Algorithmic correspondence and canonicity for non-distributive logics

Willem Conradie[1] Alessandra Palmigiano[2]

[1]University of Johannesburg

[2]ILLC, University of Amsterdam

TACL2011

(Classical) Modal Logic

Syntax

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Diamond\varphi$$

(Classical) Modal Logic

Syntax

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Diamond\varphi$$

Semantics

Relational	Algebraic
Kripke frames	BAO's
$\mathfrak{F} = (W, R)$	$\mathbb{A} = (A, \wedge, \vee, -, 1, 0, \Diamond)$
Valuations: $V : \mathrm{Var} \to \wp(W)$	Assignments: $v : \mathrm{Var} \to A$

Correspondence: An example

On models:

$$(\mathfrak{F}, V) \models p \rightarrow \Diamond p$$

Correspondence: An example

On models:

$$
\begin{array}{ll}
 & (\mathfrak{F}, V) \models p \to \Diamond p \\
\textit{iff} & (\mathfrak{F}, V) \models \forall x (P(x) \to \exists y (Rxy \wedge P(y)))
\end{array}
$$

Correspondence: An example

On models:

$$
\begin{array}{ll}
 & (\mathfrak{F}, V) \models p \to \Diamond p \\
\textit{iff} & (\mathfrak{F}, V) \models \forall x(P(x) \to \exists y(Rxy \wedge P(y)))
\end{array}
$$

On frames:

$$\mathfrak{F} \models p \to \Diamond p$$

Correspondence: An example

On models:

$$(\mathfrak{F}, V) \models p \to \Diamond p$$

$$\text{iff} \quad (\mathfrak{F}, V) \models \forall x(P(x) \to \exists y(Rxy \wedge P(y)))$$

On frames:

$$\mathfrak{F} \models p \to \Diamond p$$

$$\text{iff} \quad \mathfrak{F} \models \forall P \forall x(P(x) \to \exists y(Rxy \wedge P(y)))$$

Correspondence: An example

On models:

$$(\mathfrak{F}, V) \models p \to \Diamond p$$

$$\textit{iff} \quad (\mathfrak{F}, V) \models \forall x(P(x) \to \exists y(Rxy \wedge P(y)))$$

On frames:

$$\mathfrak{F} \models p \to \Diamond p$$

$$\textit{iff} \quad \mathfrak{F} \models \forall P \forall x(P(x) \to \exists y(Rxy \wedge P(y)))$$

$$\textit{iff} \quad \mathfrak{F} \models \forall x Rxx$$

Correspondence theory

- Given a modal formula φ, does it always have a first order correspondent?

Correspondence theory

- Given a modal formula φ, does it always have a first order correspondent? NO.

Correspondence theory

- Given a modal formula φ, does it always have a first order correspondent? NO.
- **Central question:** Which modal fmls have first-order frame correspondents? [Sahhlqvist, van Benthem, 1970's]

Correspondence theory

- Given a modal formula φ, does it always have a first order correspondent? NO.
- **Central question:** Which modal fmls have first-order frame correspondents? [Sahhlqvist, van Benthem, 1970's]
- Syntactic classes: Sahlqvist formulas [Sahlqvist], inductive formulas [Goranko, Vakarelov], Complex formulas [Vakarelov]

Correspondence theory

- Given a modal formula φ, does it always have a first order correspondent? NO.
- **Central question:** Which modal fmls have first-order frame correspondents? [Sahhlqvist, van Benthem, 1970's]
- Syntactic classes: Sahlqvist formulas [Sahlqvist], inductive formulas [Goranko, Vakarelov], Complex formulas [Vakarelov]
- Algorithms: SCAN [Gabbay, Olbach], DLS [Szalas], SQEMA [Conradie, Goranko, Vakarelov]

Correspondence theory

- Given a modal formula φ, does it always have a first order correspondent? NO.
- **Central question:** Which modal fmls have first-order frame correspondents? [Sahhlqvist, van Benthem, 1970's]
- Syntactic classes: Sahlqvist formulas [Sahlqvist], inductive formulas [Goranko, Vakarelov], Complex formulas [Vakarelov]
- Algorithms: SCAN [Gabbay, Olbach], DLS [Szalas], SQEMA [Conradie, Goranko, Vakarelov]
- Strong relationship between correspondence and completeness / canonicity.

