On /-implicative-groups and associated algebras of logic

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1. Introduction

- We have introduced and studied in 2009 the \(l\)-implicative-group as a term equivalent definition of the \(l\)-group coming from algebras of logic:

\[
\begin{align*}
\text{\(l\)-implicative groups} & \iff \text{\(l\)-groups} \\
\text{pseudo-Wajsberg algebras} & \iff \text{pseudo-MV algebras}
\end{align*}
\]
1. Introduction

- We have introduced and studied in 2009 the \( l \)-implicative-group as a term equivalent definition of the \( l \)-group coming from algebras of logic:

\[
\begin{align*}
\text{l-implicative-group} & \iff \text{l-group} \\
\iff \text{pseudo-Wajsberg algebras} & \iff \text{pseudo-MV algebras}
\end{align*}
\]

- We have studied the algebras of logic obtained by restricting the \( l \)-group/\( l \)-implicative-group operations:
  - on \( G^- \) and \( G^+ \),
  - on \([u',0] \subset G^-\) and \([0,u] \subset G^+\),
  - on \( \{-\infty\} \cup G^- \) and \( G^+ \cup \{+\infty\} \).
Now:

- we study the normal filters/ideals and the compatible deductive systems on \(-\)-group/\(-\)-implicative-group level and on corresponding algebras of logic levels and their connections,
Now:

- we study the normal filters/ideals and the compatible deductive systems on \(-\text{group}/-\text{implicative-group}\) level and on corresponding algebras of logic levels and their connections,

- we study the representability on \(-\text{group}/-\text{implicative-group}\) level and on some algebras of logic levels and their connections.
2. Preliminaries
2.1 Examples of term equivalent involutive algebras of logic:

Pseudo-Wajsberg algebras are term equivalent to pseudo-MV algebras:
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2.1 Examples of term equivalent involutive algebras of logic:

Pseudo-Wajsberg algebras are term equivalent to pseudo-MV algebras:

- left-pseudo-Wajsberg algebras \(\iff\) left-pseudo-MV algebras

\[(A^L, \rightarrow^L, \sim^L, -, \sim, 1) \quad \quad \quad (A^L, \odot, -, \sim, 0, 1)\]

\[x \odot y = (x \rightarrow^L y^-)\sim = (y \sim^L x^-)^-\]
\[0 = 1^- = 1\sim\]

\[x \rightarrow^L y = (x \odot y^-)^-\]
\[x \sim^L y = (y^- \odot x)^-\]
2. Preliminaries

2.1 Examples of term equivalent involutive algebras of logic:

**Pseudo-Wajsberg algebras** are term equivalent to pseudo-MV algebras:

- **left-pseudo-Wajsberg algebras** $\iff$ **left-pseudo-MV algebras**

  \[
  (A^L, \rightarrow^L, \bowtie^L, -, \sim, 1) \quad \rightarrow^L y = (x \bowtie y)^- \\
  x \bowtie y = (x \rightarrow^L y^-) = (y \bowtie^L x^\sim)^- \quad x \bowtie^L y = (y^- \bowtie x)^\sim \\
  0 = 1^- = 1^\sim
  \]

- **right-pseudo-Wajsberg algebras** $\iff$ **r.-pseudo-MV algebras**

  \[
  (A^R, \rightarrow^R, \bowtie^R, -, \sim, 0) \quad \rightarrow^R y = (x \bowtie y^\sim)^- \\
  x \bowtie y = (x \rightarrow^R y^-)^\sim = (y \bowtie^R x^\sim)^- \quad x \bowtie^R y = (y^- \bowtie x)^\sim \\
  1 = 0^- = 0^\sim
  \]
2.2 Examples of categorically equivalent non-commutative algebras of logic

Pseudo-BCK algebras with \( pP(\text{pseudo-product})/pS(\text{ps.-sum}) \) are categorically equivalent to porims (\( = \) partially-ordered residuated integral monoids):
2.2 Examples of categorically equivalent non-commutative algebras of logic

Pseudo-BCK algebras with \( pP \) (pseudo-product)/\( pS \) (ps.-sum) are categorically equivalent to porims (= partially-ordered residuated integral monoids):

- **left-pseudo-BCK(pP) algebras** \( \iff \) **left-porims**

\[
(A^L, \leq, \rightarrow^L, \sim^L, 1) \iff (A^L, \leq, \odot, 1)
\]

\( (pP) \) \( \exists \ x \odot y 
\]

\[
= \min \{ z \mid x \leq y \rightarrow^L z \}
\]

\[
= \min \{ z \mid y \leq x \sim^L z \}
\]

- **right-pseudo-BCK(pS) algebras** \( \iff \) **right-porims**

\[
(A^R, \leq, \rightarrow^R, \sim^R, 0) \iff (A^R, \leq, \oplus, 0)
\]

\( (pS) \) \( \exists \ x \oplus y 
\]

\[
= \max \{ z \mid x \geq y \rightarrow^R z \}
\]

\[
= \max \{ x \mid x \oplus y \geq z \}
\]

\( \exists x \sim^R z 
\]

\[
= \max \{ x \mid x \oplus y \leq z \}
\]
2.2 Examples of categorically equivalent non-commutative algebras of logic

Pseudo-BCK algebras with pP(pseudo-product)/pS(ps.-sum) are categorically equivalent to porims (= partially-ordered residuated integral monoids):

- **left-pseudo-BCK(pP) algebras** $\iff$ **left-porims**
  
  $$(A^L, \leq, \rightarrow^L, \sim^L, 1) \quad (A^L, \leq, \circ, 1)$$

  $$(pP) \exists x \circ y$$
  
  $$= \min\{z \mid x \leq y \rightarrow^L z\}$$
  
  $$= \min\{z \mid y \leq x \sim^L z\}$$

- **right-pseudo-BCK(pS) algebras** $\iff$ **right-porims**
  
  $$(A^R, \leq, \rightarrow^R, \sim^R, 0) \quad (A^R, \leq, \oplus, 0)$$

  $$(pS) \exists x \oplus y$$
  
  $$= \max\{z \mid x \geq y \rightarrow^R z\}$$
  
  $$= \max\{z \mid y \geq x \sim^R z\}$$
Remark:
• All above left-algebras of logic verify the following property of residuation, which is a Galois connection:

\[ x \circ y \leq z \iff x \leq y \rightarrow^L z \iff y \leq x \twoheadrightarrow^L z. \]
Remark:
• All above left-algebras of logic verify the following property of residuation, which is a Galois connection:

\[ x \odot y \leq z \iff x \leq y \rightarrow^L z \iff y \leq x \leftarrow^L z. \]

• All above right-algebras of logic verify the following dual property of residuation, which is a Galois connection:

\[ x \oplus y \geq z \iff x \geq y \rightarrow^R z \iff y \geq x \leftarrow^R z. \]
Remark:
• All above left-algebras of logic verify the following property of residuation, which is a Galois connection:

\[ x \odot y \leq z \iff x \leq y \rightarrow^L z \iff y \leq x \Rightarrow^L z. \]

