Ordered direct basis of a finite closure system

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Outline

Closure spaces

$\langle X, \phi \rangle$ is *a closure space*, if

- X is non-empty set (finite in this talk);
- ϕ is a closure operator on X, i.e. $\phi : B(X) \to B(X)$ with
  (1) $Y \subseteq \phi(Y)$;
  (2) $Y \subseteq Z$ implies $\phi(Y) \subseteq \phi(Z)$;
  (3) $\phi(\phi(Y)) = \phi(Y)$, for all $Y, Z \subseteq X$.
- Closed set: $A = \phi(A)$;
- Lattice of closed sets: $Cl(X, \phi)$.

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Lattices and closure spaces

Proposition. Every finite lattice L is the lattice of closed sets of some closure space $\langle X, \phi \rangle$.

- Take $X = J(L)$, the set of join-irreducible elements: $j \in J(L)$, if $j \neq 0$, and $j = a \vee b$ implies $j = a$ or $j = b$;
- define $\phi(Y) = \{j \in J(L) : j \leq \bigvee Y\}$, $Y \subseteq X$.

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Example: Building a closure space associated with lattice A_{12}.
$X = J(A_{12}) = \{1, 2, 3, 4, 5, 6\}$. $\phi(\{4, 6\}) = \{1, 3, 4, 6\}$, $\phi(\{2, 4\}) = X$ etc.

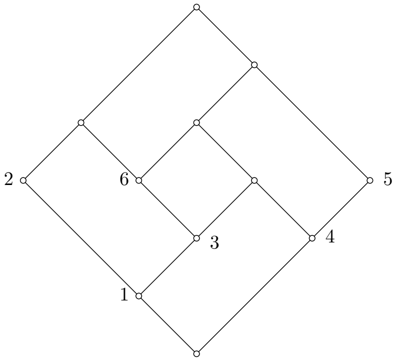

Figure: A_{12}

Closure spaces and implications

- An implication σ on X: $Y \to Z$, for $Y, Z \subseteq X$, $Z \neq \emptyset$.
- σ-closed subset A of X: if $Y \subseteq A$, then $Z \subseteq A$.
- Closure space $\langle X, \phi_\Sigma \rangle$ defined by set Σ of implications on X: A is closed, if it is σ-closed, for each $\sigma \in \Sigma$
- Every closure space $\langle X, \phi \rangle$ can be presented as $\langle X, \psi_\Sigma \rangle$, for some set Σ of implications on X.
- Example: $\Sigma = \{A \to \phi(A) : A \subseteq X, A \neq \phi(A)\}$.

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Implications and propositional Horn logic

- *Unit implication* σ on X: $Y \to z$, $Y \subseteq X, z \in X$.
- Every implication $Y \to Z$ is equivalent to the set of unit implications $\{Y \to z, z \in Z\}$: *unit expansion*.
- Logical interpretation of unit implication σ:
  $X = \{x_1, \ldots, x_n\}$, $Y = \{x_1, \ldots, x_k\}, z = x_{k+1}$
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Summarizing:

Three equivalent ways to look at closure system $\langle X, \phi \rangle$:

- lattice of closed sets $Cl(X, \phi)$;
- set of implications $\Sigma(X, \phi)$;
- definite Horn formula $\Sigma_H(X, \phi)$.

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Connections to computer science fields

- Closure operators appear: relational data bases, data-mining, knowledge structures, data analysis etc.
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Given unit basis Σ and $Y \subset X$, define
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Then $\phi_\Sigma(Y) = \pi_\Sigma(Y) \cup \pi^2_\Sigma(Y) \cup \pi^3_\Sigma(Y) \cup \dots$

A unit implicational basis is called *direct*, if
$\phi_\Sigma(Y) = \pi_\Sigma(Y)$, for all $Y \subseteq X$.

