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A topos-theoretic approach to Stone-type dualities

Olivia Caramello

University of Cambridge

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Stone-type dualities

Consider the following 'Stone-type dualities':

- Stone duality for distributive lattices (and Boolean algebras)
- Lindenbaum-Tarski duality for atomic complete Boolean algebras
- The duality between spatial frames and sober spaces
- M. Moshier and P. Jipsen's topological duality for meet-semilattices
- Alexandrov equivalence between preorders and Alexandrov spaces
- Birkhoff duality for finite distributive lattices
- The duality between algebraic lattices and sup-semilattices
- The duality between completely distributive algebraic lattices and posets

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A machinery for generating dualities

- In this talk we present a general topos-theoretic machinery for generating dualites or equivalences between categories of preordered structures and categories of posets, locales or topological spaces.
- All of the above-mentioned dualities are recovered as the result of applying the machinery to particular sets of 'ingredients', and new dualities are established.
- In fact, infinitely many new dualities can be generated through the machinery in an essentially automatic way.
- The machinery is interesting because of its inherent technical flexibility; there are essentially four degrees of freedom in choosing the ingredients.

Grothendieck topologies on preorders

Definition

Let $\mathscr{C}$ be a preorder.

(i) A (basis for a) Grothendieck topology on $\mathscr{C}$ is a function J which assigns to every element $c \in \mathscr{C}$ a family $J(c)$ of lower subsets of $(c)\downarrow$, called the J-covers on c, such that for any $S \in J(c)$ and any $c' \leq c$ the subset $S_{c'} = \{d \leq c' \mid d \in S\}$ belongs to $J(c')$.

(ii) A preorder site is a pair $(\mathscr{C}, J)$, where $\mathscr{C}$ is a preorder and J is a Grothendieck topology on $\mathscr{C}$.

(iii) A Grothendiek topology J on $\mathscr{C}$ is subcanonical if for every $c \in \mathscr{C}$ and any subset $S \in J(c)$, c is the supremum in $\mathscr{C}$ of the elements $d \in S$ (i.e., for any element c' in $\mathscr{C}$ such that for every $d \in S$ $d \leq c'$, we have $c \leq c'$).

Examples of Grothendieck topologies

- If P is a preorder, the trivial topology on P is the one in which the only covers are the maximal ones.
- If D is a distributive lattice, the coherent topology on D is the one in which the covers are exactly those which contain finite families whose join is the given element.
- If F is a frame, the canonical topology on F is the one in which the covers are exactly the families whose join is the given element.
- If D is a disjunctively distributive lattice, the disjunctive topology on D is the one in which the covers are exactly those which contain finite families of pairwise disjoint elements whose join is the given element.
- If U is a k-frame, the k-covering topology on U is the one in which the covers are the those which contain families of less than k elements whose join is the given element.

J-ideals

Definition

Given a preorder site $(\mathscr{C}, J)$, a J-ideal on $\mathscr{C}$ is a subset $I \subseteq \mathscr{C}$ such that

- for any $a, b \in \mathscr{C}$ such that $b \leq a$ in $\mathscr{C}$, $a \in I$ implies $b \in I$, and
- for any J-cover R on an element c of $\mathscr{C}$, if $a \in I$ for every $a \in R$ then $c \in I$.

We denote by $Id_J(\mathscr{C})$ the set of all the J-ideals on $\mathscr{C}$.

Theorem

Let $\mathscr{C}$ be a preorder and J be a Grothendieck topology on $\mathscr{C}$. Then $(Id_J(\mathscr{C}), \subseteq)$ is a frame.

Remark

If J is subcanonical (i.e. all the principal ideals on $\mathscr{C}$ are J-ideals) and $\mathscr{C}$ is a poset then we have an embedding $\mathscr{C} \hookrightarrow Id_J(\mathscr{C})$, which identifies $\mathscr{C}$ with the set of principal ideals on $\mathscr{C}$.