• All above right-algebras of logic verify the following dual property of residuation, which is a Galois connection:

\[ x \oplus y \geq z \iff x \geq y \rightarrow^R z \iff y \geq x \Rightarrow^R z. \]

*Pair of Galois dual algebras*
Remark:
Note that usually in group theory
and sometimes in algebras of logic theory
(as for example in the recent book on residuated lattices of
Galatos, Jipsen, Kowalski, Ono 2007)
the following operators are used:

\, /
Remark:
Note that usually in group theory
and sometimes in algebras of logic theory
(as for example in the recent book on residuated lattices of
Galatos, Jipsen, Kowalski, Ono 2007)
the following operators are used:

\ \ , / 

while we (and other authors) use the following operators:

\ \ , \ \ 

where:

\[ x \rightarrow y = y/x, \quad x \rightsquigarrow y = x\backslash y, \]
Remark:
Note that usually in group theory and sometimes in algebras of logic theory (as for example in the recent book on residuated lattices of Galatos, Jipsen, Kowalski, Ono 2007) the following operators are used:

\[ , / \]

while we (and other authors) use the following operators:

\[ \rightarrow , \leadsto \]

where:

\[ x \rightarrow y = y/x, \quad x \leadsto y = x \setminus y, \]

i.e. the implication \( \rightarrow \) is the inverse of \( / \).
Thus,
- in the **commutative** case, we have:

$$\rightarrow = \Leftrightarrow$$
Thus,
- in the **commutative** case, we have:

\[ \rightarrow = \rightsquigarrow \]

- in **left-algebras** of logic we have:
\[ x \leq y \iff x \rightarrow^L y = 1 \iff x \rightsquigarrow^L y = 1 \]
and
- in **right-algebras** of logic we have:
\[ x \geq y \iff x \rightarrow^R y = 0 \iff x \rightsquigarrow^R y = 0 \]
Thus,
- in the **commutative** case, we have:

\[ \to = \leftrightarrow \]

- in **left-algebras** of logic we have:
\[ x \leq y \iff x \to^L y = 1 \iff x \leftrightarrow^L y = 1 \] and
- in **right-algebras** of logic we have:
\[ x \geq y \iff x \to^R y = 0 \iff x \leftrightarrow^R y = 0 \]

- the operation \( \to \) is associated to the **first** argument of \( \circ \quad (\oplus) \) and
- the operation \( \leftrightarrow \) is associated to the **second** argument of \( \circ \quad (\oplus) \).
2.3 The group level: Groups, \textit{implicative-groups}

**Theorem** The following algebras are \textit{termwise equivalent}:

$$\text{implicative-groups} \iff \text{groups}$$

$$(G, \rightarrow, \leftrightarrow, 0) \quad (G, +, -, 0)$$

$$(I1), (I2), (I3), (I4) \quad (G1), (G2), (G3)$$

$$-x = x \rightarrow 0 = x \leftrightarrow 0$$

$$x + y = -(x \rightarrow (-y))$$

$$= -(y \leftrightarrow (-x))$$

$$x \rightarrow y = -(x + (-y)) = y - x,$$

$$x \leftrightarrow y = -(((-y) + x) = -x + y$$
2.3 The group level: Groups, implicative-groups

**Theorem** The following algebras are termwise equivalent:

\[
\text{implicative-groups} \iff \text{groups}
\]

\[
(G, \rightarrow, \rightsquigarrow, 0) \iff (G, +, -, 0)
\]

\[
(I1), (I2), (I3), (I4)
\]

\[
-x = x \rightarrow 0 = x \rightsquigarrow 0 \quad \quad \quad x \rightarrow y = -(x + (-y)) = y - x,
\]

\[
x + y = -(x \rightarrow (-y)) \quad \quad \quad x \rightsquigarrow y = -((-y) + x) = -x + y
\]

\[
= -(y \rightsquigarrow (-x))
\]

where:

\[
(I1) \quad y \rightarrow z = (z \rightarrow x) \rightsquigarrow (y \rightarrow x), \quad y \rightsquigarrow z = (z \rightsquigarrow x) \rightarrow (y \rightsquigarrow x),
\]

\[
(I2) \quad 0 \rightarrow x = x = 0 \rightsquigarrow x,
\]

\[
(I3) \quad x = y \iff x \rightarrow y = 0 \iff x \rightsquigarrow y = 0,
\]

\[
(I4) \quad x \rightarrow 0 = x \rightsquigarrow 0.
\]
2.4 The po-group level: po-groups, po-implicative-groups

**Theorem**  The following structures are termwise equivalent:

\[
\text{po-implicative-groups} \iff \text{po-groups}
\]

\[
(G, \leq, \rightarrow, \sim\!\rightarrow, 0) \iff (G, \leq, +, -, 0)
\]

\[
\leq \text{ partial order} \iff \leq \text{ partial order}
\]

\[
(I1),(I2),(I3),(I4) \iff (G1),(G2),(G3)
\]

\[
(I5) \iff (G4)
\]
2.4 The po-group level: po-groups, po-implicative-groups

**Theorem** The following structures are termwise equivalent:

\[
\text{po-implicative-groups} \iff \text{po-groups}
\]

\[
(G, \leq, \rightarrow, \rightsquigarrow, 0) \iff (G, \leq, +, -, 0)
\]

\[
\leq \text{ partial order} \iff \leq \text{ partial order}
\]

\[
(I1),(I2),(I3),(I4) \iff (G1),(G2),(G3)
\]

\[
(I5) \iff (G4)
\]

where:

\[
(I5) x \leq y \text{ implies } z \rightarrow x \leq z \rightarrow y \text{ and } z \rightsquigarrow x \leq z \rightsquigarrow y.
\]
Remarks:

- Groups and implicative-groups verify the residuation property (which is a Galois connection):

\[ x + y = z \iff x = y \rightarrow z \iff y = x \leadsto z, \]

(see Galatos, Jipsen, Kowalski, Ono, 2007, page 160)
Remarks:

• **Groups** and **implicative-groups** verify the **residuation property** (which is a **Galois connection**):

\[
x + y = z \iff x = y \rightarrow z \iff y = x \leadsto z,
\]

(see Galatos, Jipsen, Kowalski, Ono, 2007, page 160)

• **Po-groups** and **po-implicative-groups** verify the two **residuation properties** (which are **Galois connections**):

\[
x + y \leq z \iff x \leq y \rightarrow z \iff y \leq x \leadsto z
\]

and **dually**:

\[
x + y \geq z \iff x \geq y \rightarrow z \iff y \geq x \leadsto z.
\]
Remarks:

• 

Groups and **implicative-groups** verify

the **residuation property** *(which is a Galois connection)*:

\[ x + y = z \iff x = y \rightarrow z \iff y = x \leftarrow z, \]

(see Galatos, Jipsen, Kowalski, Ono, 2007, page 160)

• Po-groups and **po-implicative-groups** verify

the two **residuation properties** *(which are Galois connections)*:

\[ x + y \leq z \iff x \leq y \rightarrow z \iff y \leq x \leftarrow z \]

and **dually**:

\[ x + y \geq z \iff x \geq y \rightarrow z \iff y \geq x \leftarrow z. \]

*We say they are Galois dual algebras!*
2.5 Connections between the \(-\text{implicative-group}\) level \(G\) and the algebras of logic:

- on \(G^-\) and \(G^+\) level:

**Theorem**

Let \(\mathcal{G} = (G, \lor, \land, \rightarrow, \leftarrow, 0)\) be an \(-\text{implicative-group}.\)
2.5 Connections between the $l$-implicative-group level $G$ and the algebras of logic:

- on $G^-$ and $G^+$ level:

**Theorem**

Let $G = (G, \lor, \land, \rightarrow, \rightsquigarrow, 0)$ be an $l$-implicative-group.