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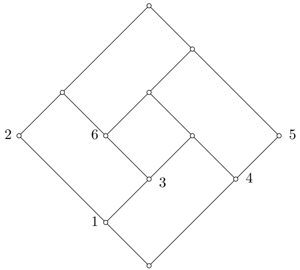
Example

Take Σ_C, the basis of 8 implications for $\langle J(A_{12}), \phi \rangle$:
$2 \to 1, 6 \to 13, 3 \to 1, 5 \to 4, 14 \to 3, 123 \to 6, 1345 \to 6, 12346 \to 5.$
Consider $Y = \{2,4\}$. Then $\pi(Y) = \{2,4,1\}$, $\pi^2(Y) = \{2,4,1,3\}$, $\pi^3(Y) = \{2,4,1,3,6\}$, $\pi^4(Y) = \{1,2,3,4,5,6\} = \phi(Y)$. This basis is *not* direct.



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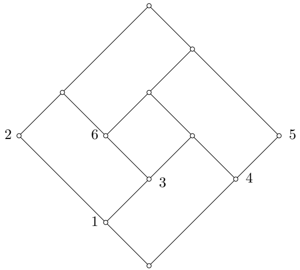

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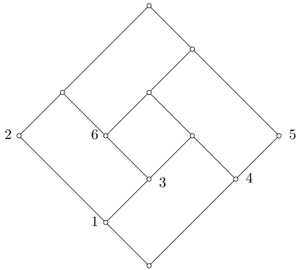

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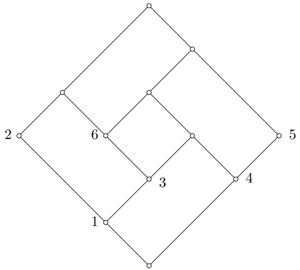

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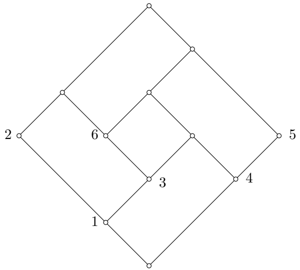

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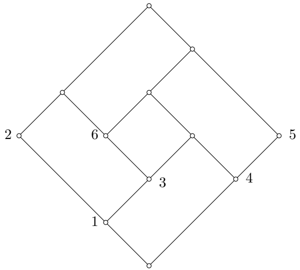

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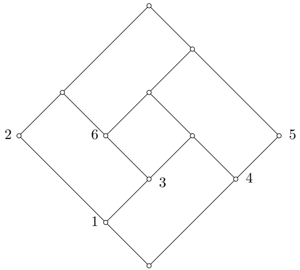

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Example: continued

Another set of implications Σ_U for $\langle J(A_{12}), \phi \rangle$:
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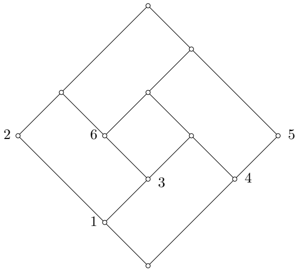

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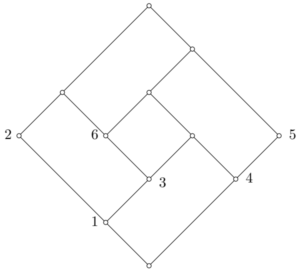

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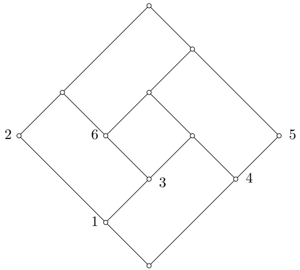

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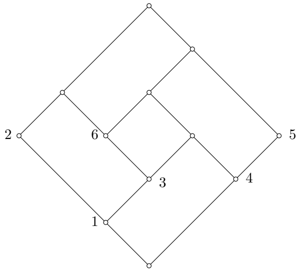

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Types of direct bases

Various unit direct bases surveyed in B-M:

- Left-minimal basis: D. Maier, The theory of relational databases, 1983
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- Dependence relation basis: B. Monjardet, Math. Soc. Sci. 1990;
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Theorem on unity

Theorem (B-M, 2010). For every finite closure system (X, ϕ), its

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Minimality

Corollary. Canonical unit basis is

- smallest
- has minimal size

among all unit direct bases for closure system (X, ϕ), ordered by inclusion.

