Categorically axiomatizing the classical quantifiers

Hyperdoctrines for classical logic

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The question

When should we consider two proofs in the classical sequent calculus identical

We consider this problem, for the logic with first-order quantifiers, using *categorical proof theory*.

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Categorical proof theory

Consider categories of formulae/proofs in a formal system.

objects = formula

morphisms = (equivalence classes of) derivations

Morphism composition: from $A \to B$ and $B \to C$ infer $A \to C$.

The equivalence classes of morphisms should characterize a natural notion of equality on proofs.

Propositional intuitionistic natural deduction

Prawitz: two ND derivations equal if they have the same $\beta\eta$-normal form.

Equational theory of a cartesian-closed category: ccc's give the "model theory" of intuitionistic natural deduction.

Propositional MLL

Two sequent MLL derivations are identical if (roughly) they have the same cut-free proof net.

Equational theory of a $*$-autonomous category (or, equivalently, a *symmetric linearly distributive category with negation*).

Propositional Classical logic

Two sequent LK derivations are identical if ?

Here we cannot use cut-elimination to define morphism equality, since it is essentially nonconfluent — if we identify derivations before and after cut-elimination, we identify *all* derivations.

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A ccc plus a dualizing negation is a poset.

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Instead, model it as *inequality*.

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Classical categories (Pym, Führmann)

A *classical category* is an *order-enriched* category $\mathcal{C}$ with

- A $*$-autonomous structure $(\mathcal{C}, \wedge, \top, (-)^{\perp})$
- Such that the defining adjunction for $(-)^{\perp}$ is an order-isomorphism
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$$\frac{A \wedge B \to C}{A \to (B \wedge C^{\perp})^{\perp}}$$

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$$\Delta : A \to A \wedge A \qquad \langle\rangle : A \to a$$

$$\Delta \circ f \leqslant (f \otimes f) \circ \Delta$$

$$\langle\rangle \circ f \leqslant \langle\rangle$$

Classical categories (Pym, Führmann)

Classical sequent proofs form a classical category, if we quotient under:

- Linear, local cut-reduction steps as equalities
- Nonlinear cut reduction steps (involving structural rules) as inequalities.

(plus some other simple identities on proofs)

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Classical categories (Pym, Führmann)

There are other non-trivial classical categories, most notably built from sets and relations.

Interpreting proofs in such categories give notions of identity on classical proofs.

Hyperdoctrines – from propositional to first-order logics.

Idea: treat the formulas/proofs over a given set of free variables as a catgeory.

Substitution/quantifiers are functors between these categories.

Key observation (Lawvere): Quantifiers arise as adjoints:

$$\frac{A \vdash B}{\exists x.A \vdash B}$$

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Hyperdoctrines for classical logic

Clear question: what is the notion of hyperdoctrine for classical sequent proofs?

Setting $\exists x \dashv x^* \dashv \forall_x$ rules out certain interpretations of the quantifiers – in particular as infinitary connectives.

Instead, we can use an "adjunction-up-to-adjunction", or "lax adjunction"...

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Classical doctrines

$$\frac{A \vdash x^* B}{\exists x. A \vdash B} \leqslant$$

$$\varepsilon : \exists x(x^* A) \to A \qquad \eta : A \to x^*(\exists x A)$$

$$f \circ \varepsilon \leqslant \varepsilon \circ \exists x(x^* f) \quad \text{and} \quad x^*(\exists x g) \circ \eta \leqslant \eta \circ g.$$

We call a morphism “strong” if these diagrams commute.

Interpreting the quantifier rules

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \exists x.A \vdash \Delta}\ \exists L$$

$$\lfloor\Gamma\rfloor \wedge \exists x.\lfloor A\rfloor \to \exists x.(\lfloor x^*\Gamma\rfloor \wedge \lfloor A\rfloor) \to \exists x.(\lfloor x^*\Delta\rfloor) \to \lfloor\Delta\rfloor$$

$$\frac{\Gamma, \vdash B, \Delta}{\Gamma, \vdash \exists x.B, \Delta}\ \exists R$$

$$\lfloor\Gamma\rfloor \to \lfloor B\rfloor \vee \lfloor\Delta\rfloor \to x^*(\exists x.\lfloor B\rfloor) \vee \lfloor\Delta\rfloor$$

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Definition

A *dual doctrine* is a functor $\mathcal{C} : \mathcal{B}^{op} \to \mathcal{C}at$ such that

- $\mathcal{B}$ has finite products.
- $\mathcal{C}(X)$ is a classical category, for each X.
- For each f a morphism in B, $\mathcal{C}(f)$ $(= f^*)$ is strong monoidal with respect to $\wedge$.
- For each projection π in $\mathcal{B}$, there is a right *lax* adjoint $\forall_\pi$, which is a symmetric monoidal functor, such that the adjunction is symmetric monoidal.
- The *Beck-Chevally condition*
- The existence of *Prenex strengths*
- The *switch morphism* is strong.

Classical doctrines

Prenex strengths The morphism

$$\mathbf{Prenex}^{\circ} = \mu_{A,x^*B} \circ (\mathbf{id} \otimes \eta) : \forall x A \vee B \to \forall x (A \vee x^* B)$$

has a right adjoint **Prenex** such that

$$\mathbf{Prenex}^{\circ} \circ \mathbf{Prenex} \leqslant \mathbf{id}$$

and

$$\mathbf{Prenex} \circ \mathbf{Prenex}^{\circ} = \mathbf{id}$$

Switch

Let $A \vee B$ be defined as $(A^\perp \wedge B^\perp)^\perp$

Then, in every $*$-autonomous category, there is a morphism

$$A \wedge (B \vee C) \rightarrow (A \wedge B) \vee C$$

called "weak/linear distributivity" (Cockett/Seely) or "switch" (Guglielmi, Lamarche, Strassburger).

Plays a key role in interpreting the cut rule. We require it to be *strong*.

Term model, soundness

We can construct a term classical category from sequent proofs in LK, by interpreting cut elimination as an inequality and then forming a quotient of proofs by an (unfortunately rather complicated) equivalence relation.

Theorem

Interpretation of proofs in a classical doctrine is sound w.r.t. cut-elimination: if Φ cut-reduces to Ψ then

$$\lfloor \Phi \rfloor \leqslant \lfloor \Psi \rfloor$$

Duality

An alternative formulation of classical categories takes the switch as basic rather than the $*$-autonomous structure. Duality is not built in, so we can axiomatize the duality of the two quantifiers (not automatic, since the adjunctions are only “up to adjunction”).

Further work

Finding concrete examples!

- We have non-syntactic example, built from families of sets and relations using an abstract GoI construction (Abramsky's Int construction).
- This at least shows the axioms do not imply collapse...
- But more would be nice!
- In particular, examples where the quantifiers arise as genuine adjoints.

First-order versions of other notions of model for classical proofs.

Fin

Thank you for your attention.