(1). Define, for all $x, y \in G^-$:

$$x \rightarrow^L y \overset{\text{def}}{=} (x \rightarrow y) \land 0,$$

$$x \rightsquigarrow^L y \overset{\text{def}}{=} (x \rightsquigarrow y) \land 0.$$ 

Then,
2.5 Connections between the $l$-implicative-group level $G$ and the algebras of logic:

- on $G^-$ and $G^+$ level:

**Theorem**

Let $G = (G, \lor, \land, \rightarrow, \Rightarrow, 0)$ be an $l$-implicative-group. 

(1). Define, for all $x, y \in G^-$:

$$x \rightarrow^L y \overset{\text{def}}{=} (x \rightarrow y) \land 0,$$

$$x \Rightarrow^L y \overset{\text{def}}{=} (x \Rightarrow y) \land 0.$$

Then,

$$G^L = (G^-, \land, \lor, \rightarrow^L, \Rightarrow^L, 1 = 0)$$

is a left-pseudo-BCK(pP) lattice with the pseudo-product $\odot = +$, lattice that is distributive, verifying conditions (pC) and (*), where: for all $x, y, z \in G^-$,
2.5 Connections between the \( l\)-implicative-group level \( G \) and the algebras of logic:

- on \( G^- \) and \( G^+ \) level:

**Theorem**

Let \( G = (G, \lor, \land, \to, \leftrightarrow, 0) \) be an \( l\)-implicative-group. 

(1). Define, for all \( x, y \in G^- \):

\[
x \to^L y \overset{\text{def}}{=} (x \to y) \land 0,
\]

\[
x \leftrightarrow^L y \overset{\text{def}}{=} (x \leftrightarrow y) \land 0.
\]

Then,

\[
G^L = (G^-, \land, \lor, \to^L, \leftrightarrow^L, \mathbf{1} = 0)
\]

is a **left-pseudo-BCK(pP)** lattice with the pseudo-product \( \odot = + \), lattice that is distributive, verifying conditions \((pC)\) and \((*)\), where: for all \( x, y, z \in G^- \),

\[
(pC) \quad x \lor y = (x \leftrightarrow^L y) \to^L y = (x \to^L y) \leftrightarrow^L y,
\]

\[
(*) \quad (x \odot z) \to^L (y \odot z) = x \to^L y, \quad (z \odot x) \leftrightarrow^L (z \odot y) = x \leftrightarrow^L y.
\]
Connections between the $l$-implicative-group level $G$ and the algebras of logic:

- On $[u', 0]$ and $[0, u]$ level:

**Corollary** (see Georgescu, A.I., 1999)

Let $\mathcal{G} = (G, \lor, \land, \to, \leftrightarrow, 0)$ be an $l$-implicative-group.
Connections between the \( l \)-implicative-group level \( G \) and the algebras of logic:

- On \([u', 0]\) and \([0, u]\) level:

**Corollary**  (see Georgescu, A.I., 1999)

Let \( G = (G, \lor, \land, \to, \multimap, 0) \) be an \( l \)-implicative-group.

(1). Let us take the interior point \( u' < 0 \) from \( G^- \) and consider the interval \([u', 0] \subset G^-\).

Then,
Connections between the $l$-implicative-group level $G$ and the algebras of logic:

- On $[u', 0]$ and $[0, u]$ level:

**Corollary** (see Georgescu, A.I., 1999)

Let $G = (G, \lor, \land, \rightarrow, \sim\rightarrow, 0)$ be an $l$-implicative-group.

(1). Let us take the interior point $u' < 0$ from $G^-$ and consider the interval $[u', 0] \subset G^-$. Then,

$$G_1^L = ([u', 0], \land, \lor, \rightarrow^L, \sim\rightarrow^L, 0 = u', 1 = 0)$$

is a bounded left-pseudo-BCK(pP) lattice with condition (pC), hence is an equivalent definition of **left-pseudo-Wajsberg algebra**.
Connections between the *l*-implicative-group level $G$ and the algebras of logic:

- **On** $\{-\infty\} \cup G^-$ and $G^+ \cup \{\infty\}$ **level:**

**Corollary** (see A. Di Nola, G. Georgescu, A.I., 2002; for the commutative case, see R. Cignoli, A. Torrens, 1997)

Let $G = (G, \lor, \land, \to, \multimap, 0)$ be an *l*-implicative-group.
Connections between the \( l \)-implicative-group level \( G \) and the algebras of logic:

- On \( \{−∞\} \cup G^- \) and \( G^+ \cup \{∞\} \) level:

**Corollary**  (see A. Di Nola, G. Georgescu, A.I., 2002; for the commutative case, see R. Cignoli, A. Torrens, 1997)
Let \( G = (G, \lor, \land, \rightarrow, \multimap, 0) \) be an \( l \)-implicative-group.

**1.** Let us consider an exterior point \( −∞ \), distinct from the elements of \( G \).
Define \( G^-_{−∞} = \{−∞\} \cup G^- \) and extend the operations from \( G^- \) to \( G^-_{−∞} \):

\[
    x \rightarrow^L y = \begin{cases} 
    (x \rightarrow y) \land 0, & \text{if } x, y \in G^- \\
    -∞, & \text{if } x \in G^-, y = −∞ \\
    0, & \text{if } x = −∞,
    \end{cases}
\]

\[
    x \multimap^L y = \begin{cases} 
    (x \multimap y) \land 0, & \text{if } x, y \in G^- \\
    -∞, & \text{if } x \in G^-, y = −∞ \\
    0, & \text{if } x = −∞,
    \end{cases}
\]
\[ x \circ y = \begin{cases} 
  x + y, & \text{if } x, y \in G^- \\
  -\infty, & \text{if } \text{otherwise.} 
\end{cases} \]

We extend \( \leq \) by putting: \( -\infty \leq x \), for any \( x \in G^- \).

Then,
\( x \odot y = \begin{cases} 
  x + y, & \text{if } x, y \in G^- \\
  -\infty, & \text{if otherwise.}
\end{cases} \)

We extend \( \leq \) by putting: \( -\infty \leq x \), for any \( x \in G_{-\infty} \).

