Topological categories versus categorically-algebraic topology

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Outline

Categorically-algebraic topology

- Lattice-valued topology is an approach to topology, which is based in lattice-valued sets of L. A. Zadeh and J. A. Goguen.
- There exist many different lattice-valued topological frameworks, e.g., categorical topological theories of S. E. Rodabaugh.
- Categorically-algebraic (catalg) topology is an approach to topology, which is based in category theory and universal algebra.
- Catalg topology provides a common setting for the majority of lattice-valued topological frameworks and gives convenient means of interaction between different topological theories.

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Universal topology

- Categorical topology has been initiated by H. Herrlich in 1971.
- Based in category theory, it is mostly concerned with the study of topological categories and their relationships to each other.
- In 1983, H. Herrlich started its branch called universal topology, to study topological categories via a 2-step approach: constructing fundamental topological categories first and then, singling out topological subcategories by topological (co-)axioms.

Main result

A concrete category is fibre-small and topological iff it is definable by a class of topological co-axioms in a functor-costructured category.

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Every fibre-small topological category is concretely isomorphic to the category of models of some topological theory.

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- There has been an attempt to compare topological theories of S. E. Rodabaugh and O. Wyler, which claimed to resolve completely the relationships between them.
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Ω-algebras and Ω-homomorphisms

Definition 1

Let $\Omega = (n_\lambda)_{\lambda\in\Lambda}$ be a class of cardinal numbers.

- An Ω-algebra is a pair $(A, (\omega_\lambda^A)_{\lambda\in\Lambda})$, comprising a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ (n_λ-ary primitive operations on A).
- An Ω-homomorphism $(A, (\omega_\lambda^A)_{\lambda\in\Lambda}) \xrightarrow{\varphi} (B, (\omega_\lambda^B)_{\lambda\in\Lambda})$ is a map $A \xrightarrow{\varphi} B$ such that $\varphi \circ \omega_\lambda^A = \omega_\lambda^B \circ \varphi^{n_\lambda}$ for every $\lambda \in \Lambda$.
- **Alg**(Ω) is the construct of Ω-algebras and Ω-homomorphisms.

- Every concrete category of this talk is supposed to have the underlying functor $|-|$ to the respective ground category.

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Varieties and their reducts

Definition 2

Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps.

- A variety of Ω-algebras is a full subcategory of $\mathbf{Alg}(\Omega)$, which is closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras) and $\mathcal{E}$-quotients (homomorphic images).
- The objects (resp. morphisms) of a variety are called algebras (resp. homomorphisms).

Definition 3

Given a variety $\mathbf{A}$, a reduct of $\mathbf{A}$ is a pair $(\|-\|, \mathbf{B})$, where $\mathbf{B}$ is a variety such that $\Omega_{\mathbf{B}} \subseteq \Omega_{\mathbf{A}}$, whereas $\mathbf{A} \xrightarrow{\|-\|} \mathbf{B}$ is a concrete functor.

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Powerset and topological theories

Definition 4

A catalg backward powerset theory (cabp-theory) in a category $\mathbf{X}$ (ground category of the theory) is a functor $\mathbf{X} \xrightarrow{P} \mathbf{A}^{op}$ to the dual category of a variety $\mathbf{A}$.

Definition 5

Let $\mathbf{X}$ be a category and let $\mathcal{T} = (P, (\|-\|, \mathbf{B}))$ comprise a cabp-theory $\mathbf{X} \xrightarrow{P} \mathbf{A}^{op}$ and a reduct $(\|-\|, \mathbf{B})$ of $\mathbf{A}$. A catalg topological theory (cat-theory) in $\mathbf{X}$ induced by $\mathcal{T}$ is the functor $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$, which is given by the composition $\mathbf{X} \xrightarrow{P} \mathbf{A}^{op} \xrightarrow{\|-\|^{op}} \mathbf{B}^{op}$.

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Catalg topological structures

Definition 6

Let T be a cat-theory in a category $\mathbf{X}$. $\mathbf{Top}(T)$ is the concrete category over $\mathbf{X}$, whose

objects (T-spaces) are pairs (X, τ), where X is an $\mathbf{X}$-object and τ is a subalgebra of TX (T-topology on X), and whose

morphisms (T-continuous $\mathbf{X}$-morphisms) $(X, \tau) \xrightarrow{f} (Y, \sigma)$ are $\mathbf{X}$-morphisms $X \xrightarrow{f} Y$ such that $(Tf)^{op}(\gamma) \in \tau$ for every $\gamma \in \sigma$.

