5 \models p \ y_6 \models \neg p$

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$x_1 \models p \ x_1 \models \Box p$

where $A_2 := \neg p \land \neg A_1 \land \Diamond A_1$
\( y_1 \models A_1 \quad y_2 \models A_2 \quad y_3 \models A_3 \)
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where \( A_3 := \alpha \land \Diamond A_2 \land \neg \Diamond A_1 \)
\[
\begin{align*}
 y_1 & \models A_1, \quad y_2 \models A_2, \quad y_3 \models A_3, \quad y_4 \models A_4, \quad y_5 \models A_5, \quad y_6 \models A_6, \\
 y_1 & \models \neg p, \quad y_2 \models \neg p, \quad y_3 \models p, \quad y_4 \models \neg p, \quad y_5 \models p, \quad y_6 \models \neg p.
\end{align*}
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where

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\begin{align*}
 A_{2n} & := \neg p \land \Diamond A_{2n-1} \land \neg \Diamond A_{2n-2} \quad \text{for } n \geq 2, \\
 A_{2n+1} & := \alpha \land \Diamond A_{2n} \land \neg \Diamond A_{2n-1} \quad \text{for } n \geq 2.
\end{align*}
\]
For any $i \geq 1$ and for any $x \in W$ the following holds:

$$\begin{align*}
    x \models A_i & \iff x = y_i \\
\end{align*}$$

Then:

$$\begin{align*}
    y_i \not\models A_i \to A_j & \text{ and } y_j \not\models A_j \to A_i \\
\end{align*}$$
For any $i \geq 1$ and for any $x \in W$ the following holds:

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Formulas $G_k$

\[ G_k := (\Box p \land \bigwedge_{i=2}^{k-1} C_i \land D_{k-1} \land E) \rightarrow \Diamond^2 A_k \] and

\[ \beta := \neg \Box p \land \Diamond \Box p, \quad \gamma := \beta \land \Diamond A_1 \land \neg \Diamond A_2 \land \neg \Diamond A_3, \]

\[ C_k := \Box^2 [A_{k-1} \rightarrow \Diamond A_k] \quad \text{for } k \geq 2, \]

\[ D_k := \Box^2 [(A_k \land \neg \Diamond A_{k+1}) \rightarrow \Diamond \varepsilon], \]

\[ E := \Box^2 (\Box p \rightarrow \Diamond \gamma), \quad \varepsilon := \beta \land \neg \Diamond A_1 \land \neg \Diamond A_2. \]
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C_k &:= \Box^2[A_{k-1} \rightarrow \Diamond A_k] \text{ for } k \geq 2, \\
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E &:= \Box^2(\Box p \rightarrow \Diamond \gamma), \\
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\end{align*}
\]
Lemma

Let $k \geq 5$ and $k$- odd number.

$\mathcal{M}_i \not⊧ G_k$ iff $i$ is divisible by $k + 2$. [Z.K. 2007]

Z. K. On the existence of a continuum of logics in

$\text{NEXT}(KTB \oplus \Box^2p \rightarrow \Box^3p)$, BSL, Vol. 36 (1), (2007), 1–7.
Each logic $L_A$ (such that $\text{card } A \geq 2$) is Halldén incomplete.

Proof. Straightforward. Let $n_1, n_2 \in A$, and $n_1, n_2$ be prime. Let $G_{n_1-2}(p)$ be the above defined formula and $G_{n_2-2}(q)$ be the appropriate formula whose variable is $q$. Then the disjunction $G_{n_1-2}(p) \lor G_{n_2-2}(q) \in L_A$ but none of $G_{n_1-2}(p)$ and $G_{n_2-2}(q)$ belongs to $L_A$. 
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Corollary

There are uncountably many logics in \( \text{NEXT}(T_2) \) which are Halldén incomplete and hence - without (CIP).
Problem

To characterize all logics in $\text{NEXT}(\text{KTB})$ (and in $\text{NEXT}(\text{T}_n)$) having CIP (for example by finding the so-called conservative formulas).
Thank you for your attention.