Then,

\[
G_2^L = (G_{-\infty}, \land, \lor, \odot, \to^L, \leadsto^L, 0 = -\infty, 1 = 0)
\]

is a left-pseudo-product algebra.
3. Normal filters/ideals, compatible deductive systems

3.1 Filters/ideals and deductive systems

- On algebras of logic level:

Proposition (see Bușneag, Rudeanu, 2010 for a more general result in the commutative case) (1). Let $A^L_r = (A^L, \leq, \odot, 1)$ be a left-porim and let $A^L_t = (A^L, \leq, \rightarrow^L, \twoheadrightarrow^L, 1)$ be the categorically equivalent left-pseudo-BCK(pP) algebra. Then,
3. Normal filters/ideals, compatible deductive systems

3.1 Filters/ideals and deductive systems

- On algebras of logic level:

**Proposition** (see Buşneag, Rudeanu, 2010 for a more general result in the commutative case)

(1). Let $\mathcal{A}_r^L = (A^L, \leq, \odot, 1)$ be a left-porim and let $\mathcal{A}_t^L = (A^L, \leq, \rightarrow^L, \leadsto^L, 1)$ be the categorically equivalent left-pseudo-BCK(pP) algebra. Then, the $(\odot)$-filters of $\mathcal{A}_r^L$ coincide with the $(\rightarrow^L, \leadsto^L)$-deductive systems of $\mathcal{A}_t^L$. 
• On po-group/po-implicative-group level:

  · In po-groups, we have the convex po-subgroup
    \((= (+)-filter-ideal)\).
  · Analogously, in po-implicative-groups, we define the convex po-subimplicative-group
    \((= (\rightarrow, \rightsquigarrow)-filter-ideal)\)
    as follows:
• On po-group/po-implicative-group level:

- In po-groups, we have the **convex po-subgroup** (= (+)-filter-ideal).
- Analogously, in po-implicative-groups, we define the **convex po-subimplicative-group** (= (→, ⇝)-filter-ideal) as follows:

**Definition**
Let \( G = (G, \leq, \rightarrow, \rightsquigarrow, 0) \) be a po-implicative-group. A **convex po-subimplicative-group** of \( G \) is a subset \( S \subseteq G \) which satisfies:

- \( 0 \in S \),
- \( x, y \in S \) imply \( x \rightarrow y, x \rightsquigarrow y \in S \),
- \( a, b \in S \) and \( a \leq x \leq b \) imply \( x \in S \).
Obviously, we have:

**Proposition**

Let $G = (G, \leq, +, -, 0)$ be a po-group and let $G_{ig} = (G, \leq, \rightarrow, \multimap, 0)$ be the term equivalent po-implicative-group. Then,
Obviously, we have:

**Proposition**

Let $G = (G, \leq, +, -, 0)$ be a po-group and let $G_{ig} = (G, \leq, \rightarrow, \leadsto, 0)$ be the term equivalent po-implicative-group.

Then, the **convex po-subgroups** of $G_g$ coincide with the **convex po-subimplicative-groups** of $G_{ig}$. 
Inspired from algebras of logic, we introduce also the following notion:
Inspired from algebras of logic, we introduce also the following notion:

**Definition**

Let $G = (G, \leq, \rightarrow, \rightsquigarrow, 0)$ be a po-implicative-group.

A **deductive system** of $G$ is a subset $S \subseteq G$ which satisfies:

- $0 \in S$;
- (a) $x \in S$, $x \rightarrow y \in S$ (or $x \rightsquigarrow y \in S$) imply $y \in S$,
  - (b) $x \in S$ implies $x \rightarrow 0 = x \rightsquigarrow 0 \in S$;
- $a, b \in S$ and $a \leq x \leq b$ imply $x \in S$. 
Proposition
Let $G_g = (G, \leq, +, -, 0)$ be a po-group and let $G_{ig} = (G, \leq, \rightarrow, \leadsto, 0)$ be the term equivalent po-implicative-group. Then,
Proposition
Let $G_g = (G, \leq, +, -, 0)$ be a po-group and let $G_{ig} = (G, \leq, \rightarrow, \rightarrow\!, 0)$ be the term equivalent po-implicative-group.
Then, the convex po-subgroups of $G_g$ coincide with the deductive systems of $G_{ig}$.
Resuming:

In po-groups/po-implicative-groups, we have:

\[
\text{convex po-} \quad \text{subgroups} \quad = \quad \text{deductive systems} \\
= \quad \text{convex po-} \quad \text{subimplicative-} \quad \text{groups}
\]
• Back to algebras of logic level:
Inspired from po-implicative-group level, we introduce the following notion:
• Back to algebras of logic level:
Inspired from po-implicative-group level, we introduce the following notion:

**Definition**

(1). Let $\mathcal{A}^L = (A^L, \leq, \rightarrow^L, \sim\rightarrow^L, 1)$ be a left-pseudo-BCK algebra. A $(\rightarrow^L, \sim\rightarrow^L)$-filter of $\mathcal{A}^L$ is a subset $F \subseteq A^L$ which satisfies:

- $1 \in F$,
- $x, y \in F$ imply $x \rightarrow^L y, x \sim\rightarrow^L y \in F$,
- $x \in F$ and $x \leq y$ imply $y \in F$.
Proposition (1). Let $\mathcal{A}^L_r = (A^L, \leq, \odot, 1)$ be a left-porim and let $\mathcal{A}^L_t = (A^L, \leq, \rightarrow^L, \rightsquigarrow^L, 1)$ be the categorically equivalent left-pseudo-BCK(pP) algebra. Then,
Proposition
(1). Let $\mathcal{A}_L^r = (A^L, \leq, \odot, 1)$ be a left-porim and let $\mathcal{A}_L^t = (A^L, \leq, \rightarrow^L, \leadsto^L, 1)$ be the categorically equivalent left-pseudo-BCK(pP) algebra. Then, any $(\odot)$-filter of $\mathcal{A}_L^r$ is a $(\rightarrow^L, \leadsto^L)$-filter of $\mathcal{A}_L^t$. The converse is not true.
Resuming:

(1). In left-porims/left-pseudo-BCK(pP) algebras, we have:

\((\bigodot)-\text{filters} = (\rightarrow^L, \leadsto^L)\)-deductive systems \(\subseteq (\rightarrow^L, \leadsto^L)-\text{filters}\)
Connections results in lattice-ordered case:

/implicative-group \iff /group

\((G, \lor, \land, \rightarrow, \sim\rightarrow, 0)\)
Connections results in lattice-ordered case:

/-implicative-group \iff/-group

\[(G, \lor, \land, \rightarrow, \sim\rightarrow, 0)\]

\[S \subseteq G\]

convex /-subimplicative-group

\[\downarrow G^- \quad G^+ \downarrow\]

\[S \cap G^- \quad S \cap G^+\]

\[\rightarrow^L, \sim\rightarrow^L\)-filter \quad \rightarrow^R, \sim\rightarrow^R\)-ideal

\[S \cap G^- \quad S \cap G^+\]

\[(\otimes\text{-filter}) \quad (\oplus\text{-ideal})\]
Connections results in lattice-ordered case:

