On scattered convex geometries

joint work with Maurice Pouzet
Université Claude-Bernard, Lyon

K. Adaricheva

Yeshiva University, New York

July 29, 2011 / TACL-2011
Marseille, France

Outline

1. Convex geometries
2. Order-scattered algebraic lattices: main problem
3. Representation of $\Omega(\eta)$ in a convex geometry
4. Semilattices of finite $\vee$-dimension: main result
5. Results for particular classes of convex geometries
6. Other results

Definition of a convex geometry

A pair (X, ϕ) of a non-empty set X and a closure operator $\phi : 2^X \to 2^X$ on X *a convex geometry*, if

- it is a zero-closed space (i.e. $\overline{\emptyset} = \emptyset$)
- ϕ satisfies *the anti-exchange axiom*:

$$x \in \overline{X \cup \{y\}} \text{ and } x \notin X \text{ imply that } y \notin \overline{X \cup \{x\}}$$
$$\text{for all } x \neq y \text{ in } A \text{ and all closed } X \subseteq A.$$

Infinite convex geometries were introduced and studied in

K.V. Adaricheva, V.A. Gorbunov, and V.I. Tumanov
Join-semidistributive lattices and convex geometries
Adv. Math., **173** (2003), 1–49.

Definition of a convex geometry

A pair (X, ϕ) of a non-empty set X and a closure operator $\phi : 2^X \to 2^X$ on X *a convex geometry*, if

- it is a zero-closed space (i.e. $\overline{\emptyset} = \emptyset$)
- ϕ satisfies *the anti-exchange axiom*:

$$x \in \overline{X \cup \{y\}} \text{ and } x \notin X \text{ imply that } y \notin \overline{X \cup \{x\}}$$
$$\text{for all } x \neq y \text{ in } A \text{ and all closed } X \subseteq A.$$

Infinite convex geometries were introduced and studied in

K.V. Adaricheva, V.A. Gorbunov, and V.I. Tumanov
Join-semidistributive lattices and convex geometries
Adv. Math., **173** (2003), 1–49.

Definition of a convex geometry

A pair (X, ϕ) of a non-empty set X and a closure operator $\phi : 2^X \to 2^X$ on X *a convex geometry*, if

- it is a zero-closed space (i.e. $\overline{\emptyset} = \emptyset$)
- ϕ satisfies *the anti-exchange axiom*:

$$x \in \overline{X \cup \{y\}} \text{ and } x \notin X \text{ imply that } y \notin \overline{X \cup \{x\}}$$
$$\text{for all } x \neq y \text{ in } A \text{ and all closed } X \subseteq A.$$

Infinite convex geometries were introduced and studied in

K.V. Adaricheva, V.A. Gorbunov, and V.I. Tumanov
Join-semidistributive lattices and convex geometries
Adv. Math., **173** (2003), 1–49.

Definition of a convex geometry

A pair (X, ϕ) of a non-empty set X and a closure operator $\phi : 2^X \to 2^X$ on X *a convex geometry*, if

- it is a zero-closed space (i.e. $\overline{\emptyset} = \emptyset$)
- ϕ satisfies *the anti-exchange axiom*:

$$x \in \overline{X \cup \{y\}} \text{ and } x \notin X \text{ imply that } y \notin \overline{X \cup \{x\}}$$
$$\text{for all } x \neq y \text{ in } A \text{ and all closed } X \subseteq A.$$

Infinite convex geometries were introduced and studied in

K.V. Adaricheva, V.A. Gorbunov, and V.I. Tumanov
Join-semidistributive lattices and convex geometries
Adv. Math., **173** (2003), 1–49.

Definition of a convex geometry

A pair (X, ϕ) of a non-empty set X and a closure operator $\phi : 2^X \to 2^X$ on X *a convex geometry*, if

- it is a zero-closed space (i.e. $\overline{\emptyset} = \emptyset$)
- ϕ satisfies *the anti-exchange axiom*:

$$x \in \overline{X \cup \{y\}} \text{ and } x \notin X \text{ imply that } y \notin \overline{X \cup \{x\}}$$
$$\text{for all } x \neq y \text{ in } A \text{ and all closed } X \subseteq A.$$

Infinite convex geometries were introduced and studied in

K.V. Adaricheva, V.A. Gorbunov, and V.I. Tumanov
Join-semidistributive lattices and convex geometries
Adv. Math., **173** (2003), 1–49.