The underlying philosophy

- In my paper

  The unification of Mathematics via Topos Theory

  I give a set of principles and methodologies which justify a view of Grothendieck toposes as 'bridges' for transferring information between distinct mathematical theories.
- This work represents a faithful implementation of this philosophy in a particular context.
- In fact, we establish our dualities precisely by 'functorializing' different representations of a given topos, which thus acts as a 'bridge' connecting the two sites:

$$(\mathscr{C}, J) \dashrightarrow \mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Sh}(Id_J(\mathscr{C})) \dashleftarrow Id_J(\mathscr{C})$$

Remark

For any preorder site $(\mathscr{C}, J)$, the J-ideals on $\mathscr{C}$ correspond precisely to the subterminal objects of the topos $\mathbf{Sh}(\mathscr{C}, J)$.

Functorialization

We can generate covariant or controvariant equivalences with categories of posets by appropriately functorializing the assignments above; we only discuss for simplicity the case of covariant equivalences with categories of frames, the other cases being conceptually similar to it.

Definition

A morphism of sites $(\mathscr{C}, J) \to (\mathscr{D}, K)$, where $\mathscr{C}$ and $\mathscr{D}$ are meet-semilattices, is a meet-semilattice homomorphism $\mathscr{C} \to \mathscr{D}$ which sends J-covers to K-covers.

Theorem

1. *A morphism of sites $f : (\mathscr{C}, J) \to (\mathscr{D}, K)$ induces, naturally in f, a frame homomorphism $\dot{f} : Id_J(\mathscr{C}) \to Id_K(\mathscr{D})$.*
   This homomorphism sends a J-ideal I on $\mathscr{C}$ to the smallest K-ideal on $\mathscr{D}$ containing the image of I under f.
2. *If J and K are subcanonical then a frame homomorphism $Id_J(\mathscr{C}) \to Id_K(\mathscr{D})$ is of the form $\dot{f}$ for some f if and only if it sends principal ideals to principal ideals; if this is the case then f is isomorphic to the restriction of $\dot{f}$ to the principal ideals.*

The general framework

Let $\mathscr{K}$ be a category of preordered structures, and suppose to have equipped each structure $\mathscr{C}$ in $\mathscr{K}$ with a Grothendieck topology $J_{\mathscr{C}}$ on $\mathscr{C}$ in such a way that every arrow $f : \mathscr{C} \to \mathscr{D}$ in $\mathscr{K}$ gives rise to a morphism of sites $f : (\mathscr{C}, J_{\mathscr{C}}) \to (\mathscr{D}, J_{\mathscr{D}})$.

These choices automatically induce a functor

$$A : \mathscr{K} \to \mathbf{Frm}$$

to the category **Frm** of frames sending any $\mathscr{C}$ in $\mathscr{K}$ to $Id_{J_{\mathscr{C}}}(\mathscr{C})$ and any $f : \mathscr{C} \to \mathscr{D}$ in $\mathscr{K}$ to the frame homomorphism $\dot{f} : Id_{J_{\mathscr{C}}}(\mathscr{C}) \to Id_{J_{\mathscr{D}}}(\mathscr{D})$.

Theorem

*With the above notation, if all the Grothendieck topologies $J_{\mathscr{C}}$ are subcanonical and the preorders in $\mathscr{K}$ are posets then the functor $A : \mathscr{K} \to$ **Frm** yields an isomorphism of categories between $\mathscr{K}$ and the subcategory of **Frm** given by the image of A.*

Recovering the structures through invariants

- The theorem just stated provides us with an infinite number of dualities. Still, it would be desirable to have a duality of $\mathscr{K}$ with a subcategory of **Frm** which is closed under isomorphisms in **Frm** (namely, the closure $ExtIm(A)$ of the image of A under isomorphisms in **Frm**) so that its objects (and arrows) could admit an intrinsic description in frame-theoretic terms.
- To achieve this, we investigate the problem of recovering a preorder $\mathscr{C}$ in $\mathscr{K}$ from the topos $\mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}})$ (equivalently, from the frame $Id_{J_{\mathscr{C}}}(\mathscr{C})$) through an invariant, functorially in $\mathscr{C}$.
- It turns out that if the topologies $J_{\mathscr{C}}$ can be 'uniformly described through an invariant' C then the principal ideals on $\mathscr{C}$ can be characterized among the elements of the frame $Id_{J_{\mathscr{C}}}(\mathscr{C})$) precisely as the ones which are C-compact.
- This enables us to define a functor on the category $ExtIm(A)$ which yields, together with A, the desired equivalence.