\(-\text{implicative-group} \iff \text{-group}\)

\((G, \lor, \land, \rightarrow, \rightsquigarrow, 0)\)

\(S \subseteq G\)

convex \(-\text{-subimplicative-group}\)

\(\downarrow G^- G^+ \downarrow\)

\(S \cap G^-\) \quad \(S \cap G^+\)

\((\rightarrow^L, \rightsquigarrow^L)\)-filter \quad \((\rightarrow^R, \rightsquigarrow^R)\)-ideal

\(S \subseteq G\)

deductive system

\(\downarrow G^- G^+ \downarrow\)

\(S \cap G^-\) \quad \(S \cap G^+\)

\((\rightarrow^L, \rightsquigarrow^L)\)-d.s. \quad \((\rightarrow^R, \rightsquigarrow^R)\)-d.s.
Resuming Theorem:
Let $G$ be an $l$-group/$l$-implicative-group.
Let $S \subseteq G$ be a convex $l$-subgroup/deductive system/
convex $l$-subimplicative-group.
Then:

(1) $S_L = S \cap G$ is in the same time:
   (⊙)-filter and ($\rightarrow_L$, $\Rightarrow_L$)-deductive system/
convex $l$-subimplicative-group.
Resuming Theorem:
Let $G$ be an \textit{l-group}/\textit{l-implicative-group}.
Let $S \subseteq G$ be a \textit{convex \textit{l}-subgroup/deductive system/convex \textit{l}-subimplicative-group}.
Then:

(1). $S^L = S \cap G^-$ is in the same time:
\textit{(\textcircled{\textit{\textbullet}})-filter and $(\rightarrow^L, \nrightarrow^L)$-deductive system and $(\rightarrow^L, \nrightarrow^L)$-filter.}
Normal filters/ideals, compatible deductive systems

3.2 Normal filters/ideals and compatible deductive systems

• On algebras of logic level

We introduce the following:

Definition

(1). Let $\mathcal{M}^L = (\mathcal{M}^L, \leq, \odot, 1)$ be a left-poim ($\leq$ partially-ordered integral left-monoid).
Normal filters/ideals, compatible deductive systems

3.2 Normal filters/ideals and compatible deductive systems

- On algebras of logic level

We introduce the following:

**Definition**

(1). Let $\mathcal{M}^L = (M^L, \leq, \odot, 1)$ be a left-poim (i.e., partially-ordered integral left-monoid).

A $(\odot)$-filter $S^L$ of $\mathcal{M}^L$ is **normal** if the following condition $(N^L)$ holds:

\[(N^L) \quad \text{for any } x \in M^L, \quad S^L \odot x = x \odot S^L.\]
Recall the following:

**Definition (see Kühr, 2007)**

(1). Let \( \mathcal{A}^L = (A^L, \leq, \rightarrow^L, \circeq^L, 1) \) be a left-pseudo-BCK algebra. A \((\rightarrow^L, \circeq^L)\)-deductive system \( S^L \) of \( \mathcal{A}^L \) is **compatible** if the following condition \((C^L)\) holds:

\[(C^L) \quad \text{for any } x, y \in A^L, \quad x \rightarrow^L y \in S^L \iff x \circeq^L y \in S^L.\]
We have obtained the following result concerning normal filters/ideals and compatible deductive systems:
We have obtained the following result concerning normal filters/ideals and compatible deductive systems:

**Theorem**

(1). Let $\mathcal{A}^L = (A^L, \wedge, \vee, \to^L, \leftrightarrow^L, 1)$ be a left-pseudo-BCK(pP) lattice with pseudo-product $\odot$, verifying $(pdiv)$:

$$(pdiv) \ (pseudo - divisibility) \ x \wedge y = (x \to^L y) \odot x = x \odot (x \leftrightarrow^L y)$$

(or let $\mathcal{A}^L_m = (A^L, \wedge, \vee, \odot, 1)$ be a left-l-rim verifying $(pdiv)$).
We have obtained the following result concerning normal filters/ideals and compatible deductive systems:

**Theorem (1).** Let $\mathcal{A}^L = (A^L, \land, \lor, \rightarrow^L, \rightsquigarrow^L, 1)$ be a left-pseudo-BCK(pP) lattice with pseudo-product $\odot$, verifying $(p\text{div})$:

\[(p\text{div}) \, (\text{pseudo - divisibility}) \quad x \land y = (x \rightarrow^L y) \odot x = x \odot (x \rightsquigarrow^L y)\]

(or let $\mathcal{A}^L_m = (A^L, \land, \lor, \odot, 1)$ be a left-$l$-rim verifying $(p\text{div})$). Let $S^L$ be a $\left(\rightarrow^L, \rightsquigarrow^L\right)$-deductive system of $\mathcal{A}^L$ (or, equivalently, a $\left(\odot\right)$-filter of $\mathcal{A}^L_m$). Then
We have obtained the following result concerning normal filters/ideals and compatible deductive systems:

**Theorem**

(1). Let $\mathcal{A}^L = (A^L, \wedge, \vee, \rightarrow^L, \leadsto^L, 1)$ be a left-pseudo-BCK(pP) lattice with pseudo-product $\odot$, verifying (pdiv):

\[(\text{pdiv}) \quad \text{(pseudo - divisibility)} \quad x \wedge y = (x \rightarrow^L y) \odot x = x \odot (x \leadsto^L y)\]

(or let $\mathcal{A}^L_m = (A^L, \wedge, \vee, \odot, 1)$ be a left-l-rim verifying (pdiv)).

Let $S^L$ be a $(\rightarrow^L, \leadsto^L)$-deductive system of $\mathcal{A}^L$ (or, equivalently, a $(\odot)$-filter of $\mathcal{A}^L_m$).

Then

$S^L$ is **compatible** if and only if is **normal**, i.e.

\[(C^L) \iff (N^L).\]
Open problem:
Find an example of left-pseudo-BCK(pP) lattice not verifying \((p\text{div})\), which has a \((\circ)\)-filter that is:
- normal but not compatible, or is
- compatible but not normal.
• On po-group/po-implicative-group level

Recall the following:
• On po-group/po-implicative-group level

Recall the following:

• Definition
Let $G_g = (G, \leq, +, -, 0)$ be a po-group.
A convex po-subgroup $S$ of $G_g$ is normal if the following condition $(N_g)$ holds:

$$(N_g) \quad \text{for any } g \in G, \ S + g = g + S.$$
On po-group/po-implicative-group level

Recall the following:

**Definition**

Let $G_g = (G, \leq, +, -, 0)$ be a po-group.

A convex po-subgroup $S$ of $G_g$ is **normal** if the following condition $(N_g)$ holds:

$$(N_g) \quad \text{for any } g \in G, \ S + g = g + S.$$ 

We introduce now the following:

**Definition**

Let $G_{ig} = (G, \leq, \rightarrow, \rightsquigarrow, 0)$ be a po-implicative-group.