- A subset $Y \subseteq X$ is called *closed*, if $Y = \phi(Y)$.
- The collection of closed sets $Cl(X, \phi)$ forms a *complete* lattice, with respect to order of containment.
- If ϕ is a *finitary* closure operator, then $Cl(X, \phi)$ is an *algebraic* lattice.
- Convex geometry may be given by $Cl(X, \phi)$.

- A subset $Y \subseteq X$ is called *closed*, if $Y = \phi(Y)$.
- The collection of closed sets $Cl(X, \phi)$ forms a *complete* lattice, with respect to order of containment.
- If ϕ is a *finitary* closure operator, then $Cl(X, \phi)$ is an *algebraic* lattice.
- Convex geometry may be given by $Cl(X, \phi)$.

- A subset $Y \subseteq X$ is called *closed*, if $Y = \phi(Y)$.
- The collection of closed sets $Cl(X, \phi)$ forms a *complete* lattice, with respect to order of containment.
- If ϕ is a *finitary* closure operator, then $Cl(X, \phi)$ is an *algebraic* lattice.
- Convex geometry may be given by $Cl(X, \phi)$.

- A subset $Y \subseteq X$ is called *closed*, if $Y = \phi(Y)$.
- The collection of closed sets $Cl(X, \phi)$ forms a *complete* lattice, with respect to order of containment.
- If ϕ is a *finitary* closure operator, then $Cl(X, \phi)$ is an *algebraic* lattice.
- Convex geometry may be given by $Cl(X, \phi)$.

Examples

- Let V be a real vector space and $X \subseteq V$. Convex geometry $Co(V, X)$ it the collection of sets $C \cap X$, where C is a convex subset of V.
- Let S be an (infinite) $\wedge$-semilattice. The convex geometry $Sub_{\wedge}(S)$ is the collection of $\wedge$-subsemilattices of S.
- For a partially ordered set $\langle P, \leq \rangle$, let $\leq^*$ denote a strict suborder of $\leq$, i.e. $\leq^* = \{(p, q) \subseteq P^2 : p \leq q \text{ and } p \neq q\}$.
  The convex geometry of suborders $O(P)$ is the lattice of transitively closed subsets of $\leq^*$.
- All three examples are *algebraic* convex geometries.

Examples

- Let V be a real vector space and $X \subseteq V$. Convex geometry $Co(V, X)$ it the collection of sets $C \cap X$, where C is a convex subset of V.
- Let S be an (infinite) $\wedge$-semilattice. The convex geometry $Sub_{\wedge}(S)$ is the collection of $\wedge$-subsemilattices of S.
- For a partially ordered set $\langle P, \leq \rangle$, let $\leq^*$ denote a strict suborder of $\leq$, i.e. $\leq^* = \{(p, q) \subseteq P^2 : p \leq q \text{ and } p \neq q\}$.
  The convex geometry of suborders $O(P)$ is the lattice of transitively closed subsets of $\leq^*$.
- All three examples are *algebraic* convex geometries.

Examples

- Let V be a real vector space and $X \subseteq V$. Convex geometry $Co(V, X)$ it the collection of sets $C \cap X$, where C is a convex subset of V.
- Let S be an (infinite) $\wedge$-semilattice. The convex geometry $Sub_{\wedge}(S)$ is the collection of $\wedge$-subsemilattices of S.
- For a partially ordered set $\langle P, \leq \rangle$, let $\leq^*$ denote a strict suborder of $\leq$, i.e. $\leq^* = \{(p, q) \subseteq P^2 : p \leq q \text{ and } p \neq q\}$.
  The convex geometry of suborders $O(P)$ is the lattice of transitively closed subsets of $\leq^*$.
- All three examples are *algebraic* convex geometries.

Examples

- Let V be a real vector space and $X \subseteq V$. Convex geometry $Co(V, X)$ it the collection of sets $C \cap X$, where C is a convex subset of V.
- Let S be an (infinite) $\wedge$-semilattice. The convex geometry $Sub_{\wedge}(S)$ is the collection of $\wedge$-subsemilattices of S.
- For a partially ordered set $\langle P, \leq \rangle$, let $\leq^*$ denote a strict suborder of $\leq$, i.e. $\leq^* = \{(p, q) \subseteq P^2 : p \leq q \text{ and } p \neq q\}$.
  The convex geometry of suborders $O(P)$ is the lattice of transitively closed subsets of $\leq^*$.
- All three examples are *algebraic* convex geometries.

