On l-implicative-groups and associated algebras of logic

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CONTENT

1. Introduction

• We have introduced and studied in 2009 the *l*-implicative-group as a term equivalent definition of the *l*-**group** coming from algebras of logic:

$$\begin{array}{ccc} \mathbf{l-implicative-groups} & \Longleftrightarrow & \mathbf{l-groups} \\ \Updownarrow & & \Updownarrow \\ \mathbf{pseudo-Wajsberg\ algebras} & \Longleftrightarrow & \mathbf{pseudo-MV\ algebras} \end{array}$$

1. Introduction

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$$\begin{array}{ccc} \mathbf{l-implicative-groups} & \Longleftrightarrow & \mathbf{l-groups} \\ \Updownarrow & & \Updownarrow \\ \mathbf{pseudo-Wajsberg\ algebras} & \Longleftrightarrow & \mathbf{pseudo-MV\ algebras} \end{array}$$

• We have studied the **algebras of logic** obtained by restricting the *l*-**group**/***l*-implicative-group** operations:
- on G^- and G^+,
- on $[u', 0] \subset G^-$ and $[0, u] \subset G^+$,
- on $\{-\infty\} \cup G^-$ and $G^+ \cup \{+\infty\}$.

Now:

- we study the normal **filters/ideals** and the compatible **deductive systems** on l-**group**/l-**implicative-group** level and on corresponding **algebras of logic** levels and their connections, Now:

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- we study the representability on l-**group**/l-**implicative-group** level and on some **algebras of logic** levels and their connections.

2. Preliminaries

2.1 Examples of term equivalent **involutive** algebras of logic:

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Pseudo-Wajsberg algebras are term equivalent to **pseudo-MV algebras:**

- **left-pseudo-Wajsberg algebras** $\iff$ **left-pseudo-MV algebras**

$(A^L, \to^L, \rightsquigarrow^L, ^-, ^\sim, 1)$

$(A^L, \odot, ^-, ^\sim, 0, 1)$

$x \odot y = (x \to^L y^-)^\sim = (y \rightsquigarrow^L x^\sim)^-$
$0 = 1^- = 1^\sim$

$x \to^L y = (x \odot y^\sim)^-$
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- **right-pseudo-Wajsberg algebras** $\Longleftrightarrow$ **r.-pseudo-MV algebras**

$(A^R, \rightarrow^R, \rightsquigarrow^R, ^-, ^\sim, 0)$ $\quad$ $(A^R, \oplus, ^-, ^\sim, 0, 1)$

$x \oplus y = (x \rightarrow^R y^-)^\sim = (y \rightsquigarrow^R x^\sim)^-$
$1 = 0^- = 0^\sim$

$x \rightarrow^R y = (x \oplus y^\sim)^-$
$x \rightsquigarrow^R y = (y^- \oplus x)^\sim$

2.2 Examples of categorically equivalent non-commutative algebras of logic

Pseudo-BCK algebras with pP(pseudo-product)/pS(ps.-sum) are categorically equivalent to porims (= partially-ordered residuated integral monoids) :

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- **left-pseudo-BCK(pP) algebras** $\Longleftrightarrow$ **left-porims**

$(A^L, \leq, \rightarrow^L, \rightsquigarrow^L, 1)$ $\quad$ $(A^L, \leq, \odot, 1)$

(pP) $\exists\ x \odot y$
$= \min\{z \mid x \leq y \rightarrow^L z\}$
$= \min\{z \mid y \leq x \rightsquigarrow^L z\}$

(pR) $\exists\ y \rightarrow^L z$
$= \max\{x \mid x \odot y \leq z\}$
$\exists\ x \rightsquigarrow^L z$
$= \max\{y \mid x \odot y \leq z\}$

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$= \min\{z \mid y \leq x \rightsquigarrow^L z\}$

(pR) $\exists\ y \to^L z$
$= \max\{x \mid x \odot y \leq z\}$
$\exists\ x \rightsquigarrow^L z$
$= \max\{y \mid x \odot y \leq z\}$