A deductive system $S$ of $G_{ig}$ is **compatible** if the following condition $(C_{ig})$ holds:

$$(C_{ig}) \quad \text{for any } x, y \in G, \ x \rightarrow y \in S \iff x \rightsquigarrow y \in S.$$
We know already that
the **convex po-subgroups** of $\mathcal{G}_g$ coincide with
the **deductive systems** of the categorically equivalent $\mathcal{G}_{ig}$.
We know already that the **convex po-subgroups** of $\mathcal{G}_g$ coincide with the **deductive systems** of the categorically equivalent $\mathcal{G}_{ig}$.

Moreover, we obtain now the following:

**Theorem**
Let $\mathcal{G}_{ig} = (G, \leq, \rightarrow, \rightsquigarrow, 0)$ be a **po-implicative-group** (or let $\mathcal{G}_g = (G, \leq, +, -, 0)$ be a **po-group**).
We know already that
the convex po-subgroups of $G_g$ coincide with
the deductive systems of the categorically equivalent $G_{ig}$.

Moreover, we obtain now the following:

**Theorem**

Let $G_{ig} = (G, \leq, \rightarrow, \leadsto, 0)$ be a po-implicative-group
(or let $G_g = (G, \leq, +, -, 0)$ be a po-group).

Let $S$ be a deductive system of $G_{ig}$
(or, equivalently, a convex po-subgroup of $G_g$).

Then,
We know already that
the convex po-subgroups of $G_g$ coincide with
the deductive systems of the categorically equivalent $G_{ig}$.

Moreover, we obtain now the following:

**Theorem**
Let $G_{ig} = (G, \leq, \rightarrow, \sim\rightarrow, 0)$ be a po-implicative-group
(or let $G_g = (G, \leq, +, -, 0)$ be a po-group).
Let $S$ be a deductive system of $G_{ig}$
(or, equivalently, a convex po-subgroup of $G_g$).
Then,
$S$ is compatible if and only if $S$ is normal, i.e.

$$(C_{ig}) \iff (N_g).$$
• On $l$-groups/$l$-implicative groups level

The result of above Theorem (formulated in partially-ordered case) remains valid in lattice-ordered case, i.e. we have:
• On \textit{l}\textsuperscript{-groups}/\textit{l}\textsuperscript{-implicative groups} level

The result of above Theorem (formulated in partially-ordered case) remains valid in lattice-ordered case, i.e. we have:

**Corollary**

Let $\mathcal{G}_{ig} = (G, \lor, \land, \to, \multimap, 0)$ be an \textit{l}\textsuperscript{-implicative-group} (or let $\mathcal{G}_g = (G, \lor, \land, +, -, 0)$ be an \textit{l}\textsuperscript{-group}).
• On \( l \)-groups/\( l \)-implicative groups level

The result of above Theorem (formulated in partially-ordered case) remains valid in lattice-ordered case, i.e. we have:

**Corollary**

Let \( \mathcal{G}_{ig} = (G, \vee, \wedge, \to, \nrightarrow, 0) \) be an \( l \)-implicative-group (or let \( \mathcal{G}_g = (G, \vee, \wedge, +, -, 0) \) be an \( l \)-group).

Let \( S \) be a **deductive system** of \( \mathcal{G}_{ig} \)

(or, equivalently, a **convex \( l \)-subgroup** of \( \mathcal{G}_g \)).

Then,
• **On \(-\text{groups}/\text{-implicative groups level****}

The result of above Theorem (formulated in partially-ordered case) remains valid in lattice-ordered case, i.e. we have:

**Corollary**
Let \( G_{ig} = (G, \lor, \land, \rightarrow, \leftrightarrow, 0) \) be an \(-\text{implicative-group}
(or let \( G_g = (G, \lor, \land, +, -, 0) \) be an \(-\text{group}).
Let \( S \) be a **deductive system** of \( G_{ig} \)
(or, equivalently, a **convex \(-\text{subgroup** of \( G_g \).**
Then,
\( S \) is **compatible** if and only if \( S \) is **normal**, i.e.

\( (C_{ig}) \iff (N_g). \)
**Normal filters/ideals, compatible deductive systems**

3.3 Connections between \(l\)-group/\(l\)-implicative-group level and algebras of logic:

- **On \(G^-\) and \(G^+\) level:**
  
  **Theorem**
  
  Let \(G_{ig} = (G, \lor, \land, \rightarrow, \sim\to, 0)\) be an \(l\)-implicative-group (or let \(G_g = (G, \lor, \land, +, -, 0)\) be an \(l\)-group).
Normal filters/ideals, compatible deductive systems

3.3 Connections between \( l\)-group/\( l\)-implicative-group level and algebras of logic:

- **On \( G^- \) and \( G^+ \) level:**

  **Theorem**
  Let \( G_{ig} = (G, \vee, \wedge, \rightarrow, \sim\rightarrow, 0) \) be an \( l\)-implicative-group (or let \( G_g = (G, \vee, \wedge, +, -, 0) \) be an \( l\)-group).
  Let \( S \) be a compatible deductive system of \( G_{ig} \) (or, equivalently, a normal convex \( l\)-subgroup of \( G_g \)).
  Then,
Normal filters/ideals, compatible deductive systems

3.3 Connections between \( l \)-group/\( l \)-implicative-group level and algebras of logic:

- **On \( G^- \) and \( G^+ \) level:**

  **Theorem**
  
  Let \( G_{ig} = (G, \lor, \land, \rightarrow, \rightsquigarrow, 0) \) be an \( l \)-implicative-group
  (or let \( G_g = (G, \lor, \land, +, -, 0) \) be an \( l \)-group).
  Let \( S \) be a compatible deductive system of \( G_{ig} \)
  (or, equivalently, a normal convex \( l \)-subgroup of \( G_g \)).
  Then,
  
  (1). \( S^L = S \cap G^- \) is a compatible \((\rightarrow^L, \rightsquigarrow^L)\)-deductive system
  of the left-pseudo-BCK(pP) lattice
  \( G^L = (G^-, \land, \lor, \rightarrow^L, \rightsquigarrow^L, 1 = 0) \)
  (or, equivalently, \( S^L \) is a normal \((\odot)\)-filter of the left-\( l \)-rim
  \( G^L_m = (G^-, \land, \lor, \odot = +, 1 = 0) \)).
Normal filters/ideals, compatible deductive systems