Examples

- Let V be a real vector space and $X \subseteq V$. Convex geometry $Co(V,X)$ it the collection of sets $C \cap X$, where C is a convex subset of V.
- Let S be an (infinite) $\wedge$-semilattice. The convex geometry $Sub_{\wedge}(S)$ is the collection of $\wedge$-subsemilattices of S.
- For a partially ordered set $\langle P, \leq \rangle$, let $\leq^*$ denote a strict suborder of $\leq$, i.e. $\leq^* = \{(p,q) \subseteq P^2 : p \leq q \text{ and } p \neq q\}$.
  The convex geometry of suborders $O(P)$ is the lattice of transitively closed subsets of $\leq^*$.
- All three examples are *algebraic* convex geometries.

A poset $(P, \leq)$ is called *order-scattered*, if the chain of rational numbers $\mathbb{Q}$ is not a sub-poset in $(P, \leq)$.

Problem. Describe order-scattered algebraic lattices.

Given algebraic lattice L, the set of its compact elements $S = S(L) \subseteq L$ forms a $\vee$-subsemilattice in L. It is well- known that $L \simeq \mathrm{Id}(S)$, where $\mathrm{Id}(S)$ is the lattice of ideals of semilattice S.

Problem. (re-visited)
Describe when algebraic lattice L is order-scattered in terms of the shape of semilattice $S(L)$ of its compact elements.

A poset $(P, \leq)$ is called *order-scattered*, if the chain of rational numbers $\mathbb{Q}$ is not a sub-poset in $(P, \leq)$.

Problem. Describe order-scattered algebraic lattices.

Given algebraic lattice L, the set of its compact elements $S = S(L) \subseteq L$ forms a $\vee$-subsemilattice in L. It is well- known that $L \simeq \mathrm{Id}(S)$, where $\mathrm{Id}(S)$ is the lattice of ideals of semilattice S.

Problem. (re-visited)
Describe when algebraic lattice L is order-scattered in terms of the shape of semilattice $S(L)$ of its compact elements.

A poset $(P, \leq)$ is called *order-scattered*, if the chain of rational numbers $\mathbb{Q}$ is not a sub-poset in $(P, \leq)$.

Problem. Describe order-scattered algebraic lattices.

Given algebraic lattice L, the set of its compact elements $S = S(L) \subseteq L$ forms a $\vee$-subsemilattice in L. It is well- known that $L \simeq \text{Id}(S)$, where $\text{Id}(S)$ is the lattice of ideals of semilattice S.

Problem. (re-visited)
Describe when algebraic lattice L is order-scattered in terms of the shape of semilattice $S(L)$ of its compact elements.

A poset $(P, \leq)$ is called *order-scattered*, if the chain of rational numbers $\mathbb{Q}$ is not a sub-poset in $(P, \leq)$.

Problem. Describe order-scattered algebraic lattices.

Given algebraic lattice L, the set of its compact elements $S = S(L) \subseteq L$ forms a $\vee$-subsemilattice in L. It is well- known that $L \simeq \text{Id}(S)$, where $\text{Id}(S)$ is the lattice of ideals of semilattice S.

Problem. (re-visited)
Describe when algebraic lattice L is order-scattered in terms of the shape of semilattice $S(L)$ of its compact elements.

Examples of “non-scattered" shapes

Example 1. Let $\mathbb{N}$ be the set of natural numbers, and let $S = \mathfrak{P}^{<\omega}(\mathbb{N})$ be the $\vee$-semilattice of its finite subsets.
Then $L = \text{Id}(S)$ is not order-scattered.

Example 2. Consider a sub-semilattice $\Omega(\eta)$ of $\mathbb{N} \times \mathbb{Q}$, where $\mathbb{Q}$ is a chain of rational numbers.
Then $L = \text{Id}(\Omega(\eta))$ is not order-scattered.

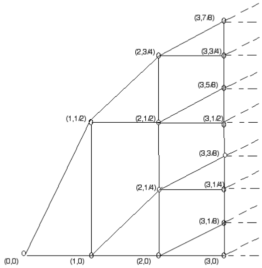

Figure: $\Omega(\eta)$

Examples of “non-scattered" shapes

Example 1. Let $\mathbb{N}$ be the set of natural numbers, and let $S = \mathfrak{P}^{<\omega}(\mathbb{N})$ be the $\vee$-semilattice of its finite subsets.
Then $L = \mathrm{Id}(S)$ is not order-scattered.

Example 2. Consider a sub-semilattice $\Omega(\eta)$ of $\mathbb{N} \times \mathbb{Q}$, where $\mathbb{Q}$ is a chain of rational numbers.
Then $L = \mathrm{Id}(\Omega(\eta))$ is not order-scattered.