Theorem 7

Given a cat-theory T, the category $\mathbf{Top}(T)$ is fibre-small and topological over $\mathbf{X}$.

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Definition 8

A topological theory in a category $\mathbf{X}$ is a functor $\mathbf{X} \xrightarrow{\mathcal{T}} \mathbf{CSLat}(\bigvee)$, where $\mathbf{CSLat}(\bigvee)$ is the variety of $\bigvee$-semilattices.

Definition 9

Let $\mathcal{T}$ be a topological theory in a category $\mathbf{X}$. $\mathbf{Top}(\mathcal{T})$ is the concrete category over $\mathbf{X}$, whose

objects ($\mathcal{T}$-models) are pairs (X, t), where X is an $\mathbf{X}$-object and t is an element of $\mathcal{T}X$, and whose

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Properties of the categories $\mathfrak{Top}(\mathcal{T})$

Theorem 10

Given a topological theory $\mathcal{T}$, the category $\mathfrak{Top}(\mathcal{T})$ is fibre-small and topological over $\mathbf{X}$.

Theorem 11

For every fibre-small topological category $(\mathbf{M}, |-|)$ over $\mathbf{X}$, there exists a topological theory $\mathcal{T}$ such that $\mathbf{M}$ is concretely isomorphic to $\mathfrak{Top}(\mathcal{T})$.

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Functor-costructured categories

Definition 12

Let $\mathbf{X}$ be a category and let $\mathbf{X}^{op} \xrightarrow{\mathfrak{T}} \mathbf{Set}$ be a functor to the category $\mathbf{Set}$ of sets. $\mathbf{Spa}(\mathfrak{T})^{op}$ is the concrete category over $\mathbf{X}$, whose

objects ($\mathfrak{T}$-spaces) are pairs (X, α), where X is an $\mathbf{X}$-object and α is a subset of $\mathfrak{T}X$, and whose

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Categories $\mathbf{Spa}(\mathfrak{T})^{op}$ are called functor-costructured categories.

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Every functor-costructured category $\mathbf{Spa}(\mathfrak{T})^{op}$ is fibre-small and topological over $\mathbf{Set}$.

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Topological co-axioms

Definition 14

Let $(\mathbf{M}, |-|)$ be a concrete category over $\mathbf{X}$.

- An $\mathbf{M}$-morphism $M_1 \xrightarrow{p} M_2$ is called identity-carried provided that $|M_1| \xrightarrow{|p|} |M_2| = X \xrightarrow{1_X} X$.
- Every identity-carried $\mathbf{M}$-morphism is called a topological co-axiom in $(\mathbf{M}, |-|)$.
- An $\mathbf{M}$-object M is said to satisfy a co-axiom $M_1 \xrightarrow{p} M_2$ provided that for every $\mathbf{M}$-morphism $M \xrightarrow{f} M_2$, there exists an $\mathbf{M}$-morphism $M \xrightarrow{g} M_1$ such that $p \circ g = f$.
- A full subcategory $\mathbf{N}$ of $\mathbf{M}$ is said to be definable by topological co-axioms in $(\mathbf{M}, |-|)$ provided that there exists a class of topological co-axioms in $(\mathbf{M}, |-|)$ such that an $\mathbf{M}$-object M satisfies each of these co-axioms iff M is an $\mathbf{N}$-object.

Properties of functor-costructured categories

Theorem 15

For a concrete category $(\mathbf{M}, |-|)$, the following are equivalent:

1. $(\mathbf{M}, |-|)$ *is fibre-small and topological;*
2. $(\mathbf{M}, |-|)$ *is concretely isomorphic to a full concretely coreflective subcategory of some functor-costructured category;*
3. $(\mathbf{M}, |-|)$ *is concretely isomorphic to a subcategory of some functor-costructured category* $\mathbf{Spa}(\mathfrak{T})^{op}$ *that is definable by topological co-axioms in* $\mathbf{Spa}(\mathfrak{T})^{op}$.