FAQ:

- How can I prove that a formula does not correspond?

FAQ:

- How can I prove that a formula does not correspond? With model-theoretic techniques (failure of Löwenheim-Skolem, compactness, etc.)

FAQ:

- How can I prove that a formula does <u>not</u> correspond? With model-theoretic techniques (failure of Löwenheim-Skolem, compactness, etc.)
- Is there a characterization of all the formulas that have a first order correspondent?

FAQ:

- How can I prove that a formula does not correspond? With model-theoretic techniques (failure of Löwenheim-Skolem, compactness, etc.)
- Is there a characterization of all the formulas that have a first order correspondent? No, and this class is an undecidable. [Chagrova]

Generalizing: Lattice based logics

Relational

RS Frames [Gehrke]

Algebraic

Lattices with operators

e.g. $\mathbb{L} = (L, \wedge, \vee, \circ, \star, \Diamond, \Box, \lhd, \rhd, \bot, \top)$

Generalizing: Lattice based logics

Relational
RS Frames [Gehrke]

Algebraic
Lattices with operators
e.g. $\mathbb{L} = (L, \wedge, \vee, \circ, \star, \Diamond, \Box, \lhd, \rhd, \bot, \top)$

$(L, \wedge, \vee, \bot, \top)$ is perfect if it is

1. complete,
2. completely join generated by its completely join irreducible elements J^{∞}, and
3. completely meet generated by its completely meet irreducible elements the set M^{∞}.

Reflexivity, again

$$\forall p[p \leq \Diamond p]$$

Reflexivity, again

$$\forall p[p \leq \Diamond p]$$

$$\forall \mathbf{j} \forall \mathbf{m} \forall p \left[\left\{ \ \mathbf{j} \leq p, \quad \Diamond p \leq \mathbf{m} \ \right\} \Rightarrow \mathbf{j} \leq \mathbf{m} \right]$$

Reflexivity, again

$$\forall p[p \leq \Diamond p]$$

$$\forall \mathbf{j} \forall \mathbf{m} \forall p \left[\left\{ \ \mathbf{j} \leq p, \quad \Diamond p \leq \mathbf{m} \ \right\} \Rightarrow \mathbf{j} \leq \mathbf{m} \right]$$

$$\forall \mathbf{j} \forall \mathbf{m} \forall p \left[\left\{ \ \Diamond \mathbf{j} \leq \mathbf{m} \ \right\} \Rightarrow \mathbf{j} \leq \mathbf{m} \right]$$

Reflexivity, again

$$\forall p[p \leq \Diamond p]$$

$$\forall \mathbf{j} \forall \mathbf{m} \forall p \left[\left\{ \ \mathbf{j} \leq p, \quad \Diamond p \leq \mathbf{m} \ \right\} \Rightarrow \mathbf{j} \leq \mathbf{m} \right]$$

$$\forall \mathbf{j} \forall \mathbf{m} \forall p \left[\left\{ \ \Diamond \mathbf{j} \leq \mathbf{m} \ \right\} \Rightarrow \mathbf{j} \leq \mathbf{m} \right]$$

$$\mathbf{j} \leq \Diamond \mathbf{j}$$

Concretely (on Kripke frames / BAO's):
$\{x\} \subseteq \{y \in W \mid Rxy\}$, i.e., Rxx.

Ackermann's Lemma

- $\mathbb{L}$ a perfect lattice with operators.
- α, β and γ terms such that
  - $p \notin \mathsf{VAR}(\alpha)$,
  - $\beta(p)$ positive in p, and
  - $\gamma(p)$ negative in p

TFAE:

Ackermann's Lemma

- $\mathbb{L}$ a perfect lattice with operators.
- α, β and γ terms such that
  - $p \notin \mathsf{VAR}(\alpha)$,
  - $\beta(p)$ positive in p, and
  - $\gamma(p)$ negative in p

TFAE:

1. $\mathbb{L}, v \models \beta(\alpha) \leq \gamma(\alpha)$
2. there exists some $v' \sim_p v$ such that $\mathbb{L}, v' \models \alpha \leq p$ and $\mathbb{L}, v' \models \beta(p) \leq \gamma(p)$.

