WELL-COMPOSED J-LOGICS AND INTERPOLATION

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July 2011

Abstract

Extensions of the Johansson minimal logic are investigated. Representation theorems for well-composed logics with the Graig interpolation property CIP, restricted interpolation property IPR and projective Beth property PBP are stated. It is proved that PBP is equivalent to IPR for any well-composed logic, and there are only finitely many well-composed logics with CIP, IPR or PBP.

Interpolation theorem proved by W.Craig in 1957 for the classical first order logic was a source of a lot of research results devoted to interpolation problem in classical and non-classical logical theories. Now interpolation is considered as a standard property of logics and calculi like consistency, completeness and so on. For the intuitionistic predicate logic and for the predicate version of Johansson's minimal logic the interpolation theorem was proved by K.Schütte (1962).

In this paper we consider several variants of the interpolation property in the minimal logic and its extension. The minimal logic introduced by I.Johansson (1937) has the same positive fragment as the intuitionistic logic but has no special axioms for negation. In contrast to the classical and intuitionistic logics, the minimal logic admits non-trivial theories containing some proposition together with its negation.

Various versions of interpolation

The original definition of interpolation admits different analogs which are equivalent in the classical logic but are not equivalent in other logics.

If $\mathbf{p}$ is a list of non-logical symbols, let $A(\mathbf{p})$ denote a formula whose all non-logical symbols are in $\mathbf{p}$, and $\mathcal{F}(\mathbf{p})$ the set of all such formulas.

Let L be a logic, $\vdash_L$ deducibility relation in L. Suppose that $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ are disjoint lists of non-logical symbols, and $A(\mathbf{p},\mathbf{q})$, $B(\mathbf{p},\mathbf{r})$ are formulas. *The Craig interpolation property CIP* and *the deductive interpolation property IPD* are defined as follows:

CIP. If $\vdash_L A(\mathbf{p},\mathbf{q}) \to B(\mathbf{p},\mathbf{r})$, then there exists a formula $C(\mathbf{p})$ such that $\vdash_L A(\mathbf{p},\mathbf{q}) \to C(\mathbf{p})$ and $\vdash_L C(\mathbf{p}) \to B(\mathbf{p},\mathbf{r})$.

IPD. If $A(\mathbf{p},\mathbf{q}) \vdash_L B(\mathbf{p},\mathbf{r})$, then there exists a formula $C(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L C(\mathbf{p})$ and $C(\mathbf{p}) \vdash_L B(\mathbf{p},\mathbf{r})$.

IPR. If $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$, then there exists a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$.

WIP. If $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$, then there exists a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$.

Beth's definability properties

have as their source the theorem on implicit definability proved by E.Beth in 1953 for the classical first order logic: Any predicate implicitly definable in a first order theory is explicitly definable. We formulate some analogs of Beth's property for propositional logics. Let $\mathbf{x}$, $\mathbf{q}$, $\mathbf{q}'$ be disjoint lists of variables not containing y and z, $A(\mathbf{x}, \mathbf{q}, y)$ a formula. We define the *projective Beth property*:

PBP. If $A(\mathbf{x}, \mathbf{q}, y), A(\mathbf{x}, \mathbf{q}', z) \vdash_L (y \leftrightarrow z)$, then $A(\mathbf{x}, \mathbf{q}, y) \vdash_L (y \leftrightarrow B(\mathbf{x}))$ for some formula $B(\mathbf{x})$.

We get a weaker version BP of the Beth property by deleting $\mathbf{q}$ in PBP.

Propositional J-logics

In all extensions of the minimal logic we have

$$\text{IPD} \iff \text{CIP} \Rightarrow \text{PBP} \Rightarrow \text{IPR} \Rightarrow \text{WIP};$$

PBP is weaker than CIP, and WIP is weaker that IPR.
All J-logics have BP.