3.3 Connections between $l$-group/$l$-implicative-group level and algebras of logic:

- **On $G^-$ and $G^+$ level:**

  **Theorem**
  Let $G_{ig} = (G, \lor, \land, \rightarrow, \sim\rightarrow, 0)$ be an $l$-implicative-group
  (or let $G_g = (G, \lor, \land, +, -, 0)$ be an $l$-group).
  Let $S$ be a compatible deductive system of $G_{ig}$
  (or, equivalently, a normal convex $l$-subgroup of $G_g$).
  Then,
  (1). $S^L = S \cap G^-$ is a compatible $(\rightarrow^L, \sim\rightarrow^L)$-deductive system
  of the left-pseudo-BCK(pP) lattice
  $G^L = (G^-, \land, \lor, \rightarrow^L, \sim\rightarrow^L, 1 = 0)$
  (or, equivalently, $S^L$ is a normal $(\circ)$-filter of the left-$l$-rim
  $G^L_m = (G^-, \land, \lor, \circ = +, 1 = 0)$),
  and $S^L$ is compatible if and only if is normal, i.e.
  $$(C^L) \iff (N^L).$$
In other words, the above Theorem says that:
In other words, the above Theorem says that:
- normality/compatibility at \( l \)-group/\( l \)-implicative-group \( G \) level is inherited by the algebras obtained by restricting the \( l \)-group/\( l \)-implicative-group operations to the negative cone \( G^- \) and to the positive cone \( G^+ \).
In other words, the above Theorem says that:
- normality/compatibility at \( l \)-group/\( l \)-implicative-group \( G \) level is inherited by the algebras obtained by restricting the \( l \)-group/\( l \)-implicative-group operations to the negative cone \( G^- \) and to the positive cone \( G^+ \).
- the equivalence

\[
(C_{ig}) \iff (N_g)
\]

(compatible if and only if normal),
existing at \( l \)-group/\( l \)-implicative-group level is preserved by the algebras obtained by restricting the \( l \)-group/\( l \)-implicative-group operations to \( G^- \) and to \( G^+ \), i.e. it induces the dual equivalences:

\[
(C^L) \iff (N^L) \quad \text{and} \quad (C^R) \iff (N^R).
\]
• On \([u', 0]\) and \([0, u]\) level:

Similar results.
• **On** $[u', 0]$ **and** $[0, u]$ **level:**

  Similar results.

• **On** $\{-\infty\} \cup G^- \text{ and } G^+ \cup \{+\infty\}$ **level:**

  Similar results.
4. Representability
4.1 Representable algebras of logic

(1). Recall (C.J. van Alten, 2002) that:
A left-pseudo-BCK(pP) lattice
$A^L = (A^L, \wedge, \vee, \rightarrow^L, \sim\rightarrow^L, 1)$ with the pseudo-product $\circ$
4. Representability

4.1 Representable algebras of logic

(1). Recall (C.J. van Alten, 2002) that:
A **left**-pseudo-BCK(pP) lattice
\( A^L = (A^L, \land, \lor, \rightarrow^L, \sim^L, 1) \) with the pseudo-product \( \odot \)
(or, equivalently,
a **non-commutative left**-residuated lattice
\( A^C = (A^L, \land, \lor, \odot, \rightarrow^L, \sim^L, 1) \))
4. Representability

4.1 Representable algebras of logic

(1). Recall (C.J. van Alten, 2002) that:

A \textbf{left}-pseudo-BCK(pP) lattice

\( \mathcal{A}^L = (A^L, \land, \lor, \to^L, \multimap^L, 1) \)

with the pseudo-product \( \odot \)

(or, equivalently,

a \textbf{non-commutative left-residuated lattice}

\( \mathcal{A}^L = (A^L, \land, \lor, \odot, \to^L, \multimap^L, 1) \))

is \textbf{representable if and only if} it satisfies the identity:

\[
(x \multimap^L y) \lor \left( \left[ \left( (y \multimap^L x) \multimap^L z \right) \multimap^L z \right] \to^L w \right) \to^L w = 1, \tag{1}
\]

or the identity

\[
(x \to^L y) \lor \left( \left[ \left( (y \to^L x) \to^L z \right) \to^L z \right] \multimap^L w \right) \multimap^L w = 1, \tag{2}
\]

for all \( x, y, z, w \in A^L \).
4.2 Representable \( l \)-groups/\( l \)-implicative-groups

Recall (M. Andersen, T. Feil, 1988, Theorem 4.1.1):
Let \( \mathcal{G} = (G, \lor, \land, +, -, 0) \) be an \( l \)-group.
The following are equivalent:
(a) \( \mathcal{G} \) is representable.
(b) \( 2(a \land b) = 2a \land 2b \);
(b\( ^d \)) \( 2(a \lor b) = 2a \lor 2b \).
(c) \( a \land (-b - a + b) \leq 0 \);
(c\( ^d \)) \( a \lor (-b - a + b) \geq 0 \).
4.2 Representable $l$-groups/$l$-implicative-groups

Recall (M. Andersen, T. Feil, 1988, Theorem 4.1.1):
Let $G = (G, \lor, \land, +, -, 0)$ be an $l$-group.
The following are equivalent:
(a) $G$ is representable.
(b) $2(a \land b) = 2a \land 2b$;
(b$^d$) $2(a \lor b) = 2a \lor 2b$.
(c) $a \land (-b - a + b) \leq 0$;
(c$^d$) $a \lor (-b - a + b) \geq 0$.
(d) Each polar subgroup is normal.
(e) Each minimal prime subgroup is normal.
(f) For each $a \in G$, $a > 0$, $a \land (-b + a + b) > 0$, for all $b \in G$;
(f$^d$) For each $a \in G$, $a < 0$, $a \lor (-b + a + b) < 0$, for all $b \in G$.
Note that $^d$ means “dual”.
Inspired from algebras of logic, we obtained the following:
Inspired from algebras of logic, we obtained the following:

**Theorem**

Let $G_g = (G, \lor, \land, +, -, 0)$ be an $l$-group (or, equivalently, let $G_{ig} = (G, \lor, \land, \rightarrow, \sim, 0)$ be the $l$-implicative-group).

The following are equivalent:

(a) $G$ is representable.
(b) $2(a \land b) = 2a \land 2b$,

(b1) $(b \rightarrow a) \land (a \sim b) \leq 0 \land [(b \sim a) \sim (b \rightarrow a)]$,

(b2) $(b \sim a) \land (a \rightarrow b) \leq 0 \land [(b \rightarrow a) \rightarrow (b \sim a)]$. 
Inspired from algebras of logic, we obtained the following:

**Theorem**

Let $\mathcal{G}_g = (G, \lor, \land, +, -, 0)$ be an $l$-group (or, equivalently, let $\mathcal{G}_{ig} = (G, \lor, \land, \rightarrow, \leadsto, 0)$ be the $l$-implicative-group).

The following are equivalent:

(a) $\mathcal{G}$ is representable.

(b) $2(a \land b) = 2a \land 2b$,

(b1) $(b \rightarrow a) \land (a \leadsto b) \leq 0 \land [(b \leadsto a) \leadsto (b \rightarrow a)]$,

(b2) $(b \leadsto a) \land (a \rightarrow b) \leq 0 \land [(b \rightarrow a) \rightarrow (b \leadsto a)]$.