- **right-pseudo-BCK(pS) algebras** $\Longleftrightarrow$ **right-porims**

$(A^R, \leq, \to^R, \rightsquigarrow^R, 0)$ $\qquad$ $(A^R, \leq, \oplus, 0)$

(pS) $\exists\ x \oplus y$
$= \max\{z \mid x \geq y \to^R z\}$
$= \max\{z \mid y \geq x \rightsquigarrow^R z\}$

(pcorR) $\exists\ y \to^R z$
$= \min\{x \mid x \oplus y \geq z\}$
$\exists\ x \rightsquigarrow^R z$
$= \min\{y \mid x \oplus y \geq z\}$

Remark:

• All above **left-algebras of logic** verify the following **property of residuation**, which is a **Galois connection**:

$$x \odot y \leq z \Longleftrightarrow x \leq y \to^{L} z \Longleftrightarrow y \leq x \leadsto^{L} z.$$

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• All above **right-algebras of logic** verify the following **dual property of residuation**, which is a **Galois connection**:

$$x \oplus y \geq z \Longleftrightarrow x \geq y \to^{R} z \Longleftrightarrow y \geq x \rightsquigarrow^{R} z.$$

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Pair of **Galois dual** *algebras*

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Note that usually in group theory
and sometimes in algebras of logic theory
(as for example in the recent book on residuated lattices of Galatos, Jipsen, Kowalski, Ono 2007)
the following operators are used:

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i.e. the implication $\rightarrow$ is the **inverse** of $/$.

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- in left-algebras of logic we have:

$x \leq y \iff x \rightarrow^L y = 1 \iff x \rightsquigarrow^L y = 1$ and

- in **right-algebras** of logic we have:

$x \geq y \iff x \rightarrow^R y = 0 \iff x \rightsquigarrow^R y = 0$

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- in left-algebras of logic we have:

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- in **right-algebras** of logic we have:

$x \geq y \Longleftrightarrow x \rightarrow^R y = 0 \Longleftrightarrow x \rightsquigarrow^R y = 0$

- the operation $\rightarrow$ is associated to the **first** argument of $\odot$ ($\oplus$) and
- the operation $\rightsquigarrow$ is associated to the **second** argument of $\odot$ ($\oplus$).

2.3 The group level: Groups, implicative-groups

Theorem The following algebras are termwise equivalent:

implicative-groups $\Longleftrightarrow$ **groups**

$(G, \rightarrow, \rightsquigarrow, 0)$ — $(G, +, -, 0)$

(I1),(I2),(I3),(I4) — (G1),(G2),(G3)

$$-x = x \rightarrow 0 = x \rightsquigarrow 0$$

$$x + y = -(x \rightarrow (-y)) = -(y \rightsquigarrow (-x))$$

$$x \rightarrow y = -(x + (-y)) = y - x,$$

$$x \rightsquigarrow y = -((-y) + x) = -x + y$$

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$-x = x \rightarrow 0 = x \rightsquigarrow 0$		$x \rightarrow y = -(x + (-y)) = y - x,$
$x + y = -(x \rightarrow (-y)) = -(y \rightsquigarrow (-x))$		$x \rightsquigarrow y = -((-y) + x) = -x + y$

where :

(I1) $y \rightarrow z = (z \rightarrow x) \rightsquigarrow (y \rightarrow x),\ y \rightsquigarrow z = (z \rightsquigarrow x) \rightarrow (y \rightsquigarrow x),$

(I2) $0 \rightarrow x = x = 0 \rightsquigarrow x,$

(I3) $x = y \Longleftrightarrow x \rightarrow y = 0 \Longleftrightarrow x \rightsquigarrow y = 0,$

(I4) $x \rightarrow 0 = x \rightsquigarrow 0.$

2.4 The po-group level: po-groups, po-implicative-groups

Theorem The following structures are termwise equivalent:

po-implicative-groups	$\Longleftrightarrow$	**po-groups**
$(G, \leq, \to, \rightsquigarrow, 0)$		$(G, \leq, +, -, 0)$
$\leq$ partial order		$\leq$ partial order
(I1),(I2),(I3),(I4)		(G1),(G2),(G3)
(I5)		**(G4)**

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$\leq$ partial order		$\leq$ partial order
(I1),(I2),(I3),(I4)		(G1),(G2),(G3)
(I5)		**(G4)**

where :

(I5) $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$.