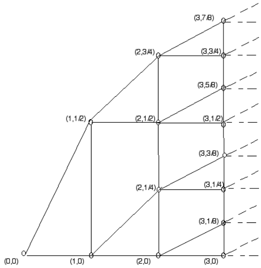

Figure: $\Omega(\eta)$

Hypothesis

Problem. (re-visited again)
For a semilattice S, show that $L = \mathrm{Id}(S)$ is order-scattered iff S is order-scattered and does not contain either $\mathfrak{P}^{<\omega}(\mathbb{N})$ or $\Omega(\eta)$ as a sub-semilattice.

Earlier result

I. Chakir, and M. Pouzet, *The length of chains in modular algebraic lattices*, Order, 24(2007), 227–247.

Theorem. Algebraic *modular* lattice L is order-scattered iff the semilattice S of its compact elements is order-scattered and does not contain $\mathfrak{P}^{<\omega}(\mathbb{N})$ as a subsemilattice.

Note. In most cases, the modular law fails in convex geometries, unless they are distributive.

Earlier result

I. Chakir, and M. Pouzet, *The length of chains in modular algebraic lattices*, Order, 24(2007), 227–247.

Theorem. Algebraic *modular* lattice L is order-scattered iff the semilattice S of its compact elements is order-scattered and does not contain $\mathfrak{P}^{<\omega}(\mathbb{N})$ as a subsemilattice.

Note. In most cases, the modular law fails in convex geometries, unless they are distributive.

Main question

Can **the problem** be solved in algebraic convex geometries?

Answer so far: YES, under some additional finitary assumption on convex geometries.

Two important components in the proof:

- representation of $\Omega(\eta)$ in convex geometry called *a multichain*
- Galvin's Theorem in infinite combinatorics

Main question

Can **the problem** be solved in algebraic convex geometries?

Answer so far: YES, under some additional finitary assumption on convex geometries.

Two important components in the proof:

- representation of $\Omega(\eta)$ in convex geometry called *a multichain*
- Galvin's Theorem in infinite combinatorics

Main question

Can **the problem** be solved in algebraic convex geometries?

Answer so far: YES, under some additional finitary assumption on convex geometries.

Two important components in the proof:

- representation of $\Omega(\eta)$ in convex geometry called *a multichain*
- Galvin's Theorem in infinite combinatorics

Multi-chains

Defining the *multi-chains*:

- Consider an infinite set E.
- Let $(\mathcal{L}_i : i \in I)$ be the set of linear orders on E.
- Build a convex geometry $C_i = \text{Id}(E, \mathcal{L}_i)$, for each $i \in I$.
- Build a closure system $C = \bigvee_{i \in I} C_i$ on E. Closed sets in C are $X = \bigcap X_i$, where X_i is closed in C_i, for each i.
- For arbitrary I, C is a convex geometry. For any finite I, C is algebraic.

Multi-chains

Defining the *multi-chains*:

- Consider an infinite set E.
- Let $(\mathcal{L}_i : i \in I)$ be the set of linear orders on E.
- Build a convex geometry $C_i = \mathrm{Id}(E, \mathcal{L}_i)$, for each $i \in I$.
- Build a closure system $C = \bigvee_{i \in I} C_i$ on E. Closed sets in C are $X = \bigcap X_i$, where X_i is closed in C_i, for each i.
- For arbitrary I, C is a convex geometry. For any finite I, C is algebraic.

Multi-chains

Defining the *multi-chains*:

- Consider an infinite set E.
- Let $(\mathcal{L}_i : i \in I)$ be the set of linear orders on E.
- Build a convex geometry $C_i = \mathrm{Id}(E, \mathcal{L}_i)$, for each $i \in I$.
- Build a closure system $C = \bigvee_{i \in I} C_i$ on E. Closed sets in C are $X = \bigcap X_i$, where X_i is closed in C_i, for each i.
- For arbitrary I, C is a convex geometry. For any finite I, C is algebraic.

Multi-chains

Defining the *multi-chains*:

- Consider an infinite set E.
- Let $(\mathcal{L}_i : i \in I)$ be the set of linear orders on E.
- Build a convex geometry $C_i = \mathrm{Id}(E, \mathcal{L}_i)$, for each $i \in I$.
- Build a closure system $C = \bigvee_{i \in I} C_i$ on E. Closed sets in C are $X = \bigcap X_i$, where X_i is closed in C_i, for each i.
- For arbitrary I, C is a convex geometry. For any finite I, C is algebraic.