From Wyler to catalg

Lemma 16

There exists a functor $\mathbf{CSLat}(\bigvee) \xrightarrow{(-)^{\vdash}} \mathbf{CSLat}(\bigvee)^{op}$ *defined by* $(A_1 \xrightarrow{\varphi} A_2)^{\vdash} = A_1^d \xrightarrow{(\varphi^{\vdash})^{op}} A_2^d$, *where* $\varphi^{\vdash}$ *is the upper adjoint of* φ *in the sense of posets and* A_i^d *is the poset dual to* A_i.

Corollary 17

Every Wyler theory $\mathbf{X} \xrightarrow{\mathcal{T}} \mathbf{CSLat}(\bigvee)$ *provides the cat-theory* $\mathbf{X} \xrightarrow{T_{\mathcal{T}}} \mathbf{CSLat}(\bigvee)^{op}$, *which is defined through the composition* $\mathbf{X} \xrightarrow{\mathcal{T}} \mathbf{CSLat}(\bigvee) \xrightarrow{(-)^{\vdash}} \mathbf{CSLat}(\bigvee)^{op}$.

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$\mathfrak{Top}(\mathfrak{T})$ versus $\mathbf{Top}(T_{\mathfrak{T}})$

Theorem 18

1. *There is a full concrete embedding* $\mathfrak{Top}(\mathfrak{T}) \overset{F}{\hookrightarrow} \mathbf{Top}(T_{\mathfrak{T}})$ *defined by* $F((X,t) \xrightarrow{f} (Y,s)) = (X, \downarrow^d t) \xrightarrow{f} (Y, \downarrow^d s)$*, where* $\downarrow^d (-)$ *stands for the lower set in the dual partial order.*
2. *There is a concrete functor* $\mathbf{Top}(T_{\mathfrak{T}}) \xrightarrow{G} \mathfrak{Top}(\mathfrak{T})$ *defined by* $G((X,\tau) \xrightarrow{f} (Y,\sigma)) = (X, \bigvee^d \tau) \xrightarrow{f} (Y, \bigvee^d \sigma)$*, where* $\bigvee^d$ *stands for the join in the dual partial order.*
3. G *is a right-adjoint-left-inverse to* F*.*

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$\mathfrak{Top}(\mathfrak{T})$ *is concretely isomorphic to a full concretely coreflective subcategory of* $\mathbf{Top}(T_{\mathfrak{T}})$*.*

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Properties of catalg topology

Proposition 20

Given a cat-theory T, every full concretely coreflective subcategory $(\mathbf{M}, |-|)$ of the category $\mathbf{Top}(T)$ is finally closed in $\mathbf{Top}(T)$.

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Given a cat-theory T, for every concrete category $(\mathbf{M}, |-|)$, the following are equivalent:

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For a concrete category $(\mathbf{M}, |-|)$, *the following are equivalent:*

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Proof.

(1) $\Rightarrow$ (2): There is a Wyler theory $\mathcal{T}$ such that $\mathbf{M}$ is concretely isomorphic to $\mathbf{Top}(\mathcal{T})$, which is concretely isomorphic to a full concretely coreflective subcategory of $\mathbf{Top}(T_{\mathcal{T}})$, i.e., $\mathbf{M}$ is concretely isomorphic to a full concretely coreflective subcategory of $\mathbf{Top}(T_{\mathcal{T}})$.

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From catalg to Wyler

Lemma 23

Given a variety $\mathbf{A}$*, there exists a functor* $\mathbf{A}^{op} \xrightarrow{(-)^{\leftarrow\!\!\!\!\!\!\!\!\;\;\varphi}} \mathbf{CSLat}(\bigvee)$ *defined by* $(A_1 \xrightarrow{\varphi} A_2)^{\leftarrow\!\!\!\!\!\!\!\!\;\;\varphi} = (Sub(A_1))^d \xrightarrow{(\varphi^{op})^{\leftarrow}} (Sub(A_2))^d$*, where* $Sub(A_i)$ *is the* $\bigwedge$*-semilattice of subalgebras of* A_i*, whereas* $(\varphi^{op})^{\leftarrow}(S) = \{a \in A_2 \mid \varphi^{op}(a) \in S\}$.