Remarks:

• **Groups** and **implicative-groups** verify
the **residuation property** (which is a **Galois connection**):

$$x + y = z \Longleftrightarrow x = y \to z \Longleftrightarrow y = x \rightsquigarrow z,$$

(see Galatos, Jipsen, Kowalski, Ono, 2007, page 160)

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• **Po-groups** and **po-implicative-groups** verify
the two **residuation properties** (which are **Galois connections**):

$$x + y \leq z \Leftrightarrow x \leq y \rightarrow z \Leftrightarrow y \leq x \rightsquigarrow z$$

and **dually**:

$$x + y \geq z \Leftrightarrow x \geq y \rightarrow z \Leftrightarrow y \geq x \rightsquigarrow z.$$

Remarks:

• **Groups** and **implicative-groups** verify
the **residuation property** (which is a **Galois connection**):

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• **Po-groups** and **po-implicative-groups** verify
the two **residuation properties** (which are **Galois connections**):

$$x + y \leq z \Leftrightarrow x \leq y \to z \Leftrightarrow y \leq x \rightsquigarrow z$$

and **dually**:

$$x + y \geq z \Leftrightarrow x \geq y \to z \Leftrightarrow y \geq x \rightsquigarrow z.$$

We say they are **Galois dual** *algebras*!

2.5 Connections between the *l*-implicative-group level G and the algebras of logic:

• on G^- and G^+ level:

Theorem

Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an *l*-implicative-group.

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Theorem

Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an *l*-implicative-group.

(1). Define, for all $x, y \in G^-$:

$$x \rightarrow^L y \overset{def}{=} (x \rightarrow y) \wedge 0,$$

$$x \rightsquigarrow^L y \overset{def}{=} (x \rightsquigarrow y) \wedge 0.$$

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Then,

$$\mathcal{G}^L = (G^-, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{1} = 0)$$

is a **left-pseudo-BCK(pP) lattice**

with the **pseudo-product** $\odot = +$**, lattice that is distributive,**

verifying conditions (pC) and (*), where: for all $x, y, z \in G^-$**,**

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is a **left-pseudo-BCK(pP) lattice**

with the **pseudo-product** $\odot = +$, **lattice that is distributive, verifying conditions (pC) and (*), where: for all** $x, y, z \in G^-$,

(pC) $\quad x \vee y = (x \rightsquigarrow^L y) \to^L y \;=\; (x \to^L y) \rightsquigarrow^L y,$

(*) $\quad (x \odot z) \to^L (y \odot z) = x \to^L y,\ (z \odot x) \rightsquigarrow^L (z \odot y) = x \rightsquigarrow^L y.$

Connections between the *l*-implicative-group level G and the algebras of logic:

- On $[u', 0]$ and $[0, u]$ level:

Corollary (see Georgescu, A.I., 1999)

Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an *l*-implicative-group.

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Corollary (see Georgescu, A.I., 1999)
Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an l-implicative-group.
(1). Let us take the interior point $u' < 0$ from G^- and consider the interval $[u', 0] \subset G^-$.
Then,

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(1). Let us take the interior point $u' < 0$ from G^- and consider the interval $[u', 0] \subset G^-$.
Then,

$$\mathcal{G}_1^L = ([u', 0], \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{0} = u', \mathbf{1} = 0)$$

is a bounded left-pseudo-BCK(pP) lattice with condition (pC), hence is an equivalent definition of **left-pseudo-Wajsberg algebra**.

Connections between the *l*-implicative-group level G and the algebras of logic:

- On $\{-\infty\} \cup G^{-}$ and $G^{+} \cup \{\infty\}$ level:

Corollary (see A. Di Nola, G. Georgescu, A.I., 2002; for the commutative case, see R. Cignoli, A. Torrens, 1997)
Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an *l*-implicative-group.

Connections between the l-implicative-group level G and the algebras of logic:

• On $\{-\infty\} \cup G^-$ and $G^+ \cup \{\infty\}$ level:

Corollary (see A. Di Nola, G. Georgescu, A.I., 2002; for the commutative case, see R. Cignoli, A. Torrens, 1997)

Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an l-implicative-group.