Multi-chains

Defining the *multi-chains*:

- Consider an infinite set E.
- Let $(\mathcal{L}_i : i \in I)$ be the set of linear orders on E.
- Build a convex geometry $C_i = \mathrm{Id}(E, \mathcal{L}_i)$, for each $i \in I$.
- Build a closure system $C = \bigvee_{i \in I} C_i$ on E. Closed sets in C are $X = \bigcap X_i$, where X_i is closed in C_i, for each i.
- For arbitrary I, C is a convex geometry. For any finite I, C is algebraic.

Multi-chains

Defining the *multi-chains*:

- Consider an infinite set E.
- Let $(\mathcal{L}_i : i \in I)$ be the set of linear orders on E.
- Build a convex geometry $C_i = \mathrm{Id}(E, \mathcal{L}_i)$, for each $i \in I$.
- Build a closure system $C = \bigvee_{i \in I} C_i$ on E. Closed sets in C are $X = \bigcap X_i$, where X_i is closed in C_i, for each i.
- For arbitrary I, C is a convex geometry. For any finite I, C is algebraic.

Duplex

The multi-chain $C = \bigvee_{i \in I} \mathrm{Id}(E, \mathcal{L}_i)$ is called *a duplex*, if

- E is a countable set;
- $|I| = 2$;
- $(E, \mathcal{L}_1)$ is isomorphic to a chain of natural numbers;
- $(E, \mathcal{L}_2)$ has a sub-chain of rational numbers.

Duplex

The multi-chain $C = \bigvee_{i \in I} \mathrm{Id}(E, \mathcal{L}_i)$ is called *a duplex*, if

- E is a countable set;
- $|I| = 2$;
- $(E, \mathcal{L}_1)$ is isomorphic to a chain of natural numbers;
- $(E, \mathcal{L}_2)$ has a sub-chain of rational numbers.

Duplex

The multi-chain $C = \bigvee_{i \in I} \mathrm{Id}(E, \mathcal{L}_i)$ is called *a duplex*, if

- E is a countable set;
- $|I| = 2$;
- $(E, \mathcal{L}_1)$ is isomorphic to a chain of natural numbers;
- $(E, \mathcal{L}_2)$ has a sub-chain of rational numbers.

Duplex

The multi-chain $C = \bigvee_{i \in I} \mathrm{Id}(E, \mathcal{L}_i)$ is called *a duplex*, if

- E is a countable set;
- $|I| = 2$;
- $(E, \mathcal{L}_1)$ is isomorphic to a chain of natural numbers;
- $(E, \mathcal{L}_2)$ has a sub-chain of rational numbers.

Representation

Lemma. For any duplex $C = \text{Id}(E, \mathcal{L}_1) \vee \text{Id}(E, \mathcal{L}_2)$, $\Omega(\eta)$ is a sub-semilattice of the semilattice of compact elements of C.

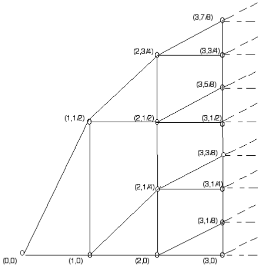

Figure: $\Omega(\eta)$

Galvin's Theorem

Theorem (F. Galvin, unpublished)
Suppose the pairs of rationals are divided into finitely many classes $A_1, \ldots, A_n$. Fix the ordering on the rationals with order type Ω. Then there is a subset X of rationals of order type η and indices i, j (with possibly $i = j$) such that all pairs of X on which two orders coincide belong to A_i, and all pairs of X on which the two orders disagree belong to A_j.

The proof is available in:
R. Fraïssé, *The theory of relations*, North-Holland Pub.Co., Amsterdam, 2000.

Galvin's Theorem

Theorem (F. Galvin, unpublished)
Suppose the pairs of rationals are divided into finitely many classes $A_1, \ldots, A_n$. Fix the ordering on the rationals with order type Ω. Then there is a subset X of rationals of order type η and indices i, j (with possibly $i = j$) such that all pairs of X on which two orders coincide belong to A_i, and all pairs of X on which the two orders disagree belong to A_j.

The proof is available in:
R. Fraïssé, *The theory of relations*, North-Holland Pub.Co., Amsterdam, 2000.

Definition of $\vee$-dimension

Finitary condition needed for the main theorem.

We say that a semilattice S with 0 has $\vee$-*dimension* $dim_{\vee}(S) = \kappa$, if

- κ is the smallest cardinal for which
- there exist κ chains C_i, $i < \kappa$, with minimal element 0_i
- and injective map $f : S \to \Pi C_i$ satisfying
- $f(a \vee b) = f(a) \vee f(b)$
- $f(0) = (0_i, i < \kappa)$.

Compare: for the definition of the *order dimension* of S, f is simply order-preserving map.