Adjunction

$\mathbb{A}$ and $\mathbb{B}$ complete lattices.

$f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{B} \to \mathbb{A}$.

Adjunction

$\mathbb{A}$ and $\mathbb{B}$ complete lattices.

$f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{B} \to \mathbb{A}$.

$f \dashv g$ if $f(a) \leq b$ iff $a \leq g(b)$

iff $f(\bigvee S) = \bigvee_{s \in S} f(s)$ and $g(\bigwedge S) = \bigwedge_{s \in S} g(s)$

Adjunction

$\mathbb{A}$ and $\mathbb{B}$ complete lattices.

$f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{B} \to \mathbb{A}$.

$$f \dashv g \quad \text{if} \quad f(a) \leq b \text{ iff } a \leq g(b)$$

$$\text{iff} \quad f(\bigvee S) = \bigvee_{s \in S} f(s) \text{ and } g(\bigwedge S) = \bigwedge_{s \in S} g(s)$$

Example

$\Diamond^{-1} \dashv \Box$ and $\Diamond \dashv \Box^{-1}$

Confluence: $\Diamond\Box p \rightarrow \Box\Diamond p$

$$\forall p[\Diamond\Box p \leq \Box\Diamond p]$$

Confluence: $\Diamond\Box p \rightarrow \Box\Diamond p$

$$\forall p[\Diamond\Box p \leq \Box\Diamond p]$$

$$\forall p[\Diamond \bigvee\{j \in J^{\infty} \mid j \leq \Box p\} \leq \bigwedge\{m \in M^{\infty} \mid \Box\Diamond p \leq m\}]$$

Confluence: $\Diamond\Box p \rightarrow \Box\Diamond p$

$$\forall p[\Diamond\Box p \leq \Box\Diamond p]$$

$$\forall p[\Diamond \bigvee\{j \in J^{\infty} \mid j \leq \Box p\} \leq \bigwedge\{m \in M^{\infty} \mid \Box\Diamond p \leq m\}]$$

$$\forall p[\bigvee\{\Diamond j \mid j \in J^{\infty} \ \&\ j \leq \Box p\} \leq \bigwedge\{m \in M^{\infty} \mid \Box\Diamond p \leq m\}]$$

Confluence: $\Diamond\Box p \rightarrow \Box\Diamond p$

$$\forall p[\Diamond\Box p \leq \Box\Diamond p]$$

$$\forall p[\Diamond \bigvee \{j \in J^{\infty} \mid j \leq \Box p\} \leq \bigwedge \{m \in M^{\infty} \mid \Box\Diamond p \leq m\}]$$

$$\forall p[\bigvee \{\Diamond j \mid j \in J^{\infty} \;\&\; j \leq \Box p\} \leq \bigwedge \{m \in M^{\infty} \mid \Box\Diamond p \leq m\}]$$

$$\forall p \forall j \forall m[(j \leq \Box p \;\&\; \Box\Diamond p \leq m) \Rightarrow \Diamond j \leq m]$$

Confluence: $\Diamond\Box p \to \Box\Diamond p$

$$\forall p[\Diamond\Box p \leq \Box\Diamond p]$$

$$\forall p[\Diamond \bigvee\{j \in J^\infty \mid j \leq \Box p\} \leq \bigwedge\{m \in M^\infty \mid \Box\Diamond p \leq m\}]$$

$$\forall p[\bigvee\{\Diamond j \mid j \in J^\infty \ \&\ j \leq \Box p\} \leq \bigwedge\{m \in M^\infty \mid \Box\Diamond p \leq m\}]$$

$$\forall p \forall j \forall m[(j \leq \Box p \ \&\ \Box\Diamond p \leq m) \Rightarrow \Diamond j \leq m]$$

$$\forall p \forall j \forall m[(\blacklozenge j \leq p \ \&\ \Box\Diamond p \leq m) \Rightarrow \Diamond j \leq m]$$

