Stone duality for skew Boolean algebras

Ganna Kudryavtseva

Ljubljana University

TACL, 2011

Table of contents

Notation

- BA — the category of Boolean algebras
- BS — the category of Boolean spaces
- LCBS — the category of locally compact Boolean spaces
- GBA — the category of generalized Boolean algebras
- ESLCBS — the category of étale spaces over LCBS whose morphisms are étale spaces cohomomorphisms over morphisms in LCBS
- LSBA — the category of left-handed skew Boolean algebras and SBA morphisms over morphisms of GBA

Skew Boolean algebras

A skew lattice S
is an algebra $(S; \wedge, \vee)$, such that $\wedge$ and $\vee$ are associative, idempotent and satisfy the absorption identities $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ and $(y \vee x) \wedge x = x = (y \wedge x) \vee x$.

The natural partial order $\leq$ on S
is defined by $x \leq y$ if and only if $x \wedge y = y \wedge x = x$, or equivalently, $x \vee y = y \vee x = y$.

A skew lattice S is called Boolean, provided that
$x \vee y = y \vee x$ if and only if $x \wedge y = y \wedge x$, S has a zero element and each principal subalgebra $\lceil x \rceil = \{u \in S : u \leq x\} = x \wedge S \wedge x$ forms a Boolean lattice. $(S; \wedge, \vee, \setminus, 0)$ is called a skew Boolean algebra.

Relation $\mathcal{D}$

Let $\mathcal{D}$ be the equivalence relation on a skew lattice S defined by $x\mathcal{D}y$ if and only if $x \wedge y \wedge x = x$ and $y \wedge x \wedge y = y$.

Theorem(Leech)

The relation $\mathcal{D}$ on a skew lattice S is a congruence, the $\mathcal{D}$-classes are maximal rectangular subalgebras, the quotient algebra $S/\mathcal{D}$ forms the maximal lattice image of S. If S is a skew Boolean algebra, then $S/\mathcal{D}$ is the maximal generalized Boolean algebra image of S.

Left-handed and primitive SBAs

A skew lattice S is left-handed,
if the rectangular subalgebras are flat in the sense that $x\mathcal{D}y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

A skew Boolean algebra S is called primitive, if:

- it has only one non-zero $\mathcal{D}$-class, or, equivalently,
- $S/\mathcal{D}$ is the Boolean algebra $\mathbf{2}$.

Finite primitive left-handed skew Boolean algebras:
$\mathbf{n}+\mathbf{2} = \{0, 1, \ldots, n+1\}$, $n \geq 0$, the operations are determined by lefthandedness: $i \wedge j = i$, $i \vee j = j$ for $i \neq j$ and $i, j \neq 0$.

From an étale space to skew Boolean algebra

Construction

Let $X \in \mathrm{Ob}(\mathsf{LCBS})$ and (E, f, X) be an étale space. Let $E^\star$ be the set of sections of E whose base sets are compact and clopen. Fix $s, t \in E^\star$ and assume $s \in E(U)$, $t \in E(V)$. Define the quasi-union $s \underline{\cup} t \in E(U \cup V)$:

$$(s \underline{\cup} t)(x) = \begin{cases} t(x), & \text{if } x \in V, \\ s(x), & \text{if } x \in U \setminus V, \end{cases}$$

and the quasi-intersection $s \overline{\cap} t \in E(U \cap V)$:

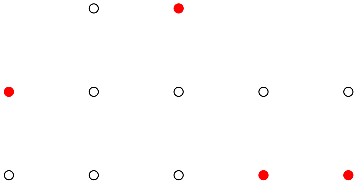
$$(s \overline{\cap} t)(x) = s(x) \text{ for all } x \in U \cap V.$$

Proposition

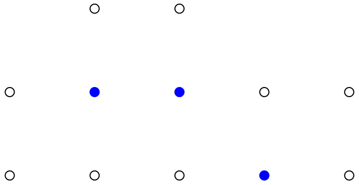
$(E^\star, \underline{\cup}, \overline{\cap}, \setminus, \varnothing)$ (where $\varnothing$ is the section of the empty set of X) is a left-handed skew Boolean algebra.