Ordered iteration

Suppose the set of implications Σ are put into some linear order:

$$\Sigma = \langle s_1, s_2, \dots, s_n \rangle.$$

A mapping $\rho_\Sigma : P(X) \to P(X)$ associated with this ordering is called an *ordered iteration* of Σ:

- For any $Y \subseteq S$, let $Y_0 = Y$.
- If Y_k is computed and implication s_{k+1} is $A \to b$, then

$$Y_{k+1} = \begin{cases} Y_k \cup \{b\}, & \text{if } A \subseteq Y_k, \\ Y_k, & \text{otherwise.} \end{cases}$$

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$$Y_{k+1} = \begin{cases} Y_k \cup \{b\}, & \text{if } A \subseteq Y_k, \\ Y_k, & \text{otherwise.} \end{cases}$$

- Finally, $\rho_\Sigma(Y) = Y_n$.

Such iteration is utilized in *forward chaining* algorithm in logic programming.

Ordered iteration

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Example

Take Σ_C, the set of implications for $\langle J(A_{12}), \phi\rangle$, in its original order:
$2 \to 1, 6 \to 13, 3 \to 1, 5 \to 4, 14 \to 3, 123 \to 6, 1345 \to 6, 12346 \to 5$.

Consider $Y = \{2, 4\}$.

Then $\pi(Y) = \{2, 4, 1\}$, while
$\rho(Y) = \{2, 4, 1, 3, 6, 5\} = \phi(Y)$.

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Ordered direct basis

An implicational basis of $\langle X, \phi\rangle$, together with its order: $\Sigma = \langle s_1, \ldots, s_n\rangle$ is called *ordered direct*, if $\rho(Y) = \phi(Y)$, for every $Y \subseteq X$.

D-basis

OD-graph of a finite lattice:
J.B.Nation *An approach to lattice varieties of finite height*, Algebra Universalis **27** (1990), 521–543.

The full information about a finite lattice L can be compactly recorded in

- partially ordered set of join-irreducible elements $\langle J(L), \leq \rangle$;
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Example

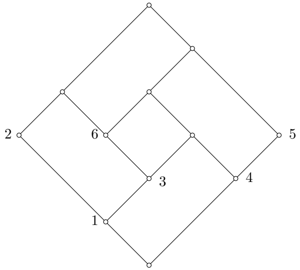

Figure: A_{12}

For lattice A_{12}, the poset of join-irreducible elements is:
$\langle J(A_{12}), \leq \rangle = \langle \{1, 2, 3, 4, 5, 6, \}, 1 \leq 2, 1 \leq 3 \leq 6, 4 \leq 5 \rangle$.

Example:continued

We say join-irreducible elements $j_1, \ldots, j_k$ form a *minimal join-cover* for $j \in J(L)$, if

- $j \leq j_1 \vee \cdots \vee j_k$,
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For example, $3 \leq 1 \vee 4$ is a minimal cover.
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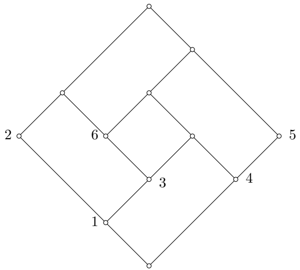

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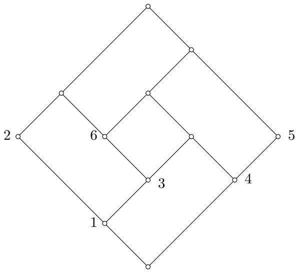

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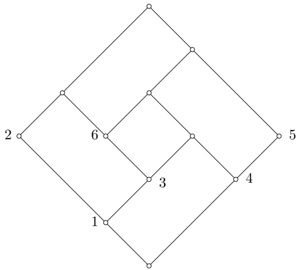

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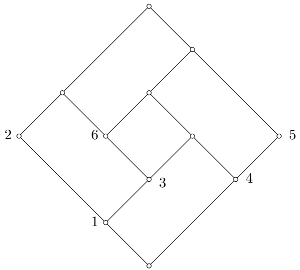

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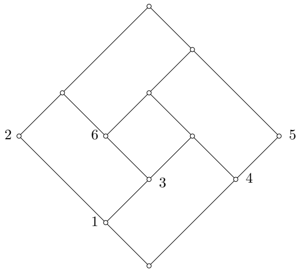