The language of the logic J contains $\&, \vee, \rightarrow, \bot, \top$ as primitive; negation is defined by $\neg A = A \rightarrow \bot$; $(A \leftrightarrow B) = (A \rightarrow B)\&(B \rightarrow A)$. A formula is said to be *positive* if contains no occurrences of $\bot$. The logic J can be axiomatized by the calculus, which has the same axiom schemes as the positive intuitionistic calculus Int^{+}, and the only rule of inference is modus ponens. By a J-*logic* we mean an arbitrary set of formulas containing all the axioms of J and closed under modus ponens and substitution rules. We denote

$$\mathrm{Int} = \mathrm{J} + (\bot \rightarrow p),\ \mathrm{Cl} = \mathrm{Int} + (p \vee \neg p),\ \mathrm{Neg} = \mathrm{J} + \bot,$$

$$\mathrm{JX} = \mathrm{J} + (\bot \rightarrow A) \vee (A \rightarrow \bot).$$

A logic is *non-trivial* if it differs from the set of all formulas. A J-logic is *superintuitionistic* if it contains the intuitionistic logic Int, and *negative* if contains the logic Neg; L is *paraconsistent* if contains neither Int nor Neg. L is *well-composed* if it contains JX. For any J-logic L we denote by $E(L)$ the family of all J-logics containing L.

There are only finitely many s.i.logics with CIP, IPR or PBP [M77, M2000]. A similar result holds for positive and negative logics [M2003]. All superintuitionistic and negative logics possess WIP.

Theorem (M2010)

IPR and PBP are equivalent over Int *and* Neg.

Theorem

CIP, IPR and PBP are decidable over Int *and* Neg*, i.e. there are algorithms which, given a finite set* Ax *of axiom schemes, decide if the logic* $\mathrm{Int} + Ax$ *(or* $\mathrm{Neg} + Ax$*) has CIP, IPR or PBP.*

There is a continuum of J-logics with WIP and a continuum of J-logics without WIP.

Theorem (M2011)

WIP is decidable over $\mathbf{J}$.

Algebraic interpretation

For extensions of the minimal logic the algebraic semantics is built with using so-called J-*algebras*, i.e. algebras $\mathbf{A} =< A; \&, \vee, \rightarrow, \perp, \top >$ satisfying the conditions:
$< A; \&, \vee, \rightarrow, \perp, \top >$ is a lattice with respect to $\&, \vee$ having a greatest element $\top$, where
$z \leq x \rightarrow y \iff z \& x \leq y$,
$\perp$ is an arbitrary element of A.

A formula B is said to be *valid* in a J-algebra $\mathbf{A}$ if the identity $B = \top$ is satisfied in $\mathbf{A}$.

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A formula B is said to be *valid* in a J-algebra **A** if the identity $B = \top$ is satisfied in **A**.

A J-algebra is called a *Heyting algebra* if $\perp$ is the least element of A, and a *negative algebra* if $\perp$ is the greatest element of A. A one-element J-algebra is said to be *degenerate*; it is the only J-algebra, which is both a negative algebra and a Heyting algebra. A J-algebra **A** is *non-degenerate* if it contains at least two elements; **A** is said to be *well connected* (or *strongly compact*) if for all $x, y \in$ **A** the condition $x \vee y = \top \Leftrightarrow (x = \top$ or $y = \top)$ is satisfied. An element a of **A** is called an *opremum of* **A** if it is the greatest among the elements of **A** different from $\top$. By B_0 we denote the two-element Boolean algebra $\{\perp, \top\}$.

In this section we find algebraic equivalents of the interpolation properties.

It is well known that the family of all J-algebras forms a variety, i.e. can be determined by identities. There exists a one-to-one correspondence between logics extending the logic J and varieties of J-algebras. If A is a formula and $\mathbf{A}$ is an algebra, we say that A is valid in $\mathbf{A}$ and write $\mathbf{A} \models A$ if the identity $A = \top$ is satisfied in $\mathbf{A}$. We write $\mathbf{A} \models L$ instead of $(\forall A \in L)(\mathbf{A} \models A)$.