Example

s is the section colored in red

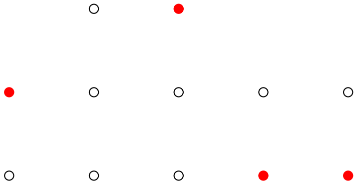


t is the section colored in blue

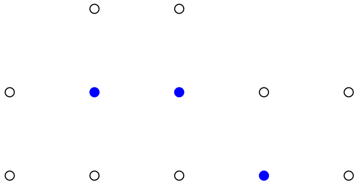


Example

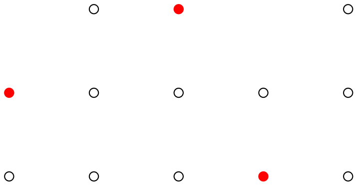
s is the section colored in red



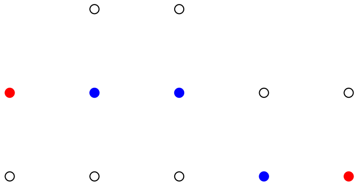
t is the section colored in blue



The set $s \overline{\cap} t$



The set $s \underline{\cup} t$



From cohomomorphisms of étale spaces to homomorphisms of SBAs

Definition

Let $(\mathcal{A}, g, X)$ and $(\mathcal{B}, h, Y)$ be étale spaces and $f : X \to Y$ be in $\mathrm{Hom}_{\mathsf{LCBS}}(X, Y)$. An *$f$-cohomomorphism* $k : \mathcal{B} \rightsquigarrow \mathcal{A}$ is a collection of maps $k_x : \mathcal{B}_{f(x)} \to \mathcal{A}_x$ for each $x \in X$ such that for every section $s \in \mathcal{B}(U)$ the function $x \mapsto k_x(s(f(x)))$ is a section of $\mathcal{A}$ over $f^{-1}(U)$.

The functor SB

Let (E, e, X) and (G, g, Y) be étale spaces, $f : X \to Y$ be in $\mathrm{Hom}_{\mathsf{GBA}}(X, Y)$ and $k : G \rightsquigarrow E$ be an f-cohomomorphism. k preserves 0, $\overline{\cap}$ and $\underline{\cup}$ for sections in $E^{\star}$, so that one can look at k as to an element of $\mathrm{Hom}_{\mathsf{LSBA}}(G^{\star}, E^{\star})$. We have constructed the functor $\mathbf{SB} : \mathsf{ESLCBS} \to \mathsf{LSBA}$ given by $\mathbf{SB}(E, f, X) = (E, f, X)^{\star}$ and $\mathbf{SB}(k) = k$.

Filters and prime filters of skew Boolean algebras

Definition

$S \in \mathrm{Ob}(\mathsf{LSBA})$. A subset $U \subseteq S$ is called a filter provided that:

1. for all $a, b \in S$: $a \in U$ and $b \geq a$ implies $b \in U$;
2. for all $a, b \in S$: $a \in U$ and $b \in U$ imply $a \wedge b \in U$.

Definition

$U \subseteq S$ is a preprime filter if U is a filter and there is a prime filter F of $S/\mathcal{D}$ such that $\alpha(U) = F$ (where $\alpha : S \to S/\mathcal{D}$ is the projection of S onto $S/\mathcal{D}$). Denote by $\mathcal{PU}_F$ the set of all preprime filters contained in $\alpha^{-1}(F)$. Minimal elements of $\mathcal{PU}_F$ form the set $\mathcal{U}_F$ and are called prime filters of S. Prime filters are exactly minimal nonempty preimages of 1 under the morphisms $S \to \mathbf{3}$.

The spectrum of a skew Boolean algebra

The spectrum $S^{\star}$ of S is defined as the set of all SBA-prime filters of S.

The spectrum of a skew Boolean algebra

The spectrum $S^{\star}$ of S is defined as the set of all SBA-prime filters of S.
Topology on $S^{\star}$?

The spectrum of a skew Boolean algebra

The spectrum $S^\star$ of S is defined as the set of all SBA-prime filters of S.
Topology on $S^\star$?

- For $a \in S$ we define the set

$$M(a) = \{F \in S^\star : a \in F\}.$$

- Topology on $S^\star$: its subbase is formed by the sets $M(a)$, $a \in S$.

The spectrum of a skew Boolean algebra

The spectrum $S^{\star}$ of S is defined as the set of all SBA-prime filters of S.
Topology on $S^{\star}$?

- For $a \in S$ we define the set

$$M(a) = \{F \in S^{\star} : a \in F\}.$$

- Topology on $S^{\star}$: its subbase is formed by the sets $M(a)$, $a \in S$.

Proposition

Let $f : S^{\star} \to (S/\mathcal{D})^{\star}$ be the map, given by $U \mapsto F$, whenever $U \in \mathcal{U}_F$.
Then $(S^{\star}, f, (S/\mathcal{D})^{\star})$ is an étale space.