Confluence: $\Diamond\Box p \rightarrow \Box\Diamond p$

$$\forall p[\Diamond\Box p \leq \Box\Diamond p]$$

$$\forall p[\Diamond \bigvee\{j \in J^\infty \mid j \leq \Box p\} \leq \bigwedge\{m \in M^\infty \mid \Box\Diamond p \leq m\}]$$

$$\forall p[\bigvee\{\Diamond j \mid j \in J^\infty \ \&\ j \leq \Box p\} \leq \bigwedge\{m \in M^\infty \mid \Box\Diamond p \leq m\}]$$

$$\forall p \forall j \forall m[(j \leq \Box p \ \&\ \Box\Diamond p \leq m) \Rightarrow \Diamond j \leq m]$$

$$\forall p \forall j \forall m[(\blacklozenge j \leq p \ \&\ \Box\Diamond p \leq m) \Rightarrow \Diamond j \leq m]$$

$$\forall j \forall m[(\Box\Diamond\blacklozenge j \leq m) \Rightarrow \Diamond j \leq m]$$

Confluence: $\Diamond\Box p \to \Box\Diamond p$

$$\forall p[\Diamond\Box p \leq \Box\Diamond p]$$

$$\forall p[\Diamond \bigvee\{j \in J^\infty \mid j \leq \Box p\} \leq \bigwedge\{m \in M^\infty \mid \Box\Diamond p \leq m\}]$$

$$\forall p[\bigvee\{\Diamond j \mid j \in J^\infty \;\&\; j \leq \Box p\} \leq \bigwedge\{m \in M^\infty \mid \Box\Diamond p \leq m\}]$$

$$\forall p\forall j\forall m[(j \leq \Box p \;\&\; \Box\Diamond p \leq m) \Rightarrow \Diamond j \leq m]$$

$$\forall p\forall j\forall m[(\blacklozenge j \leq p \;\&\; \Box\Diamond p \leq m) \Rightarrow \Diamond j \leq m]$$

$$\forall j\forall m[(\Box\Diamond\blacklozenge j \leq m) \Rightarrow \Diamond j \leq m]$$

$$\forall j[\Diamond j \leq \Box\Diamond\blacklozenge j]$$

Confluence: $\Diamond\Box p \rightarrow \Box\Diamond p$ — Translation

On Kripke frames

$$\forall j[\Diamond j \leq \Box\Diamond\blacklozenge j]$$

Confluence: $\Diamond\Box p \rightarrow \Box\Diamond p$ — Translation

On Kripke frames

$$\forall j[\Diamond j \leq \Box\Diamond\blacklozenge j]$$

$$\forall y[\Diamond\{y\} \leq \Box\Diamond\Diamond^{-1}\{y\}]$$

Confluence: $\Diamond\Box p \to \Box\Diamond p$ — Translation

On Kripke frames

$$\forall j[\Diamond j \leq \Box\Diamond\blacklozenge j]$$

$$\forall y[\Diamond\{y\} \leq \Box\Diamond\Diamond^{-1}\{y\}]$$

$$\forall y[Rxy \to \forall z(Rxz \to \exists u(Rzu \wedge Ryu))]$$

Confluence: $\Diamond\Box p \rightarrow \Box\Diamond p$ — Translation

On Kripke frames

$$\forall j[\Diamond j \leq \Box\Diamond\blacklozenge j]$$

$$\forall y[\Diamond\{y\} \leq \Box\Diamond\Diamond^{-1}\{y\}]$$

$$\forall y[Rxy \rightarrow \forall z(Rxz \rightarrow \exists u(Rzu \wedge Ryu))]$$

On DML-frames / intuitionistic-ML frames

$$\forall y[\Diamond\{y\}\uparrow \leq \Box\Diamond\blacklozenge\{y\}\uparrow]$$

ALBA algorithm

Algorithm / calculus:

- based on Ackermann, approximation, and residuation rules.

ALBA algorithm

Algorithm / calculus:

- based on Ackermann, approximation, and residuation rules.
- eliminates propositional variables from inequalities for sake of special variables ranging over J^{∞} and M^{∞}.

ALBA algorithm

Algorithm / calculus:

- based on Ackermann, approximation, and residuation rules.
- eliminates propositional variables from inequalities for sake of special variables ranging over J^∞ and M^∞.

Theorem

All ALBA-reducible inequalities are elementary on the relational semantics.

ALBA algorithm

Algorithm / calculus:

- based on Ackermann, approximation, and residuation rules.
- eliminates propositional variables from inequalities for sake of special variables ranging over J^{∞} and M^{∞}.