(1). Let us consider an exterior point $-\infty$, distinct from the elements of G. Define $G^-_{-\infty} = \{-\infty\} \cup G^-$ and extend the operations from G^- to $G^-_{-\infty}$:

$$x \rightarrow^L y = \begin{cases} (x \rightarrow y) \wedge 0, & \text{if} \quad x, y \in G^- \\ -\infty, & \text{if} \quad x \in G^-,\ y = -\infty \\ 0, & \text{if} \quad x = -\infty, \end{cases}$$

$$x \rightsquigarrow^L y = \begin{cases} (x \rightsquigarrow y) \wedge 0, & \text{if} \quad x, y \in G^- \\ -\infty, & \text{if} \quad x \in G^-,\ y = -\infty \\ 0, & \text{if} \quad x = -\infty, \end{cases}$$

$$x \odot y = \begin{cases} x + y, & \text{if} \quad x, y \in G^- \\ -\infty, & \text{if} \quad otherwise. \end{cases}$$

We extend $\leq$ by puting: $-\infty \leq x$, for any $x \in G^-_{-\infty}$.

Then,

$$x \odot y = \begin{cases} x + y, & \text{if} \quad x, y \in G^- \\ -\infty, & \text{if} \quad otherwise. \end{cases}$$

We extend $\leq$ by puting: $-\infty \leq x$, for any $x \in G^-_{-\infty}$.
Then,

$$\mathcal{G}_2^L = (G^-_{-\infty}, \wedge, \vee, \odot, \rightarrow^L, \rightsquigarrow^L, \mathbf{0} = -\infty, \mathbf{1} = 0)$$

is a **left-pseudo-product algebra**.

3.**Normal** filters/ideals, **compatible** deductive systems

3.1 Filters/ideals and deductive systems

- **On algebras of logic level:**

Proposition (see Bușneag, Rudeanu, 2010 for a more general result in the commutative case)
(1). Let $\mathcal{A}_r^L = (A^L, \leq, \odot, 1)$ be a left-porim and let $\mathcal{A}_t^L = (A^L, \leq, \rightarrow^L, \rightsquigarrow^L, 1)$ be the categorically equivalent left-pseudo-BCK(pP) algebra.
Then,

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Then,
the $(\odot)$-**filters** of $\mathcal{A}_r^L$ **coincide** with
the $(\rightarrow^L, \rightsquigarrow^L)$**-deductive systems** of $\mathcal{A}_t^L$.

- **On po-group/po-implicative-group level:**

· In **po-groups**, we have the **convex po-subgroup** (= (+)-filter-ideal).
· Analogously, in po-implicative-groups, we define the convex po-subimplicative-group (= $(\rightarrow, \rightsquigarrow)$-filter-ideal) as follows:

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· In **po-groups**, we have the **convex po-subgroup** (= (+)-filter-ideal).
· Analogously, in po-implicative-groups, we define the convex po-subimplicative-group (= $(\to, \rightsquigarrow)$-filter-ideal) as follows:

Definition

Let $\mathcal{G} = (G, \leq, \to, \rightsquigarrow, 0)$ be a po-implicative-group.
A **convex po-subimplicative-group** of $\mathcal{G}$ is a subset $S \subseteq G$ which satisfies:
· $0 \in S$,
· $x, y \in S$ imply $x \to y,\ x \rightsquigarrow y \in S$,
· $a, b \in S$ and $a \leq x \leq b$ imply $x \in S$.

Obviously, we have:

Proposition

Let $\mathcal{G}_g = (G, \leq, +, -, 0)$ be a **po-group** and let $\mathcal{G}_{ig} = (G, \leq, \rightarrow, \rightsquigarrow, 0)$ be the term equivalent po-implicative-group.

Then,

Obviously, we have:

Proposition

Let $\mathcal{G}_g = (G, \leq, +, -, 0)$ be a **po-group** and let $\mathcal{G}_{ig} = (G, \leq, \to, \rightsquigarrow, 0)$ be the term equivalent po-implicative-group.

Then,

the **convex po-subgroups** of $\mathcal{G}_g$ coincide with the **convex po-subimplicative-groups** of $\mathcal{G}_{ig}$.

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Definition

Let $\mathcal{G} = (G, \leq, \rightarrow, \rightsquigarrow, 0)$ be a po-implicative-group.