The spectrum of a skew Boolean algebra

The spectrum $S^\star$ of S is defined as the set of all SBA-prime filters of S.
Topology on $S^\star$?

- For $a \in S$ we define the set

$$M(a) = \{F \in S^\star : a \in F\}.$$

- Topology on $S^\star$: its subbase is formed by the sets $M(a)$, $a \in S$.

Proposition

Let $f : S^\star \to (S/\mathcal{D})^\star$ be the map, given by $U \mapsto F$, whenever $U \in \mathcal{U}_F$.
Then $(S^\star, f, (S/\mathcal{D})^\star)$ is an étale space.

Proposition

$\mathcal{U}_F = S^\star_F$, $F \in (S/\mathcal{D})^\star$.

From SBA-homomorphisms to étale space cohomomorphisms

Let S, T and $k : T \to S$ be in LSBA. Let $\overline{k} : T/\mathcal{D} \to S/\mathcal{D}$ be the induced morphism in GBA. Let $F \in (S/\mathcal{D})^\star$ and $V \in S_F^\star$.

Proposition

The set $k^{-1}(V)$, if nonempty, is some $U' \in \mathcal{PU}_{\overline{k}^{-1}(F)}$. In other words, preimage under k of an SBA-prime filter, if nonempty, is a set of SBA-prime filters; k is agreed with $\overline{k}$.

Definition

We set $\tilde{k}_F(U) = V$, provided that $k^{-1}(V) \supseteq U$. The maps $k_F : T^\star_{\overline{k}^{-1}(F)} \to S_F^\star$ constitute a $\overline{k}^{-1}$-cohomomorphism $\tilde{k} : T^\star \rightsquigarrow S^\star$.

We have defined the functor **ES** : LSBA $\to$ ESLCBS by setting $\mathbf{ES}(S) = S^\star$ and $\mathbf{ES}(k) = \tilde{k}$.

Refinement of Stone duality to skew Boolean algebras

Theorem (K,2011)

The functors **SB** *and* **ES** *establish an equivalence between the categories* ESLCBS *and* LSBA*, where the natural isomorphism* $\beta : 1_{\mathsf{LSBA}} \to \mathbf{SB} \cdot \mathbf{ES}$ *and* $\gamma : 1_{\mathsf{ESLCBS}} \to \mathbf{ES} \cdot \mathbf{SB}$ *are given by*

$$\beta_S(a) = M(a) = \{F \in S^\star : a \in F\}, S \in \mathrm{Ob}(\mathsf{LSBA}), a \in S;$$

$$\gamma_E(A) = N_A = \{N \in E^\star : A \in N\}, E \in \mathrm{Ob}(\mathsf{ESLCBS}), A \in E.$$

This theorem generalizes the classical Stone duality viewed as an equivalence between the categories LCBS^{op} and GBA.

Skew Boolean ∩-algebras

Definition

A skew Boolean algebra S has finite intersections, if any finite set $\{s_1, \dots s_k\}$ of elements in S has the greatest lower bound with respect to $\leq$. A skew Boolean algebra S with finite intersections, considered as an algebra $(S; \wedge, \vee, \setminus, \cap, 0)$, where $\cap$ is the binary operation on S sending (a, b) to $a \cap b$, is called a skew Boolean ∩-algebra.

Example

All skew Boolean algebras S such that $S/\mathcal{D}$ is finite have finite intersections.

LSBIA — the category of left-handed skew Boolean ∩-algebras and skew Boolean ∩-algebra morphisms.

The category ESLCBSE

Equalizer condition

Let $X \in \mathrm{Ob}(\mathsf{LCBS})$. Call an étale space (E, π, X) an étale space with compact clopen equalizers if for every U, V compact clopen in X and any $A \in E(U)$, $B \in E(V)$, the intersection $A \cap B$ belongs to $E(W)$ for some compact clopen set W of X.

A characterization of equalizer condition

(E, π, X) is an étale space with compact clopen equalizers if and only if E is Hausdorff.

Injective étale space cohomomorphisms

Let $k \in \mathrm{Hom}_{\mathsf{ESLCBS}}(E_1, E_2)$. We call k injective if all its components k_F are injective maps.

ESLCBSE: the category of étale spaces satisfying the equalizer condition and injective étale space cohomomorphisms.

Refinement of Stone duality to skew Boolean ∩-algebras

Theorem (K,2011)

The restrictions functors **SB** *and* **ES** *to the categories* ESLCBSE *and* LSBIA*, respectively, establish an equivalence between the categories* ESLCBSE *and* LSBIA*.*

A different view of this duality is due to Bauer and Cvetko-Vah (2011).

