On Normal-Valued Basic Pseudo Hoops

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The Romanian algebraic school during the last decade contributed a lot to noncommutative generalizations of many-valued reasoning which generalizes MV-algebras by C.C. Chang. They introduced pseudo MV-algebras (independently introduced also by J.Rachunek as generalized MV-algebras), pseudo BL-algebras, pseudo hoops. We recall that pseudo BL-algebras are also a noncommutative generalization of P. Hájek's BL-algebras: a variety that is an algebraic counterpart of fuzzy logic. Therefore, a pseudo BL-algebras is an algebraic presentation of a non-commutative generalization of fuzzy logic. These structures are studied also in the area of quantum structures.

However, as it was recently recognized, many of these notions have a very close connections with notions introduced already by B. Bosbach in his pioneering papers on various classes of semigroups: among others he introduced complementary semigroups (today known as pseudo-hoops). A deep investigation of these structures can be found in his papers. Nowadays, all these structures can be also studied under one common roof, as residuated lattices.

The main aim is to continue in the study of pseudo hoops, focusing on normal-valued ones. We present an equational basis of normal-valued basic pseudo hoops. In addition, we show that every pseudo hoop satisfies the Riesz Decomposition Property (RDP) and we present also a Holland's type representation of basic pseudo hoops.

We recall that a *pseudo hoop* is an algebra $(M; \odot, \rightarrow, \rightsquigarrow, 1)$ of type $\langle 2, 2, 2, 0 \rangle$ such that, for all $x, y, z \in M$,

(i) $x \odot 1 = x = 1 \odot x$;

(ii) $x \rightarrow x = 1 = x \rightsquigarrow x$;

(iii) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$;

(iv) $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$;

(v) $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$.

If $\odot$ is commutative (equivalently $\rightarrow = \rightsquigarrow$), M is said to be a *hoop*. If we set $x \leq y$ iff $x \rightarrow y = 1$ (this is equivalent to $x \rightsquigarrow y = 1$), then $\leq$ is a partial order such that $x \wedge y = (x \rightarrow y) \odot x$ and M is a $\wedge$-semilattice.

We say that a pseudo hoop M

(i) is *bounded* if there is a least element 0, otherwise, M is *unbounded*,

(ii) satisfies *prelinearity* if, given $x, y \in M$, $(x \to y) \vee (y \to x)$ and $(x \rightsquigarrow y) \vee (y \rightsquigarrow x)$ are defined in M and they are equal 1,

(iii) is *cancellative* if $x \odot y = x \odot z$ and $s \odot x = t \odot x$ imply $y = z$ and $s = t$,

(iv) is a *pseudo BL-algebra* if M is a bounded lattice satisfying prelinearity.

Many examples of pseudo hoops can be made from ℓ-groups. Now let G be an ℓ-group (written multiplicatively and with a neutral element e). On the negative cone $G^- = \{g \in G : \; g \leq e\}$ we define: $x \odot y := xy$, $x \to y := (yx^{-1}) \wedge e$, $x \rightsquigarrow y := (x^{-1}y) \wedge e$, for $x, y \in G^-$. Then $(G^-; \odot, \to, \rightsquigarrow, e)$ is an unbounded (whenever $G \neq \{e\}$) cancellative pseudo hoop. Conversely, every cancellative pseudo hoop is isomorphic to some $(G^-; \odot, \to, \rightsquigarrow, e)$ (G. Georgescu, L. Leuștean, V. Preoteasa).

A pseudo hoop M is said to be *basic* if, for all $x, y, z \in M$,

(B1) $(x \to y) \to z \leq ((y \to x) \to z) \to z$;

(B2) $(x \rightsquigarrow y) \rightsquigarrow z \leq ((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z$.

Every basic pseudo hoop is a distributive lattice with prelinearity (G. Georgescu, L. Leuştean, V. Preoteasa).

Theorem

If M is a pseudo hoop with prelinearity, then M is basic, M is a lattice, and

$$((x \rightsquigarrow y) \to y) \wedge ((y \rightsquigarrow x) \to x) = x \vee y =$$

$$((x \to y) \rightsquigarrow y) \wedge ((y \to x) \rightsquigarrow x) \tag{3.1}$$

for all $x, y \in M$.

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Theorem

The class of bounded pseudo hoops with prelinearity is termwise equivalent to the variety of pseudo BL-algebras.
Moreover, basic pseudo hoops are just subreducts of pseudo BL-algebras.

A subset F of a pseudo hoop is said to be a *filter* if

(i) $x, y \in F$ implies $x \odot y \in F$,

(ii) $x \leq y$ and $x \in F$ imply $y \in F$.