To any logic $L \in E(\mathrm{J})$ there corresponds a variety

$$V(L) = \{\mathbf{A} | \mathbf{A} \models L\}.$$

Every logic L is characterized by the variety $V(L)$.
If $L \in E(\mathrm{Int})$, then $V(L)$ is a variety of Heyting algebras, and if $L \in E(\mathrm{Neg})$, then a variety of negative algebras.
Recall [M2003] that a J-logic has the Craig interpolation property if and only if $V(L)$ has the amalgamation property AP.

We recall the definitions. A class V has *Amalgamation Property* if it satisfies
AP: For each $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$, there exist an algebra $\mathbf{D}$ in V and monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$.
Super-Amalgamation Property (SAP) is AP with extra conditions:

$$\delta(x) \leq \varepsilon(y) \Leftrightarrow (\exists z \in \mathbf{A})(x \leq z \text{ and } z \leq y),$$

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Restricted Amalgamation Property (RAP) and *Weak Amalgamation Property (WAP)* are defined as follows:
RAP: for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$, there exist an algebra $\mathbf{D}$ in V and homomorphisms $g : \mathbf{B} \to \mathbf{D}$ and $h : \mathbf{C} \to \mathbf{D}$ such that $g(x) = h(x)$ for all $x \in \mathbf{A}$ and the restriction of g onto $\mathbf{A}$ is a monomorphism.

WAP: For each $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that **A** is a common subalgebra of **B** and **C**, there exist an algebra **D** in V and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$, and $\bot \neq \top$ in **D** whenever $\bot \neq \top$ in **A**.

A class V has *Strong Epimorphisms Surjectivity* if it satisfies SES: For each $\mathbf{A}, \mathbf{B}$ in V, for every monomorphism $\alpha : \mathbf{A} \to \mathbf{B}$ and for every $x \in \mathbf{B} - \alpha(\mathbf{A})$ there exist $\mathbf{C} \in V$ and homomorphisms $\beta : \mathbf{B} \to \mathbf{C}$, $\gamma : \mathbf{B} \to \mathbf{C}$ such that $\beta\alpha = \gamma\alpha$ and $\beta(x) \neq \gamma(x)$.

Theorem

[M2005] *For any* J*-logic L:*
(1) L has CIP iff V(L) has SAP iff V(L) has AP,
(2) L has IPR iff V(L) has RAP,
(3) L has WIP iff V(L) has WAP,
(4) L has PBP iff V(L) has SES.

In varieties of J-algebras:

$$\mathrm{SAP} \iff \mathrm{AP} \Rightarrow \mathrm{SES} \Rightarrow \mathrm{RAP} \Rightarrow \mathrm{WAP}.$$

Interpolation in well-composed J-logics

A J-algebra is *well-composed* if every its element is comparable with $\perp$. For any well-composed J-algebra **A**, the set $\mathbf{A}^l = \{x | \ x \leq \perp\}$ forms a negative algebra, and the set $\mathbf{A}^u = \{x | \ x \geq \perp\}$ forms a Heyting algebra.

If **B** is a negative algebra and **C** is a Heyting algebra, we denote by $\mathbf{B} \uparrow \mathbf{C}$ a well-composed algebra **A** such that $\mathbf{A}^l$ is isomorphic to **B** and $\mathbf{A}^u$ to **C**.

For a negative algebra **B**, we denote by $\mathbf{B}^\Lambda$ a J-algebra arisen from **B** by adding a new greatest element $\top$.

A J-algebra **A** is *finitely indecomposable* if for all $x, y \in \mathbf{A}$: $x \vee y = \top \Leftrightarrow (x = \top \text{ or } y = \top)$.

For $L_1 \in E(\mathrm{Neg})$, $L_2 \in E(\mathrm{Int})$ we denote by $L_1 \uparrow L_2$ a logic characterized by all algebras of the form $\mathbf{A} \uparrow \mathbf{B}$, where $\mathbf{A} \models L_1$, $\mathbf{B} \models L_2$; a logic characterized by all algebras $\mathbf{A} \uparrow \mathbf{B}$, where $\mathbf{A}$ is a finitely decomposable algebra in $V(L_1)$ and $\mathbf{B} \in V(L_2)$, is denoted by $L_1 \Uparrow L_2$.
In [M2005] an axiomatization was found for logics $L_1 \uparrow L_2$ and $L_1 \Uparrow L_2$, where L_1 is a negative and L_2 an s.i. logic.