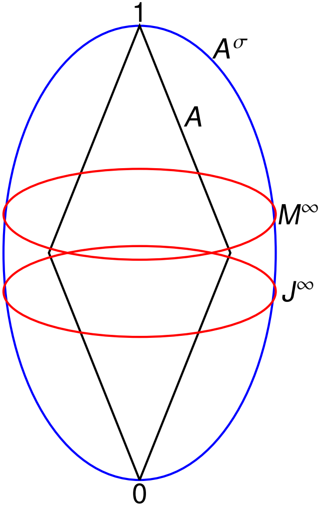
Theorem

All ALBA-reducible inequalities are elementary on the relational semantics.

Theorem

All ALBA-reducible inequalities are canonical.

Canononical Extension



Canonicity

Admissible assignment v for $\mathcal{L}^+$ on $\mathbb{A}^\sigma$:
$v : \mathrm{PROP} \to A$,
$v : \mathrm{J} \to J^\infty(\mathbb{A}^\sigma)$, and
$v : \mathrm{M} \to M^\infty(\mathbb{A}^\sigma)$.

Canonicity

Admissible assignment v for $\mathcal{L}^+$ on $\mathbb{A}^\sigma$:
v : PROP $\rightarrow A$,
v : J $\rightarrow J^\infty(\mathbb{A}^\sigma)$, and
v : M $\rightarrow M^\infty(\mathbb{A}^\sigma)$.

Notation: Validity under admissible assignments: $\mathbb{A}^\sigma \models_{\mathbb{A}} \varphi \leq \psi$.

Canonicity

Admissible assignment v for $\mathcal{L}^+$ on $\mathbb{A}^\sigma$:
$v : \mathsf{PROP} \to A$,
$v : \mathsf{J} \to J^\infty(\mathbb{A}^\sigma)$, and
$v : \mathsf{M} \to M^\infty(\mathbb{A}^\sigma)$.

Notation: Validity under admissible assignments: $\mathbb{A}^\sigma \models_{\mathbb{A}} \varphi \leq \psi$.

Outline of the canonicity proof:

$$
\begin{array}{ccc}
\mathbb{A} \models \varphi \leq \psi & & \mathbb{A}^\sigma \models \varphi \leq \psi \\
\Updownarrow & & \\
\mathbb{A}^\sigma \models_{\mathbb{A}} \varphi \leq \psi & & \Updownarrow \\
\Updownarrow & & \\
\mathbb{A}^\sigma \models_{\mathbb{A}} \mathrm{ALBA}(\varphi \leq \psi) & \Leftrightarrow & \mathbb{A}^\sigma \models \mathrm{ALBA}(\varphi \leq \psi)
\end{array}
$$

Justifying the Ackermann rule

We need an Ackermann lemma which says:

TFAE:

1. $\mathbb{A}^\sigma, v \models \beta(\alpha) \leq \gamma(\alpha)$
2. there exists some $v' \sim_p v$ such that $\mathbb{A}^\sigma, v' \models \alpha \leq p$ and $\mathbb{L}, v' \models \beta(p) \leq \gamma(p)$.

for admissible assignments v and v', where

- $\alpha, \beta, \gamma \in \mathcal{L}^+$,
- with positivity/negativity as before.

Justifying the Ackermann rule

We need an Ackermann lemma which says:

TFAE:

1. $\mathbb{A}^\sigma, v \models \beta(\alpha) \leq \gamma(\alpha)$
2. there exists some $v' \sim_p v$ such that $\mathbb{A}^\sigma, v' \models \alpha \leq p$ and $\mathbb{L}, v' \models \beta(p) \leq \gamma(p)$.

for admissible assignments v and v', where

- $\alpha, \beta, \gamma \in \mathcal{L}^+$,
- with positivity/negativity as before.

PROBLEM: $v(\alpha) \notin \mathbb{A}$.