(b1') $2(a \lor b) = 2a \lor 2b$,

(b1') $(b \rightarrow a) \lor (a \leadsto b) \geq 0 \lor [(b \leadsto a) \leadsto (b \rightarrow a)]$,

(b2') $(b \leadsto a) \lor (a \rightarrow b) \geq 0 \lor [(b \rightarrow a) \rightarrow (b \leadsto a)]$. 
Inspired from algebras of logic, we obtained the following:

**Theorem**

Let $G = (G, \lor, \land, +, -, 0)$ be an $l$-group (or, equivalently, let $G_{ig} = (G, \lor, \land, \rightarrow, \leadsto, 0)$ be the $l$-implicative-group). The following are equivalent:

(a) $G$ is representable.
(b) $2(a \land b) = 2a \land 2b$,
(b$_1$) $(b \rightarrow a) \land (a \leadsto b) \leq 0 \land [(b \leadsto a) \leadsto (b \rightarrow a)]$,
(b$_2$) $(b \leadsto a) \land (a \rightarrow b) \leq 0 \land [(b \rightarrow a) \rightarrow (b \leadsto a)]$.
(b$^d$) $2(a \lor b) = 2a \lor 2b$,
(b$_1^d$) $(b \rightarrow a) \lor (a \leadsto b) \geq 0 \lor [(b \leadsto a) \leadsto (b \rightarrow a)]$,
(b$_2^d$) $(b \leadsto a) \lor (a \rightarrow b) \geq 0 \lor [(b \rightarrow a) \rightarrow (b \leadsto a)]$.
(c) $a \land (-b - a + b) \leq 0$,
(c$_1$) $(x \leadsto y) \land (((((y \leadsto x) \leadsto z) \leadsto z) \rightarrow w) \rightarrow w) \leq 0$,
(c$_2$) $(x \rightarrow y) \land (((((y \rightarrow x) \rightarrow z) \rightarrow z) \leadsto w) \leadsto w) \leq 0$. 
Inspired from algebras of logic, we obtained the following:

**Theorem**

Let $G = (G, \lor, \land, +, -, 0)$ be an $l$-group (or, equivalently, let $G_{ig} = (G, \lor, \land, \rightarrow, \sim, 0)$ be the $l$-implicative-group).

The following are equivalent:

(a) $G$ is representable.

(b) $2(a \land b) = 2a \land 2b$,
(b1) $(b \rightarrow a) \land (a \sim b) \leq 0 \land [(b \sim a) \sim (b \rightarrow a)]$,
(b2) $(b \sim a) \land (a \rightarrow b) \leq 0 \land [(b \rightarrow a) \rightarrow (b \sim a)]$.

(b) $2(a \lor b) = 2a \lor 2b$,
(b1) $(b \rightarrow a) \lor (a \sim b) \geq 0 \lor [(b \sim a) \sim (b \rightarrow a)]$,
(b2) $(b \sim a) \lor (a \rightarrow b) \geq 0 \lor [(b \rightarrow a) \rightarrow (b \sim a)]$.

(c) $a \land (-b - a + b) \leq 0$,
(c1) $(x \sim y) \land (((y \sim x) \sim z) \sim z) \rightarrow w) \rightarrow w) \leq 0$,
(c2) $(x \rightarrow y) \land (((y \rightarrow x) \rightarrow z) \rightarrow z) \sim w) \sim w) \leq 0$.

(c) $a \lor (-b - a + b) \geq 0$,
(c1) $(x \sim y) \lor (((y \sim x) \sim z) \sim z) \rightarrow w) \rightarrow w) \geq 0$,
(c2) $(x \rightarrow y) \lor (((y \rightarrow x) \rightarrow z) \rightarrow z) \sim w) \sim w) \geq 0$. 

4.3 Connections between the /-group level and the algebras of logic:

- **On \( G^- \) and \( G^+ \) level**

We obtained the following results:

**Theorem**

Let \( G = (G, \lor, \land, \rightarrow, \sim\rightarrow, 0) \) be a **representable /-implicative-group**. Then,
4.3 Connections between the \( l \)-group level and the algebras of logic:

- On \( G^- \) and \( G^+ \) level

We obtained the following results:

**Theorem**

Let \( \mathcal{G} = (G, \lor, \land, \rightarrow, \sim \rightarrow, 0) \) be a representable \( l \)-implicative-group. Then,

(1). \( G^L = (G^-, \land, \lor, \rightarrow^L, \sim \rightarrow^L, 1 = 0) \) is a representable left-pseudo-BCK(pP) lattice (with the pseudo-product \( \odot = + \)).
**Theorem**

Let $G = (G, \lor, \land, \rightarrow, \Rightarrow, 0)$ be a **representable** $l$-implicative-group. Then,
Theorem

Let $G = (G, \lor, \land, \rightarrow, \twoheadrightarrow, 0)$ be a representable $l$-implicative-group. Then,

(1) the representable left-pseudo-BCK(pP) lattice $G^L = (G^-, \land, \lor, \rightarrow^L, \twoheadrightarrow^L, 1 = 0)$ with the pseudo-product $\odot = +$ verifies also the following conditions: for all $a, b \in G^-$,
Theorem
Let $\mathcal{G} = (G, \lor, \land, \rightarrow, \nrightarrow, 0)$ be a representable $l$-implicative-group. Then,

(i) the representable left-pseudo-BCK(pP) lattice $\mathcal{G}^L = (G^-, \land, \lor, \rightarrow^L, \nrightarrow^L, 1 = 0)$ with the pseudo-product $\circ = +$ verifies also the following conditions: for all $a, b \in G^-$,

(i) $(a \lor b)^2 = a^2 \lor b^2$, i.e. $(a \lor b) \circ (a \lor b) = (a \circ a) \lor (b \circ b)$,
(ii) Condition (i) is equivalent with condition

\[ [b \rightarrow^L (a \nrightarrow^L (a \circ a))] \lor [a \nrightarrow^L (b \rightarrow^L (b \circ b))] = 1. \tag{3} \]

(iii) $(b \rightarrow^L a) \lor (a \nrightarrow^L b) = 1$,
(iv) Condition (iii) implies condition (3).
Proposition
Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \sim, 0)$ be an $l$-implicative-group. (1). If $\mathcal{G}$ verifies the condition:
(b1''') for all $a, b \in G$, $(b \rightarrow a) \vee (a \sim b) \geq 0$,
then
Proposition
Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \sim\rightarrow, 0)$ be an $l$-implicative-group.

(1). If $\mathcal{G}$ verifies the condition:

(b1d”) for all $a, b \in G$, $(b \rightarrow a) \lor (a \sim\rightarrow b) \geq 0$,

then

the left-pseudo-BCK(pP) lattice $\mathcal{G}^L = (G^-, \wedge, \vee, \rightarrow^L, \sim\rightarrow^L, 1 = 0)$ verifies the condition (iii) from above Theorem, namely:

(iii) for all $a, b \in G^-$, $(b \rightarrow^L a) \lor (a \sim\rightarrow^L b) = 1 = 0$.
Thank you for your attention!