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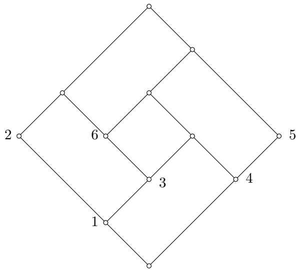

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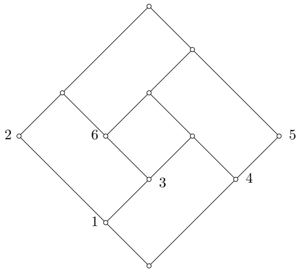

D-basis

Definition. Let $\langle X, \phi \rangle$ be a canonical closure system with $L = Cl(X, \phi)$. The set of implications Σ_D is called a *D-basis* of $\langle X, \phi \rangle$, if it is made of two parts:

- $\{a \to b : b \in \phi(\{a\})\}$; equivalently, $b \leq a$ in $\langle J(L), \leq \rangle$. This part is called *a binary part* of the basis.
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- Σ_D generates $\langle X, \phi \rangle$, i.e., *D*-basis is indeed a basis of this closure system.
- Σ_D is a subset of the canonical unit basis.
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If Σ is a any unit direct basis of $\langle X, \phi \rangle$ of size $s(\Sigma) = S$ with m implications, then

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Comparison

Unit canonical basis Σ_U for $\langle J(A_{12}), \phi \rangle$ has 13 implications.
$2 \to 1, 6 \to 1, 6 \to 3, 3 \to 1, 5 \to 4, 14 \to 3, 24 \to 3, 15 \to 3,$
$23 \to 6, 15 \to 6, 25 \to 6, 24 \to 5, 24 \to 6.$

D-basis has 9 implications.
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Canonical basis

J.L. Guiques, V. Duquenne, *Familles minimales d'implications informatives résultant d'une tables de données binares*, Math. Sci. Hum. **95** (1986), 5–18.

- Defined *critical subsets of X* for any given closure system $\langle X, \phi \rangle$.
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Computer testing

Proposition. There is no possibility, in general, to establish the order on canonical basis that would turn it into ordered direct.

Example. Canonical basis of Duquenne-Guigues on 6-element set that cannot be ordered:
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Lattice of the counterexample

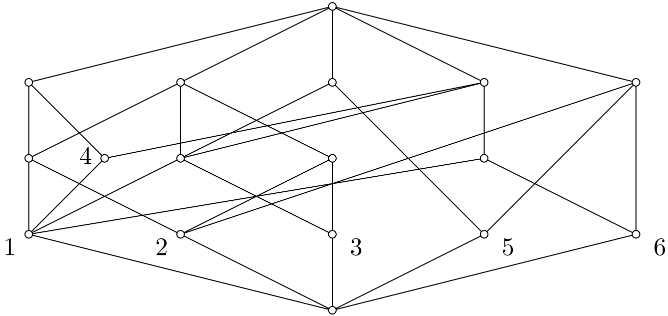
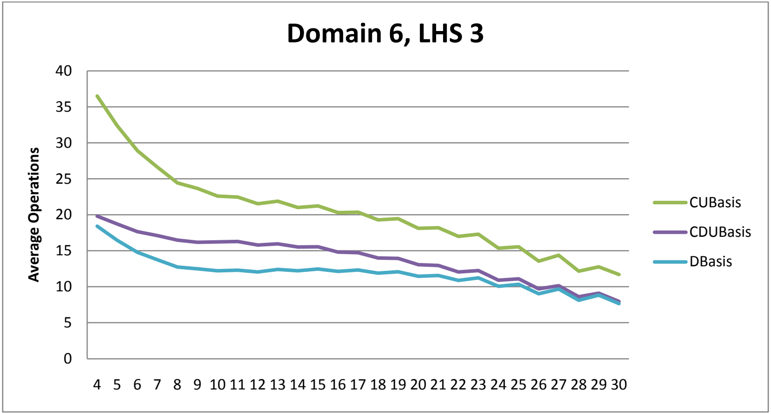


Figure: Lattice for D-G non-orderable

Three bases comparison



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Regards from Yeshiva College, New York

Figure: Yeshiva college graduating students, 2011

Regards from across America: New York-Hawai’i

Figure: Hiking in Catskill mountains, New York State