A **deductive system** of $\mathcal{G}$ is a subset $S \subseteq G$ which satisfies:

· $0 \in S$;

·(a) $x \in S$, $x \rightarrow y \in S$ (or $x \rightsquigarrow y \in S$) imply $y \in S$,

(b) $x \in S$ implies $x \rightarrow 0 = x \rightsquigarrow 0 \in S$;

· $a, b \in S$ and $a \leq x \leq b$ imply $x \in S$.

Proposition

Let $\mathcal{G}_g = (G, \leq, +, -, 0)$ be a **po-group** and let $\mathcal{G}_{ig} = (G, \leq, \rightarrow, \rightsquigarrow, 0)$ be the term equivalent **po-implicative-group**.

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Proposition
Let $\mathcal{G}_g = (G, \leq, +, -, 0)$ be a **po-group** and let $\mathcal{G}_{ig} = (G, \leq, \rightarrow, \rightsquigarrow, 0)$ be the term equivalent **po-implicative-group**.
Then,
the **convex po-subgroups** of $\mathcal{G}_g$ coincide with the **deductive systems** of $\mathcal{G}_{ig}$.

Resuming:

In **po-groups**/**po-implicative-groups**, we have:

$$\textbf{convex po} - \textbf{subgroups} = \textbf{deductive systems}$$
$$= \textbf{convex po} - \textbf{subimplicative} - \textbf{groups}$$

- **Back to algebras of logic level:**

Inspired from po-implicative-group level, we introduce the following notion:

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Definition

(1). Let $\mathcal{A}^L = (A^L, \leq, \rightarrow^L, \rightsquigarrow^L, 1)$ be a left-pseudo-BCK algebra.
A **$(\rightarrow^L, \rightsquigarrow^L)$-filter** of $\mathcal{A}^L$ is a subset $F \subseteq A^L$ which satisfies:

· $1 \in F$,

· $x, y \in F$ imply $x \rightarrow^L y, x \rightsquigarrow^L y \in F$,

· $x \in F$ and $x \leq y$ imply $y \in F$.

Proposition

(1). Let $\mathcal{A}_r^L = (A^L, \leq, \odot, 1)$ be a **left-porim** and let $\mathcal{A}_t^L = (A^L, \leq, \rightarrow^L, \rightsquigarrow^L, 1)$ be the categorically equivalent **left-pseudo-BCK(pP) algebra**.

Then,

Proposition

(1). Let $\mathcal{A}_r^L = (A^L, \leq, \odot, 1)$ be a **left-porim** and let $\mathcal{A}_t^L = (A^L, \leq, \to^L, \rightsquigarrow^L, 1)$ be the categorically equivalent **left-pseudo-BCK(pP) algebra**.

Then,

any $(\odot)$-**filter** of $\mathcal{A}_r^L$ is a $(\to^L, \rightsquigarrow^L)$**-filter** of $\mathcal{A}_t^L$.

The converse is not true.

Resuming:

(1). In left-porims/left-pseudo-BCK(pP) algebras, we have:

($\odot$)-filters $=$ ($\rightarrow^L, \rightsquigarrow^L$)-deductive systems $\subseteq$ ($\rightarrow^L, \rightsquigarrow^L$)-filters

Connections results in **lattice-ordered** case:

l-implicative-group $\Longleftrightarrow$ **l-group**

$(G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ $\qquad$ $(G, \vee, \wedge, +, -, 0)$

Connections results in **lattice-ordered** case:

***l*-implicative-group**		$\Longleftrightarrow$	***l*-group**
$(G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$			$(G, \vee, \wedge, +, -, 0)$
$S \subseteq G$			$S \subseteq G$
convex *l*-subimplicative-group			convex *l*-subgroup
$\Downarrow G^-$	$G^+ \Downarrow$	$\Downarrow G^-$	$G^+ \Downarrow$
$S \cap G^-$	$S \cap G^+$	$S \cap G^-$	$S \cap G^+$
$(\rightarrow^L, \rightsquigarrow^L)$-filter	$(\rightarrow^R, \rightsquigarrow^R)$-ideal	$(\odot)$-filter	$(\oplus)$-ideal

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$S \subseteq G$	
deductive system	
$\Downarrow G^-$	$G^+ \Downarrow$
$S \cap G^-$	$S \cap G^+$
$(\rightarrow^L, \rightsquigarrow^L)$-d.s.	$(\rightarrow^R, \rightsquigarrow^R)$-d.s.