We denote by $\mathcal{F}(M)$ the set of all filters of M. A subset F is a filter iff

(i) $1 \in F$,

(ii) $x, x \to y \in F$ implies $y \in F$ (or equivalently $x, x \rightsquigarrow y \in F$ implies $y \in F$).

Thus, F is a *deductive system*.
A filter F is normal if $x \to y \in F$ iff $x \rightsquigarrow y \in F$. This is equivalent $a \odot F = F \odot a$ for any $a \in M$; We define $x\theta_F y$ iff $x \to y \in F$ and $y \to x \in F$. The relation θ_F is a lattice congruence and, moreover, if F is normal, then θ_F is a congruence on M.

We are saying that a pseudo hoop M satisfies the *Riesz decomposition property* ((RDP) for short) if $a \geq b \odot c$ implies that there are two elements $b_1 \geq b$ and $c_1 \geq c$ such that $a = b_1 \odot c_1$.

Theorem

Every pseudo hoop M satisfies (RDP).

Theorem

The system of all filters, $\mathcal{F}(M)$, of a pseudo hoop M is a distributive lattice under the set-theoretical inclusion. In addition, $F \cap \bigvee_i F_i = \bigvee_i (F \cap F_i)$.

Let F be a filter of a basic pseudo hoop M. Then all statements (i)–(viii) are equivalent.

(i) F is prime.

(ii) If $f \vee g = 1$, then $f \in F$ or $g \in F$.

(iii) For all $f, g \in M$, $f \to g \in F$ or $g \to f \in F$.

(iii') For all $f, g \in M$, $f \rightsquigarrow g \in F$ or $g \rightsquigarrow f \in F$.

(iv) If $f \vee g \in F$, then $f \in F$ or $g \in F$.

(v) If $f, g \in M$, then there is $c \in F$ such that $c \odot f \leq g$ or $c \odot g \leq f$.

(vi) If F_1 and F_2 are two filters of M containing F, then $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$.

(vii) If F_1 and F_2 are two filters of M such that $F \subsetneq F_1$ and $F \subsetneq F_2$, then $F \subsetneq F_1 \cap F_2$.

(viii) If $f, g \notin F$, then $f \vee g \notin F$.

Lemma

Let M be a basic pseudo hoop. If A is a lattice ideal of M and F is a filter of M such that $F \cap A = \emptyset$, then there is a prime filter P of M containing F and disjoint with A.

(1) The *value* of an element $g \in M \setminus \{1\}$ is any filter V of M that is maximal with respect to the property $g \notin V$. Due to previous Lemma, a value V exists and it is prime. Let $\mathrm{Val}(g)$ be the set of all values of $g < 1$. The filter V^* generated by a value V of g and by the element g is said to be the *cover* of V.

(2) We recall that a filter F is *finitely meet-irreducible* if, for each two filters F_1, F_2 such that $F \subsetneq F_1$ and $F \subsetneq F_2$, we have $F \subsetneq F_1 \cap F_2$. The finite meet-irreducibility is a sufficient and necessary condition for a filter F to be prime.

We say that a basic pseudo-hoop M is *normal-valued* if every value V of M is normal in its cover V^*.
According to Wolfenstein, an ℓ-group G is normal-valued iff every $a, b \in G^-$ satisfy $b^2a^2 \leq ab$, or in our language

$$b^2 \odot a^2 \leq a \odot b. \tag{6.1}$$

Hence, every cancellative pseudo hoop M is normal-valued iff (6.1) holds for all $a, b \in M$. Moreover, every representable pseudo hoop satisfies (6.1).
Similarly, a pseudo MV-algebra is normal-valued iff (6.1) holds.

Theorem

Any pseudohoops with no non-trivial filters is commutative.

Lemma

Let M be a basic pseudo hoop and $a, b, x \in M$ be such that $V(a \odot b) \leq Vx$ for any $V \in \mathrm{Val}(x)$. Then $a^2 \odot b^2 \leq x$.

Theorem

Let M be a normal-valued basic pseudo hoop, then the following inequalities hold.

(i) $x^2 \odot y^2 \leq y \odot x$.

(ii) $((x \to y)^n \rightsquigarrow y)^2 \leq (x \rightsquigarrow y)^{2n} \to y$ *for any* $n \in \mathbb{N}$.

(iii) $((x \rightsquigarrow y)^n \to y)^2 \leq (x \to y)^{2n} \rightsquigarrow y$ *for any* $n \in \mathbb{N}$.