Definition of $\vee$-dimension

Finitary condition needed for the main theorem.
We say that a semilattice S with 0 has $\vee$-*dimension* $dim_{\vee}(S) = \kappa$, if

- κ is the smallest cardinal for which
- there exist κ chains C_i, $i < \kappa$, with minimal element 0_i
- and injective map $f : S \to \Pi C_i$ satisfying
- $f(a \vee b) = f(a) \vee f(b)$
- $f(0) = (0_i, i < \kappa)$.

Compare: for the definition of the *order dimension* of S, f is simply order-preserving map.

Definition of $\vee$-dimension

Finitary condition needed for the main theorem.
We say that a semilattice S with 0 has $\vee$-*dimension* $dim_\vee(S) = \kappa$, if

- κ is the smallest cardinal for which
- there exist κ chains C_i, $i < \kappa$, with minimal element 0_i
- and injective map $f : S \to \Pi C_i$ satisfying
- $f(a \vee b) = f(a) \vee f(b)$
- $f(0) = (0_i, i < \kappa)$.

Compare: for the definition of the *order dimension* of S, f is simply order-preserving map.

Definition of $\vee$-dimension

Finitary condition needed for the main theorem.
We say that a semilattice S with 0 has $\vee$-*dimension* $dim_\vee(S) = \kappa$, if

- κ is the smallest cardinal for which
- there exist κ chains C_i, $i < \kappa$, with minimal element 0_i
- and injective map $f : S \to \Pi C_i$ satisfying
- $f(a \vee b) = f(a) \vee f(b)$
- $f(0) = (0_i, i < \kappa)$.

Compare: for the definition of the *order dimension* of S, f is simply order-preserving map.

Definition of $\vee$-dimension

Finitary condition needed for the main theorem.
We say that a semilattice S with 0 has $\vee$-*dimension* $dim_{\vee}(S) = \kappa$, if

- κ is the smallest cardinal for which
- there exist κ chains C_i, $i < \kappa$, with minimal element 0_i
- and injective map $f : S \to \Pi C_i$ satisfying
- $f(a \vee b) = f(a) \vee f(b)$
- $f(0) = (0_i, i < \kappa)$.

Compare: for the definition of the *order dimension* of S, f is simply order-preserving map.

Definition of $\vee$-dimension

Finitary condition needed for the main theorem.
We say that a semilattice S with 0 has $\vee$-*dimension* $dim_{\vee}(S) = \kappa$, if

- κ is the smallest cardinal for which
- there exist κ chains C_i, $i < \kappa$, with minimal element 0_i
- and injective map $f : S \to \Pi C_i$ satisfying
- $f(a \vee b) = f(a) \vee f(b)$
- $f(0) = (0_i, i < \kappa)$.

Compare: for the definition of the *order dimension* of S, f is simply order-preserving map.

Definition of $\vee$-dimension

Finitary condition needed for the main theorem.
We say that a semilattice S with 0 has $\vee$-*dimension* $dim_\vee(S) = \kappa$, if

- κ is the smallest cardinal for which
- there exist κ chains C_i, $i < \kappa$, with minimal element 0_i
- and injective map $f : S \to \Pi C_i$ satisfying
- $f(a \vee b) = f(a) \vee f(b)$
- $f(0) = (0_i, i < \kappa)$.

Compare: for the definition of the *order dimension* of S, f is simply order-preserving map.

Definition of $\vee$-dimension

Finitary condition needed for the main theorem.
We say that a semilattice S with 0 has $\vee$-*dimension* $dim_\vee(S) = \kappa$, if

- κ is the smallest cardinal for which
- there exist κ chains C_i, $i < \kappa$, with minimal element 0_i
- and injective map $f : S \to \Pi C_i$ satisfying
- $f(a \vee b) = f(a) \vee f(b)$
- $f(0) = (0_i, i < \kappa)$.

Compare: for the definition of the *order dimension* of S, f is simply order-preserving map.

M_3 example

M_3 has the order dimension 2.

If a, b, c are atoms, then $f : M_3 \to C_1 \times C_2$, where
$C_1 = 0_1 < a_1 < b_1 < c_1 < 1_1$,
$C_2 = 0_2 < c_2 < b_2 < a_2 < 1_2$, and $f(x) = (x_1, x_2)$. f does not preserve the join operation.

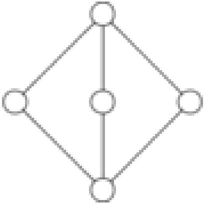

Figure: M_3

On the other hand, one can make ∨-embedding with three chains: $C_x = 0_x < x < 1_x$, $x = a, b, c$. Thus, $dim_\vee(M_3) = 3$.