Resuming Theorem:
Let $\mathcal{G}$ be an *l*-**group**/*l*-**implicative-group**.
Let $S \subseteq G$ be a **convex** *l*-**subgroup/deductive system/**
convex *l*-**subimplicative-group**.
Then:

Resuming Theorem:
Let $\mathcal{G}$ be an *l*-**group**/***l*-implicative-group**.
Let $S \subseteq G$ be a **convex *l*-subgroup/deductive system/ convex *l*-subimplicative-group**.
Then:
(1). $S^L = S \cap G^-$ is in the same time:
$(\odot)$-filter and $(\rightarrow^L, \rightsquigarrow^L)$-deductive system and $(\rightarrow^L, \rightsquigarrow^L)$-filter.

Normal filters/ideals, **compatible** deductive systems

- **On algebras of logic level**

We introduce the following:

Definition

(1). Let $\mathcal{M}^L = (M^L, \leq, \odot, 1)$ be a left-poim (= partially-ordered integral left-monoid).

Normal filters/ideals, **compatible** deductive systems

3.2 Normal filters/ideals and compatible deductive systems

- **On algebras of logic level**

We introduce the following:

Definition

(1). Let $\mathcal{M}^L = (M^L, \leq, \odot, 1)$ be a left-poim
(= partially-ordered integral left-monoid).
A $(\odot)$-filter S^L of $\mathcal{M}^L$ is **normal** if the following condition (N^L) holds:

$$(N^L) \qquad \textit{for any } x \in M^L, \;\; S^L \odot x = x \odot S^L.$$

Recall the following:

Definition (see Kühr, 2007)

(1). Let $\mathcal{A}^L = (A^L, \leq, \rightarrow^L, \rightsquigarrow^L, 1)$ be a left-pseudo-BCK algebra. A $(\rightarrow^L, \rightsquigarrow^L)$-**deductive system** S^L of $\mathcal{A}^L$ is **compatible** if the following condition (C^L) holds:

$$(C^L) \qquad \textit{for any } x, y \in A^L, \;\; x \rightarrow^L y \in S^L \Longleftrightarrow x \rightsquigarrow^L y \in S^L.$$

We have obtained the following result concerning normal filters/ideals and compatible deductive systems:

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Theorem

(1). Let $\mathcal{A}^L = (A^L, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, 1)$ be a left-pseudo-BCK(pP) lattice with **pseudo-product** $\odot$, verifying (pdiv):

$$(pdiv)\ (pseudo-divisibility)\ \ x \wedge y = (x \rightarrow^L y) \odot x = x \odot (x \rightsquigarrow^L y)$$

(or let $\mathcal{A}^L_m = (A^L, \wedge, \vee, \odot, 1)$ be a **left-*l*-rim** verifying (pdiv)).

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(or, equivalently, a **$(\odot)$-filter** of $\mathcal{A}^L_m$).
Then
S^L is **compatible** if and only if is **normal**, i.e.

$$(C^L) \Longleftrightarrow (N^L).$$

Open problem:

Find an example of left-pseudo-BCK(pP) lattice **not verifying** (pdiv), which has a $(\odot)$-filter that is:

- **normal** but not **compatible**, or is
- **compatible** but not **normal**.

- **On po-group/po-implicative-group level**

Recall the following:

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Let $\mathcal{G}_g = (G, \leq, +, -, 0)$ be a **po-group**.
A **convex po-subgroup** S of $\mathcal{G}_g$ is **normal** if the following condition (N_g) holds:

$$(N_g) \qquad \textit{for any } g \in G, \;\; S + g = g + S.$$

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We introduce now the following:

- **Definition**

Let $\mathcal{G}_{ig} = (G, \leq, \rightarrow, \rightsquigarrow, 0)$ be a po-implicative-group.
A **deductive system** S of $\mathcal{G}_{ig}$ is **compatible** if the following condition (C_{ig}) holds:

$$(C_{ig}) \qquad \textit{for any } x, y \in G, \;\; x \rightarrow y \in S \Longleftrightarrow x \rightsquigarrow y \in S.$$

We know already that
the **convex po-subgroups** of $\mathcal{G}_g$ coincide with
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Moreover, we obtain now the following:

Theorem

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The result of above Theorem (formulated in partially-ordered case) remains valid in lattice-ordered case, i.e. we have:

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Let $\mathcal{G}_{ig} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an l-implicative-group (or let $\mathcal{G}_g = (G, \vee, \wedge, +, -, 0)$ be an l-group).