Moreover, if a basic pseudo hoop satisfies inequalities (i)–(iii), *then it is normal-valued.*

Holland's Representation

Finally, we will visualize basic pseudo hoops in a Holland's Representation Theorem type which says that every ℓ-group can be embedded into the system of automorphisms of a linearly ordered set. This was generalized in for some ℓ-monoids. We show that this result can be extended also for basic pseudo hoops.

Theorem

Let M be a basic pseudo hoop. Then there is a linearly ordered set Ω and a subsystem $\mathrm{M}(M)$ of $\mathrm{Mon}(\Omega)$ such that $\mathrm{M}(M)$ is a sublattice of $\mathrm{Mon}(\Omega)$ containing e and each element of it is residuated. Moreover, $\mathrm{M}(M)$ can be converted into a basic pseudo hoop and is isomorphic to M with the $\odot$-operation corresponding to composition of functions.

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- P. Aglianò and F. Montagna, *Varieties of BL-algebras I: general properties,* J. Pure Appl. Algebra **181** (2003), 105–129.
- M. Anderson, C.C. Edwards, *A representation theorem for distributive ℓ-monoids,* Canad. Math. Bull. **27** (1984), 238–240.
- K. Blount, C. Tsinakis, *The structure of residuated lattices,* Inter. J. Algebra Comput. **13** (2003), 437–461.
- B. Bosbach, *Komplementäre Halbgruppen. Axiomatik und Arithmetik,* Fund. Math. **64** (1966), 257–287.
- B. Bosbach, *Komplementäre Halbgruppen. Kongruenzen and Quotienten,* Fund. Math. **69** (1970), 1–14.
- B. Bosbach, *Residuation groupoids - again* Results in Math. **53** (2009), 27–51.
- B. Bosbach, *Divisibility groupoids - again,* Results in Math. **57** (2010), 257–285.

- C.C. Chang, *Algebraic analysis of many valued logics,* Trans. Amer. Math. Soc. **88** (1958), 467–490.
- M.R. Darnel, *"Theory of Lattice-Ordered Groups"*, Marcel Dekker, Inc., New York, 1995.
- A. Di Nola, G. Georgescu, and A. Iorgulescu, *Pseudo-BL algebras I,* Multiple Val. Logic **8** (2002), 673–714.
- A. Di Nola, G. Georgescu, and A. Iorgulescu, *Pseudo-BL algebras II,* Multiple Val. Logic **8** (2002), 715–750.
- A. Dvurečenskij, *States on pseudo MV-algebras,* Studia Logica **68** (2001), 301–327.
- A. Dvurečenskij, *Pseudo MV-algebras are intervals in ℓ-groups,* J. Austral. Math. Soc. **70** (2002), 427–445.
- A. Dvurečenskij, *Holland's theorem for pseudo-effect algebras,* Czechoslovak Math. J. **56** (2006), 47–59.
- A. Dvurečenskij, *Every linear pseudo BL-algebra admits a state,* Soft Computing **11** (2007), 495–501.

A. Dvurečenskij, *Aglianò–Montagna type decomposition of linear pseudo hoops and its applications,* J. Pure Appl. Algebra **211** (2007), 851–861.

A. Dvurečenskij, R. Giuntini, and T. Kowalski, *On the structure of pseudo BL-algebras and pseudo hoops in quantum logics,* Found. Phys. **40** (2010), 1519–1542. DOI:10.1007/s10701-009-9342-5

N. Galatos, C. Tsinakis, *Generalized MV-algebras.* J. Algebra **283** (2005), 254–291.

G. Georgescu, A. Iorgulescu, *Pseudo-MV algebras,* Multi-Valued Logic **6** (2001), 95–135.

G. Georgescu, L. Leuștean, V. Preoteasa, *Pseudo-hoops,* J. Mult.-Val. Log. Soft Comput. **11** (2005), 153–184.

P. Hájek, *"Metamathematics of Fuzzy Logic"*, Trends in Logic - Studia Logica Library, Volume 4, Kluwer Academic Publishers, Dordrecht, 1998.

W.C. Holland, *The lattice-ordered group of automorphisms of an ordered set,* Michigan Math. J. **10** (1963), 399–408.

P. Jipsen and F. Montagna, *On the structure of generalized BL-algebras,* Algebra Universalis **55** (2006), 226–237.

R. Mesiar, O. Nánásiová, Z. Riečanová, J. Paseka, *Special issue - Quantum structures: Theory and applications,* Inform. Sci. **179** (2009), 475–477.

J. Rachůnek, *A non-commutative generalization of MV-algebras,* Czechoslovak Math. J. **52** (2002), 255–273.