M_3 example

M_3 has the order dimension 2.
If a, b, c are atoms, then $f : M_3 \to C_1 \times C_2$, where
$C_1 = 0_1 < a_1 < b_1 < c_1 < 1_1$,
$C_2 = 0_2 < c_2 < b_2 < a_2 < 1_2$, and $f(x) = (x_1, x_2)$. f does not preserve the join operation.

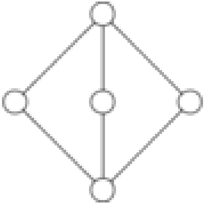

Figure: M_3

On the other hand, one can make ∨-embedding with three chains: $C_x = 0_x < x < 1_x$, $x = a, b, c$. Thus, $dim_\vee(M_3) = 3$.

M_3 example

M_3 has the order dimension 2.
If a, b, c are atoms, then $f : M_3 \to C_1 \times C_2$, where
$C_1 = 0_1 < a_1 < b_1 < c_1 < 1_1$,
$C_2 = 0_2 < c_2 < b_2 < a_2 < 1_2$, and $f(x) = (x_1, x_2)$. f does not preserve the join operation.

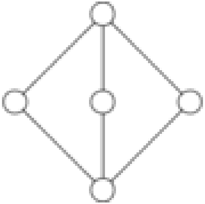

Figure: M_3

On the other hand, one can make ∨-embedding with three chains: $C_x = 0_x < x < 1_x$, $x = a, b, c$. Thus, $dim_\vee(M_3) = 3$.

M_3 example

M_3 has the order dimension 2.
If a, b, c are atoms, then $f : M_3 \to C_1 \times C_2$, where
$C_1 = 0_1 < a_1 < b_1 < c_1 < 1_1$,
$C_2 = 0_2 < c_2 < b_2 < a_2 < 1_2$, and $f(x) = (x_1, x_2)$. f does not preserve the join operation.

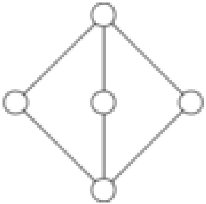

Figure: M_3

On the other hand, one can make $\vee$-embedding with three chains: $C_x = 0_x < x < 1_x$, $x = a, b, c$. Thus, $dim_\vee(M_3) = 3$.

M_3 example

M_3 has the order dimension 2.
If a, b, c are atoms, then $f : M_3 \to C_1 \times C_2$, where
$C_1 = 0_1 < a_1 < b_1 < c_1 < 1_1$,
$C_2 = 0_2 < c_2 < b_2 < a_2 < 1_2$, and $f(x) = (x_1, x_2)$. f does not preserve the join operation.

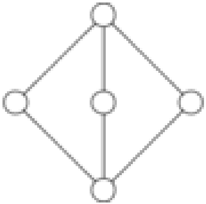

Figure: M_3

On the other hand, one can make ∨-embedding with three chains: $C_x = 0_x < x < 1_x$, $x = a, b, c$. Thus, $dim_\vee(M_3) = 3$.

M_3 example

M_3 has the order dimension 2.
If a, b, c are atoms, then $f : M_3 \to C_1 \times C_2$, where
$C_1 = 0_1 < a_1 < b_1 < c_1 < 1_1$,
$C_2 = 0_2 < c_2 < b_2 < a_2 < 1_2$, and $f(x) = (x_1, x_2)$. f does not preserve the join operation.

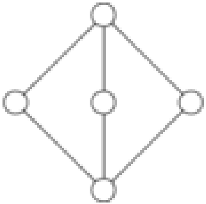

Figure: M_3

On the other hand, one can make $\vee$-embedding with three chains:
$C_x = 0_x < x < 1_x$, $x = a, b, c$. Thus, $dim_\vee(M_3) = 3$.

Main result

Theorem 1. Let S be the semilattice of compact elements of algebraic convex geometry $C = \mathrm{Id}(S)$. If $dim_{\vee} S = n < \omega$, then C is order scattered iff S is order scattered and $\Omega(\eta)$ is not a subsemilattice of S.

Note: $\mathfrak{P}^{<\omega}(\mathbb{N})$ cannot appear as a sub-semilattice of any semilattice S with $dim_{\vee} S = n < \omega$.

Main result

Theorem 1. Let S be the semilattice of compact elements of algebraic convex geometry $C = \mathrm{Id}(S)$. If $dim_{\vee} S = n < \omega$, then C is order scattered iff S is order scattered and $\Omega(\eta)$ is not a subsemilattice of S.