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Normal filters/ideals, **compatible** deductive systems

3.3 Connections between l-group/l-implicative-group level and algebras of logic:

- **On G^- and G^+ level:**

Theorem

Let $\mathcal{G}_{ig} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an l-implicative-group (or let $\mathcal{G}_g = (G, \vee, \wedge, +, -, 0)$ be an l-**group**).

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Then,
(1). $S^L = S \cap G^-$ is a **compatible $(\rightarrow^L, \rightsquigarrow^L)$-deductive system**
of the left-pseudo-BCK(pP) lattice
$\mathcal{G}^L = (G^-, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{1} = 0)$
(or, equivalently, S^L is a **normal** $(\odot)$-filter of the left-l-rim
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In other words, the above **Theorem** says that:

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- normality/compatibility at l-group/l-implicative-group G level is **inherited** by the algebras obtained by restricting the l-group/l-implicative-group operations to the negative cone G^- and to the positive cone G^+.

In other words, the above **Theorem** says that:
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- the equivalence

$$(C_{ig}) \Longleftrightarrow (N_g)$$

(**compatible** if and only if **normal**),
existing at l-group/l-implicative-group level is **preserved** by the algebras obtained by restricting the l-group/l-implicative-group operations to G^- and to G^+, i.e. it induces the dual equivalences:

$$(C^L) \Longleftrightarrow (N^L) \qquad \text{and} \qquad (C^R) \Longleftrightarrow (N^R).$$

- **On $[u', 0]$ and $[0, u]$ level:**

$$\textit{Similar results}.$$

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$$\textit{Similar results}.$$

- **On $\{-\infty\} \cup G^-$ and $G^+ \cup \{+\infty\}$ level:**

$$\textit{Similar results}.$$

4. Representability

4.1 Representable algebras of logic

(1). Recall (C.J. van Alten, 2002) that:
A **left**-pseudo-BCK(pP) lattice
$\mathcal{A}^L = (A^L, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, 1)$ with the pseudo-product $\odot$

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$\mathcal{A}^{\mathcal{L}} = (A^L, \wedge, \vee, \odot, \rightarrow^L, \rightsquigarrow^L, 1)$)
is **representable if and only if** it satisfies the identity:

$$(x \rightsquigarrow^L y) \vee (([((y \rightsquigarrow^L x) \rightsquigarrow^L z) \rightsquigarrow^L z] \rightarrow^L w) \rightarrow^L w) = 1, \quad (1)$$

or the identity

$$(x \rightarrow^L y) \vee (([((y \rightarrow^L x) \rightarrow^L z) \rightarrow^L z] \rightsquigarrow^L w) \rightsquigarrow^L w) = 1, \quad (2)$$

for all $x, y, z, w \in A^L$.

4.2 Representable l-groups/l-implicative-groups

Recall (M. Andersen, T. Feil, 1988, Theorem 4.1.1):
Let $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$ be an l-**group**.
The following are equivalent:
(a) $\mathcal{G}$ is **representable**.
(b) $2(a \wedge b) = 2a \wedge 2b$;
(b^d) $2(a \vee b) = 2a \vee 2b$.
(c) $a \wedge (-b - a + b) \leq 0$;
(c^d) $a \vee (-b - a + b) \geq 0$.

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(c) $a \wedge (-b - a + b) \leq 0$;
(c^d) $a \vee (-b - a + b) \geq 0$.
(d) Each polar subgroup is normal.
(e) Each minimal prime subgroup is normal.
(f) For each $a \in G$, $a > 0$, $a \wedge (-b + a + b) > 0$, for all $b \in G$;
(f^d) For each $a \in G$, $a < 0$, $a \vee (-b + a + b) < 0$, for all $b \in G$.
Note that ^{d} means "dual".