Corollary 24

Every cat-theory $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op}$ *provides a Wyler theory* $\mathbf{X} \xrightarrow{\mathcal{T}_T} \mathbf{CSLat}(\bigvee)$ *defined by the composition* $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op} \xrightarrow{(-)^{\leftarrow\!\!\!\!\!\!\!\!\;\;\varphi}} \mathbf{CSLat}(\bigvee)$.

Theorem 25

The categories $\mathbf{Top}(T)$ *and* $\mathfrak{Top}(\mathcal{T}_T)$ *are equal.*

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Every cat-theory $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op}$ *provides a Wyler theory* $\mathbf{X} \xrightarrow{\mathcal{T}_T} \mathbf{CSLat}(\bigvee)$ *defined by the composition* $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op} \xrightarrow{(-)^{\leftarrow\!\!\!\!\varphi}} \mathbf{CSLat}(\bigvee)$.

Theorem 25

The categories $\mathbf{Top}(T)$ *and* $\mathfrak{Top}(\mathcal{T}_T)$ *are equal.*

From functor-costructured to catalg

Remark 26

Given a functor $\mathbf{X}^{op} \xrightarrow{\mathfrak{T}} \mathbf{Set}$, *there exists the functor* $\mathbf{X} \xrightarrow{T_{\mathfrak{T}}} \mathbf{Set}^{op}$ *defined as* $\mathbf{X} \xrightarrow{\mathfrak{T}^{op}} \mathbf{Set}^{op}$.

Theorem 27

The categories $\mathbf{Spa}(\mathfrak{T})^{op}$ *and* $\mathbf{Top}(T_{\mathfrak{T}})$ *are equal.*

From functor-costructured to catalg

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Given a functor $\mathbf{X}^{op} \xrightarrow{\mathfrak{T}} \mathbf{Set}$, *there exists the functor* $\mathbf{X} \xrightarrow{T_{\mathfrak{T}}} \mathbf{Set}^{op}$ *defined as* $\mathbf{X} \xrightarrow{\mathfrak{T}^{op}} \mathbf{Set}^{op}$.

Theorem 27

The categories $\mathbf{Spa}(\mathfrak{T})^{op}$ *and* $\mathbf{Top}(T_{\mathfrak{T}})$ *are equal.*

From catalg to functor-costructured

Theorem 28

1. *There is a full concrete embedding* $\mathbf{Top}(T) \overset{F}{\hookrightarrow} \mathbf{Spa}(\mathfrak{T}_T)^{op}$ *defined by* $F((X,\tau) \xrightarrow{f} (Y,\sigma)) = (X,|\tau|) \xrightarrow{f} (Y,|\sigma|)$.
2. *There is a concrete functor* $\mathbf{Spa}(\mathfrak{T}_T)^{op} \xrightarrow{G} \mathbf{Top}(T)$ *defined by* $G((X,\alpha) \xrightarrow{f} (Y,\beta)) = (X,\langle\alpha\rangle) \xrightarrow{f} (Y,\langle\beta\rangle)$, *where* $\langle S\rangle$ *stands for the subalgebra generated by a set* S.
3. G *is a right-adjoint-left-inverse to* F.

Corollary 29

$\mathbf{Top}(T)$ *is concretely isomorphic to a full concretely coreflective subcategory of* $\mathbf{Spa}(\mathfrak{T}_T)^{op}$.

From catalg to functor-costructured

Theorem 28

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$\mathbf{Top}(T)$ *is concretely isomorphic to a full concretely coreflective subcategory of* $\mathbf{Spa}(\mathfrak{T}_T)^{op}$.

Final remarks

- Following the rapid development of both catalg topology and categorical topology, this talk clarified the relationships between these two approaches to the study of topological structures.
- The setting of topological theories of O. Wyler is more general than the catalg one, in the sense that every category of the form $\mathbf{Top}(T)$ can be reconstructed completely through a suitable category of the form $\mathfrak{Top}(\mathcal{T})$, whereas the converse way requires the application of some topological co-axioms, whose ultimate description in each case can be problematic.
- In concrete applications, catalg framework appears to be more suitable, since it provides the underlying algebraic structures of the topological structures, whereas topological theories of O. Wyler contain the information on their ground category only.

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Thank you for your attention!