Note: $\mathfrak{P}^{<\omega}(\mathbb{N})$ cannot appear as a sub-semilattice of any semilattice S with $dim_{\vee} S = n < \omega$.

Main result

Theorem 1. Let S be the semilattice of compact elements of algebraic convex geometry $C = \mathrm{Id}(S)$. If $dim_\vee S = n < \omega$, then C is order scattered iff S is order scattered and $\Omega(\eta)$ is not a subsemilattice of S.

Note: $\mathfrak{P}^{<\omega}(\mathbb{N})$ cannot appear as a sub-semilattice of any semilattice S with $dim_\vee S = n < \omega$.

Convex sets of vector spaces

Theorem 2. Convex geometry $C = Co(V, X)$ is order scattered iff the semilattice S of compact elements of C is order scattered and does not have $\mathfrak{P}^{<\omega}(\mathbb{N})$ as a subsemilattice.

Subsemilattices and suborders

Theorem 3. Let P be an infinite $\wedge$-semilattice, then the lattice $Sub_\wedge(P)$ of subsemilattices of P always has a copy of $\mathbb{Q}$. Thus, $Sub_\wedge(P)$ is order-scattered iff P is finite.

Theorem 4. Let $(P, \leq)$ be a partially ordered set, and $\leq^* = \leq \setminus \{(p, p) : p \in P\}$. The lattice of suborders $O(P)$ is order-scattered iff $\leq^*$ is finite.

Subsemilattices and suborders

Theorem 3. Let P be an infinite $\wedge$-semilattice, then the lattice $Sub_\wedge(P)$ of subsemilattices of P always has a copy of $\mathbb{Q}$. Thus, $Sub_\wedge(P)$ is order-scattered iff P is finite.

Theorem 4. Let $(P, \leq)$ be a partially ordered set, and $\leq^* = \leq \setminus \{(p, p) : p \in P\}$. The lattice of suborders $O(P)$ is order-scattered iff $\leq^*$ is finite.

Other results

- Algebraic convex geometries have *the geometric description*: per L. Santocanale and F. Wehrung, *Varieties of lattices with geometric description*, http://arxiv.org/abs/1102.2195
- Example of algebraic distributive lattice which is not a convex geometry.
- Convex geometry $Co(V, X)$ is order-scattered iff it is topologically scattered. (Analogue of Theorem of M. Mislov for algebraic distributive lattices.)

Other results

- Algebraic convex geometries have *the geometric description*: per L. Santocanale and F. Wehrung, *Varieties of lattices with geometric description*, http://arxiv.org/abs/1102.2195
- Example of algebraic distributive lattice which is not a convex geometry.
- Convex geometry $Co(V, X)$ is order-scattered iff it is topologically scattered. (Analogue of Theorem of M. Mislov for algebraic distributive lattices.)

Other results

- Algebraic convex geometries have *the geometric description*: per L. Santocanale and F. Wehrung, *Varieties of lattices with geometric description*, http://arxiv.org/abs/1102.2195
- Example of algebraic distributive lattice which is not a convex geometry.
- Convex geometry $Co(V, X)$ is order-scattered iff it is topologically scattered. (Analogue of Theorem of M. Mislov for algebraic distributive lattices.)

Other results

- Algebraic convex geometries have *the geometric description*: per L. Santocanale and F. Wehrung, *Varieties of lattices with geometric description*, http://arxiv.org/abs/1102.2195
- Example of algebraic distributive lattice which is not a convex geometry.
- Convex geometry $Co(V, X)$ is order-scattered iff it is topologically scattered. (Analogue of Theorem of M. Mislov for algebraic distributive lattices.)

Maurice Pouzet

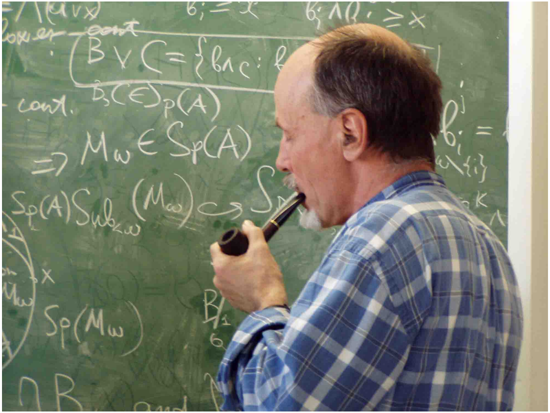

Figure: At the moment of thought

Greetings from New York State

Thank you ! Mercy ! Spasibo !

Figure: Manhattan from Bear Mountain