Continuum of extensions of the fusion of non-tabular and non-maximal modal logics over S4

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Definitions

$$S4 = K \oplus \{\Box p \to p, \Box p \to \Box\Box p\}$$

Fusion $\Lambda_1 \otimes \Lambda_2$ of two 1-modal logics Λ_1, Λ_2 is the least 2-modal logic, which contains axioms of Λ_1 for the first modality and axioms of Λ_2 for the second.

Normal modal logic is called tabular, if it is determined with a finite Kripke frame.

Normal modal logic is called maximal, if it is maximal by inclusion consistent logic.

$$NExt(L) = \{\Lambda | L \subseteq \Lambda \text{ and } \Lambda \text{ is normal modal logic}\}$$

Theorem

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For any two logics L_1, L_2 over $S4$, if L_1 is not tabular and L_2 is not maximal, then the power of $NExt(L_1 \otimes L_2)$ is continuum.

Pretabular logics

A logic is called pretabular, if it is not tabular, but all its normal proper consistent extensions are tabular.

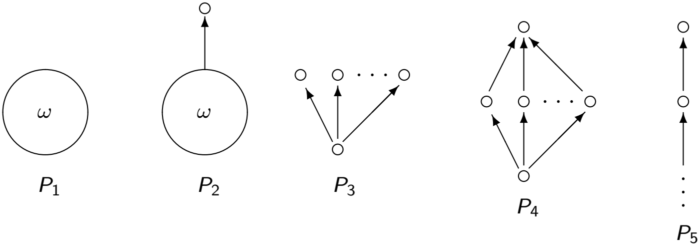
Lemma

Any non-tabular logic is contained in a pretabular logic.

Esakia-Meshki Theorem

Theorem (Esakia, Meshki)

There are exactly five pretabular logics in $NExt(S4)$. They are determined with the following Kripke frames.



Premaximal logics

Lemma

Every non-maximal logic over S4 is contained in one of the following logics: $Log(\circ \rightarrow \circ)$ *и* $Log(\circ \leftrightarrow \circ)$.

Observation

Thus, to prove the theorem, we should consider 10 cases: the combinations of 5 pretabular and 2 non-maximal logics over S4.

Idea of the proof

Consider the case $Log(P_1) \otimes Log(\circ\leftrightarrow\circ)$

Construct such formulas ϕ_i, $i \in \mathbb{N}$ that:

for logics $\Lambda_I := Log(P_1) \otimes Log(\circ\leftrightarrow\circ) + \{\phi_i | i \in I\}$, where $I \subseteq \mathbb{N}$,

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Then the power of $\{\Lambda_I | I \subseteq \mathbb{N}\}$ is continuum.

Table

Dependence of $NExt(L_1 \otimes L_2)$ on $NExt(L_1)$ and $NExt(L_2)$

$NExt(L_2)$ \ $NExt(L_1)$	**2**	**Finite**	**Countable**	**Continuum**
2	2	Finite	Countable	Continuum
Finite	Finite	?	**Continuum**	Continuum
Countable	Countable	**Continuum**	Continuum	Continuum
Continuum	Continuum	Continuum	Continuum	Continuum

Wolter's result

In 1994 Wolter built an injective embedding of $NExt(T)$ into the extensions of $S5 \otimes Log(\circ \!\rightarrow\! \circ)$.
This embedding reflected many good properties such as decidability amd FMP.

Corollary: the power of NExt($S5 \otimes Log(\circ \!\rightarrow\! \circ)$) is continuum.