Remark

This duality (only for ES with finite stalks) also follows from a universal algebra result due to Keimel and Werner (1974), because finite primitive skew Boolean ∩-algebras are quasi-primal.

The Hom-set construction of $\lambda_n(X)$

Let $n \geq 0$ be fixed from now on.
Let $X \in \mathrm{Ob}(\mathsf{LCBS})$. $\lambda_n(X)$: the set of all continuous maps $f : X \to \{0, \dots, n+1\}$, such that $f^{-1}(1), \dots f^{-1}(n+1)$ are compact sets.

Define $\wedge$, $\vee$ and 0 on $\lambda_n(X)$ as the induced operations of $\wedge$, $\vee$ and 0 on the primitive skew Boolean algebra $\mathbf{n+2}$: for f, g in $\lambda_n(X)$

$$(f \wedge g)(x) = f(x) \wedge g(x), (f \vee g)(x) = f(x) \vee g(x)$$

and set the zero of $\lambda_n(X)$ to be the zero function on X.
This turns $\lambda_n(X)$ into skew Boolean algebras.

Representation of $\lambda_n(X)$ as a set of $(n+1)$ tuples

Let $f : X \to \{0, 1, \dots, n+1\}$. Via the bijection

$$f \mapsto (f^{-1}(1, \dots, n+1), f^{-1}(1, \dots, n), \dots, f^{-1}(1)).$$

we have

$$\lambda_n(X) = \{(A_{n+1}, A_n, \dots, A_1) : X \supseteq A_{n+1} \supseteq \dots \supseteq A_1, \\ A_i \text{ is compact and clopen for all } 1 \leq i \leq n+1\}.$$

Operations on $(n+1)$-tuples:

$$(A_i)_{n+1 \geq i \geq 1} \wedge (B_i)_{n+1 \geq i \geq 1} = (A_i \cap B_{n+1})_{n+1 \geq i \geq 1},$$
$$(A_i)_{n+1 \geq i \geq 1} \vee (B_i)_{n+1 \geq i \geq 1} = ((A_i \setminus B_{n+1}) \cup B_i)_{n+1 \geq i \geq 1}.$$

The zero of $\lambda_n(X)$ is the $(n+1)$-tuple $(\varnothing, \dots, \varnothing)$.

The functors λ_n, $n \geq 0$

λ_n on morphisms

Let $f : X_2 \to X_1$ be in $\mathrm{Hom}_{LCBS^{op}}(X_1, X_2)$. For $(A_i)_{n+1\geq i\geq 1} \in \lambda_n(X_1)$ we set

$$\lambda_n(f)((A_i)_{n+1\geq i\geq 1}) = (f^{-1}(A_i))_{n+1\geq i\geq 1}.$$

Remark

The functor ω : GBA $\to$ LSBA, $\omega(B) = \lambda_1(B^\star)$, is the “twisted product” functor introduced by Leech and Spinks (2008). λ_1 provides a natural setting to ω.

The étale space $(\lambda_n(X))^\star$

Proposition

Each stalk of $(\lambda_n(X))^\star$ contains $(n+1)$ elements. For $F \in X$ denote

$$(\lambda_n(X))^\star_F = \{F_{(1)}, \ldots, F_{(n+1)}\}.$$

The topology on $(\lambda_n(X))^\star$ is given by the base consisting of the sets $\{F_{(i)} : F \in X\}$, $1 \leq i \leq n+1$, X runs through compact clopen sets of X. The element $(A_i)_{n+1 \geq i \geq 1} \in \lambda_n(X)$ is represented in $(\lambda_n(X))^\star$ by the section

$$(\cup_{F \in A_1} F_{(1)}) \cup (\cup_{F \in A_2 \setminus A_1} F_{(2)}) \cup \cdots \cup (\cup_{F \in A_{n+1} \setminus A_n} F_{(n+1)}).$$

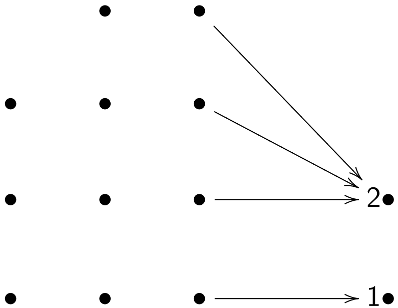
The elements of $\Lambda_n(S)$

Let $S \in \mathrm{Ob}(\mathsf{LSBA})$ and $n \geq 0$. Denote by $\Lambda_n(S)$ be extended n-spectrum of S: the set of all non-zero homomorphisms from S to $\mathbf{n}+\mathbf{2}$.