Justifying the Ackermann rule (2)

This would we possible if we could prove the following equivalent:

$$\beta(v(\alpha)) \quad \leq \quad \gamma(v(\alpha))$$

Justifying the Ackermann rule (2)

This would we possible if we could prove the following equivalent:

$$\begin{aligned} \beta(v(\alpha)) &\leq \gamma(v(\alpha)) \\ \beta(\bigwedge\{a \in \mathbb{A} \mid v(\alpha) \leq a\}) &\leq \gamma(\bigwedge\{a \in \mathbb{A} \mid v(\alpha) \leq a\}) \end{aligned}$$

Justifying the Ackermann rule (2)

This would we possible if we could prove the following equivalent:

$$
\begin{array}{rcl}
\beta(v(\alpha)) & \leq & \gamma(v(\alpha)) \\
\beta(\bigwedge\{a \in \mathbb{A} \mid v(\alpha) \leq a\}) & \leq & \gamma(\bigwedge\{a \in \mathbb{A} \mid v(\alpha) \leq a\}) \\
\bigwedge\{\beta(a) \mid v(\alpha) \leq a \in \mathbb{A}\} & \leq & \bigvee\{\gamma(a) \mid v(\alpha) \leq a \in \mathbb{A}\})
\end{array}
$$

Justifying the Ackermann rule (2)

This would we possible if we could prove the following equivalent:

$$\begin{array}{rcl}
\beta(v(\alpha)) & \leq & \gamma(v(\alpha)) \\
\beta(\bigwedge\{a \in \mathbb{A} \mid v(\alpha) \leq a\}) & \leq & \gamma(\bigwedge\{a \in \mathbb{A} \mid v(\alpha) \leq a\}) \\
\bigwedge\{\beta(a) \mid v(\alpha) \leq a \in \mathbb{A}\} & \leq & \bigvee\{\gamma(a) \mid v(\alpha) \leq a \in \mathbb{A}\}) \\
\beta(a_0) & \leq & \gamma(a_0)
\end{array}$$

Justifying the Ackermann rule (2)

This would we possible if we could prove the following equivalent:

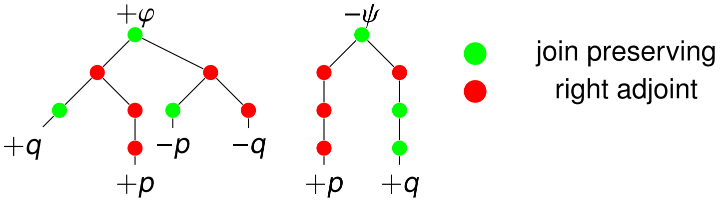
$$\begin{array}{rcl} \beta(v(\alpha)) & \leq & \gamma(v(\alpha)) \\ \beta(\bigwedge\{a \in \mathbb{A} \mid v(\alpha) \leq a\}) & \leq & \gamma(\bigwedge\{a \in \mathbb{A} \mid v(\alpha) \leq a\}) \\ \bigwedge\{\beta(a) \mid v(\alpha) \leq a \in \mathbb{A}\} & \leq & \bigvee\{\gamma(a) \mid v(\alpha) \leq a \in \mathbb{A}\}) \\ \beta(a_0) & \leq & \gamma(a_0) \end{array}$$

Use classification of $\mathcal{L}^+$-terms as syntactically open / closed.

Sahlqvist inequalities

$$\varphi \leq \psi$$

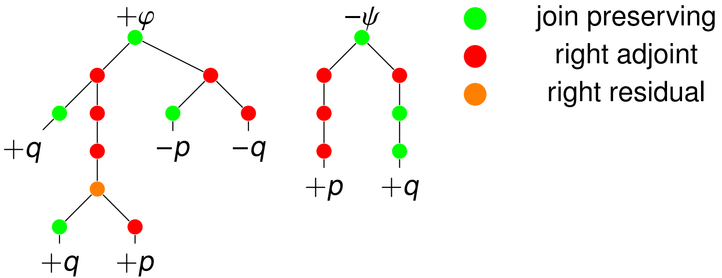
with $\epsilon_p = 1$ and $\epsilon_q = \partial$.



Inductive Inequalities

$$\varphi \leq \psi$$

with $\epsilon_p = 1$, and $\epsilon_q = \partial$, and
$q <_\Omega p$.



Completeness for inductive inequalities

Theorem

ALBA successfully reduces all inductive (and hence Sahlqvist) inequalities.

Corollary

All inductive (and hence Sahlqvist) inequalities are elementary and canonical.


The End