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Theorem

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(b^d) $2(a \vee b) = 2a \vee 2b$,

($b1^d$) $(b \rightarrow a) \vee (a \rightsquigarrow b) \geq 0 \vee [(b \rightsquigarrow a) \rightsquigarrow (b \rightarrow a)]$,

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($b2^d$) $(b \rightsquigarrow a) \vee (a \rightarrow b) \geq 0 \vee [(b \rightarrow a) \rightarrow (b \rightsquigarrow a)]$.

(c) $a \wedge (-b - a + b) \leq 0$,

(c1) $(x \rightsquigarrow y) \wedge (([((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z] \rightarrow w) \rightarrow w) \leq 0$,

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Let $\mathcal{G}_g = (G, \vee, \wedge, +, -, 0)$ be an l-**group** (or, equivalently, let $\mathcal{G}_{ig} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be the l-**implicative-group**).

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(b1) $(b \to a) \wedge (a \rightsquigarrow b) \leq 0 \wedge [(b \rightsquigarrow a) \rightsquigarrow (b \to a)]$,

(b2) $(b \rightsquigarrow a) \wedge (a \to b) \leq 0 \wedge [(b \to a) \to (b \rightsquigarrow a)]$.

(b^d) $2(a \vee b) = 2a \vee 2b$,

($\mathrm{b1}^d$) $(b \to a) \vee (a \rightsquigarrow b) \geq 0 \vee [(b \rightsquigarrow a) \rightsquigarrow (b \to a)]$,

($\mathrm{b2}^d$) $(b \rightsquigarrow a) \vee (a \to b) \geq 0 \vee [(b \to a) \to (b \rightsquigarrow a)]$.

(c) $a \wedge (-b - a + b) \leq 0$,

(c1) $(x \rightsquigarrow y) \wedge (([((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z] \to w) \to w) \leq 0$,

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4.3 Connections between the *l*-group level and the algebras of logic:

• On G^- and G^+ level

We obtained the following results:

Theorem

Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be a **representable** *l*-implicative-group.
Then,

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Let $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be a **representable** l-implicative-group.
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(1). $\mathcal{G}^L = (G^-, \wedge, \vee, \to^L, \rightsquigarrow^L, \mathbf{1} = 0)$ is a **representable** left-pseudo-BCK(pP) lattice (with the pseudo-product $\odot = +$).

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(1). the **representable** left-pseudo-BCK(pP) lattice $\mathcal{G}^L = (G^-, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{1} = 0)$ with the pseudo-product $\odot = +$ verifies also the following conditions: for all $a, b \in G^-$,

Theorem
Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be a **representable** l-implicative-group.
Then,
(1). the **representable** left-pseudo-BCK(pP) lattice
$\mathcal{G}^L = (G^-, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{1} = 0)$ with the pseudo-product $\odot = +$
verifies also the following conditions: for all $a, b \in G^-$,

(i) $(a \vee b)^2 = a^2 \vee b^2$, i.e. $(a \vee b) \odot (a \vee b) = (a \odot a) \vee (b \odot b)$,
(ii) Condition (i) is equivalent with condition

$$[b \rightarrow^L (a \rightsquigarrow^L (a \odot a))] \vee [a \rightsquigarrow^L (b \rightarrow^L (b \odot b))] = \mathbf{1}. \qquad (3)$$

(iii) $(b \rightarrow^L a) \vee (a \rightsquigarrow^L b) = \mathbf{1}$,
(iv) Condition (iii) implies condition (3).

Proposition

Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an *l*-implicative-group.

(1). If $\mathcal{G}$ verifies the condition:

(b1$^{d''}$) for all $a, b \in G$, $(b \rightarrow a) \vee (a \rightsquigarrow b) \geq 0$,

then

Proposition
Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an l-implicative-group.
(1). If $\mathcal{G}$ verifies the condition:
(b1^{d}") for all $a, b \in G$, $(b \rightarrow a) \vee (a \rightsquigarrow b) \geq 0$,
then
the left-pseudo-BCK(pP) lattice $\mathcal{G}^L = (G^-, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{1} = 0)$
verifies the condition (iii) from above Theorem, namely:
(iii) for all $a, b \in G^-$, $(b \rightarrow^L a) \vee (a \rightsquigarrow^L b) = \mathbf{1} = 0$.

Thank you for your attention !