Proposition

There is a bijective correspondence between the elements of $\Lambda_n(S)$ and the functions $f \in \{1, \dots, n+1\}^{S_F^\star}$, where F runs through $(S/\mathcal{D})^\star$.

Example:



The object part of Λ_n

The sets $L(s, i)$

Let $x \in S^\star = (S^\star, \pi, (S/\mathcal{D})^\star)$ and let $F = \pi(x)$. For $1 \leq i \leq n+1$ we set

$$p_i(x) = \{f \in \{1, \dots, n+1\}^{S^\star_F} : f(x) = i\},$$

$$L(s, i) = \cup_{x \in M(s)} p_i(x).$$

Topology on $\Lambda_n(S)$

Turn $\Lambda_n(S)$ into a topological space by proclaiming the sets $L(s, i)$, $s \in S$, $1 \leq i \leq n+1$, to form a subbase of the topology.

Theorem

$\Lambda_n(S)$ is a locally compact Boolean space. $\Lambda_n(S)$ is a Boolean space if and only if $(S/D)^\star$ is a Boolean space.

Morphism part of Λ_n

Suppose $h : S_1 \to S_2$ is in $\mathrm{Hom}_{LSBA}(S_1, S_2)$. Let $f \in \Lambda_n(S_2)$, $f \in \{1, \dots, n+1\}^{S_F^\star}$. We set $h'(f) = f\tilde{h}_F$. We prove that $h' \in \mathrm{Hom}(\mathsf{LCBS}^{op})$ and set $\Lambda_n(h) = h'$.

Example. $S_1 = S_2 = \mathbf{3}$,

$$\Lambda_1(S_1) = \Lambda_1(S_2) = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \right\}$$

Suppose $\tilde{h}_F = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$.

Then $h'\left(\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $h'\left(\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$,
$h'\left(\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $h'\left(\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$.

The adjunctions $\Lambda_n \dashv \lambda_n$

Theorem (K,2011)

Let $n \geq 0$. The functor $\Lambda_n : \mathsf{LSBA} \to \mathsf{LCBS}^{op}$ is a left adjoint to the functor $\lambda_n : \mathsf{LCBS}^{op} \to \mathsf{LSBA}$. The unit of the adjunction $\eta : 1_{\mathsf{LSBA}} \to \lambda_n \Lambda_n$ is given by $\eta_S(a) = (\cup_{i=1}^{k} L(a,i))_{n+1 \geq k \geq 1}$, $S \in \mathrm{Ob}(\mathsf{LSBA})$, $a \in S$.

The adjunctions $\Lambda_n \dashv \lambda_n$

Theorem (K,2011)

Let $n \geq 0$. The functor $\Lambda_n : \mathsf{LSBA} \to \mathsf{LCBS}^{op}$ is a left adjoint to the functor $\lambda_n : \mathsf{LCBS}^{op} \to \mathsf{LSBA}$. The unit of the adjunction $\eta : 1_{\mathsf{LSBA}} \to \lambda_n\Lambda_n$ is given by $\eta_S(a) = (\cup_{i=1}^k L(a,i))_{n+1 \geq k \geq 1}$, $S \in \mathrm{Ob}(\mathsf{LSBA})$, $a \in S$.

Remark

The constructed adjunctions are induced by objects $\{0, \dots, n+1\}$, $n \geq 0$, sitting in two categories: LCBS and LSBA in a similar fashion as the Stone duality is induced by $\{0, 1\}$ considered as sitting in LCBS and in GBA.

Remark

The above result answers the question posed by Leech and Spinks (2008)

More results

We describe the structure of algebras of the monads induced by the adjunctions $\Lambda_n \dashv \lambda_n$ and prove that these adjunctions are monadic for all $n \geq 0$.

Our approach leads to new structure results about skew Boolean algebras: congruences, minimal skew Boolean covers, new examples and counterexamples..., it also allowed to answer all of the open questions posed by Leech and Spinks in their 2008 paper.

Further work

- Skew version of Priestley duality

Further work

- Skew version of Priestley duality
- Canonical extensions of skew Boolean algebras

Further work

- Skew version of Priestley duality
- Canonical extensions of skew Boolean algebras
- Connection of skew Boolean algebras with groupoids

Further work

- Skew version of Priestley duality
- Canonical extensions of skew Boolean algebras
- Connection of skew Boolean algebras with groupoids
- Your suggestions....