It is known that there are only finitely many s.i. and negative logics with CIP, IPR and PBP [GM,M2005,M2010]. We give the list of all negative logics with CIP:

$$\mathrm{Neg},\ \mathrm{NC} = \mathrm{Neg} + (p \to q) \vee (q \to p),$$

$$\mathrm{NE} = \mathrm{Neg} + p \vee (p \to q),\ \mathrm{For} = \mathrm{Neg} + p.$$

For any J-logic L define

$$L_{neg} = L + \bot.$$

The following theorem describes all well-composed logics with CIP.

Theorem

Let L be a well-composed logic. Then L has CIP if and only if L coincides with one of the logics:
(1) $L_1 \cap L_2$*, where* $L_1 = L_{neg}$ *is a negative logic with CIP and* L_2 *is a superintuitionistic logic with CIP;*
(2) $L_1 \cap (L_3 \Uparrow L_2)$*, where* $L_1 = L_{neg}$ *is a negative logic with CIP,* L_2 *is a consistent s.i. logic with CIP and* $L_3 \in \{\mathrm{Neg}, \mathrm{NC}, \mathrm{NE}\}$*;*
(3) $L_1 \cap (L_3 \uparrow L_2)$*, where* L_1*,* L_2*,* L_3 *are the same as in (2).*

The following two theorems give a full description of well-composed logics with IPR and PBP.
It is proved in [M2011] that WIP is decidable over J, i.e. there is an algorithm which, given a finite set *Ax* of axiom schemes, decides if the logic J+*Ax* has WIP. A crucial role in the description of J-logics with WIP belongs to the following list of eight logics:

$$\mathrm{SL} = \{\mathrm{For}, \mathrm{Cl}, (\mathrm{NE} \uparrow \mathrm{Cl}), (\mathrm{NC} \uparrow \mathrm{Cl},$$

$$(\mathrm{Neg} \uparrow \mathrm{Cl}), (\mathrm{NE} \Uparrow \mathrm{Cl}), (\mathrm{NC} \Uparrow \mathrm{Cl}), (\mathrm{Neg} \Uparrow \mathrm{Cl})\}.$$

For a negative algebra **B**, we denote by $\mathbf{B}^{\Lambda}$ a J-algebra arisen from **B** by adding a new greatest element $\top$.
Let $\Lambda(L) = \{\mathbf{B}^{\Lambda} | \mathbf{B}^{\Lambda} \in V(L)\}$.

Theorem

Let L be a well-composed logic, the logic L_{neg} have IPR and

$$L = L_{neg} \cap L_0 \cap L_1,$$

where $L_0 \in SL$, $\Lambda(L_0) \supseteq \Lambda(L_1)$, $L_1 \in \{\mathrm{For}, (L_2 \uparrow L_3), (L_2 \Uparrow L_3)\}$, L_2 is a negative logic with CIP, and L_3 is a superintuitionistic logic with IPR. Then L has IPR and, moreover, L has PBP.

Theorem

Let a well-composed logic L have IPR. Then the logic L_{neg} has IPR, and L is representable as

$$L = L_{neg} \cap L_0 \cap L_1,$$

where $L_0 \in SL$, $\Lambda(L_0) \supseteq \Lambda(L_1)$, $L_1 \in \{\mathrm{For}, (L_2 \uparrow L_3), (L_2 \Uparrow L_3)\}$, L_2 is a negative logic with CIP, and L_3 is a superintuitionistic logic with IPR.

Corollary

1. *There are only finitely many well-composed logics with IPR; all of them are finitely axiomatizable.*
2. *IPR and PBP are equivalent on the class of well-composed logics.*

Problem 1. How many J-logics have CIP, IPR or PBP?
Problem 2. Are IPR and PBP equivalent over J?
Problem 3. Are CIP, IPR and/or PBP decidable over J? The same question for the class of well-composed logics.

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