Topologies defined through invariants

Definition

Let C be a frame-theoretic invariant property of families of elements of a frame (for example: to be finite, to be a singleton, to be of cardinality at most k for some cardinal k, to be formed by elements which are pairwise disjoint, to be directed etc.)

- Given a structure $\mathscr{C}$ in $\mathscr{K}$, the Grothendieck topology $J_{\mathscr{C}}$ is said to be C-induced if for any J^F_{can}-dense monotone embedding $i : \mathscr{C} \hookrightarrow F$ into a frame F (where J^F_{can} is the canonical topology on F), possibly satisfying some invariant property P which is known to hold for the canonical embedding $\mathscr{C} \hookrightarrow Id_{J_{\mathscr{C}}}(\mathscr{C})$, such that the $J_{\mathscr{C}}$-covers on $\mathscr{C}$ are sent by i to covers in F, for any family $\mathscr{A}$ of elements in $\mathscr{C}$ there exists a $J_{\mathscr{C}}$-cover on an element $c \in \mathscr{C}$ such that the elements $a \in \mathscr{A}$ such that $a \leq c$ generate S if and only if the image $i(\mathscr{A})$ of the family $\mathscr{A}$ in F has a refinement satisfying C made of elements of the form $i(c')$ (for $c' \in \mathscr{C}$);
- An element u of a frame F is said to be C-compact if every covering of u in F has a refinement satisfying C.

The main result

Theorem

If all the Grothendieck topologies $J_{\mathscr{C}}$ associated to the structures $\mathscr{C}$ in $\mathscr{K}$ are C-induced and the invariant C satisfies the property that for any structure $\mathscr{C}$ in $\mathscr{K}$ and for any family $\mathscr{F}$ of principal $J_{\mathscr{C}}$-ideals on $\mathscr{C}$, $\mathscr{F}$ has a refinement satisfying C (if and) only if it has a refinement satisfying C made of principal $J_{\mathscr{C}}$-ideals on $\mathscr{C}$ then the functor $\mathrm{ExtIm}(A) \to \mathscr{K}$ sending a frame F in $\mathrm{ExtIm}(A)$ to the poset of C-compact elements of F and acting on the arrows accordingly is a categorical inverse to A.

The target categories of frames

Theorem

- *The frames in ExtIm(A) are precisely the frames F with a basis B_F of C-compact elements which, regarded as a poset with the induced order, belongs to $\mathscr{K}$, and such that the embedding $B_F \hookrightarrow F$ satisfies property P, the property that every covering in F of an element of B_F is refined by a covering made of elements of B_F which satisfies the invariant C, and the property that the J_{B_F}-covering sieves are sent by the embedding $B_F \hookrightarrow F$ into covering families in F (where J_{B_F} is the Grothendieck topology with which B_F comes equipped as a structure in $\mathscr{K}$).*
- *The arrows $F \to F'$ in ExtIm(A) are the frame homomorphisms which send C-compact elements to C-compact elements in such a way that their restriction to the subsets of C-compact elements can be identified with an arrow in $\mathscr{K}$.*

The subterminal topology

The following notion provides a way for endowing a given set of points of a topos with a natural topology.

Definition

Let $\xi : X \to \mathrm{P}$ be an indexing of a set P of points of a Grothendieck topos $\mathscr{E}$ by a set X. We define the subterminal topology $\tau_{\xi}^{\mathscr{E}}$ as the image of the function $\phi_{\mathscr{E}} : \mathrm{Sub}_{\mathscr{E}}(1) \to \mathscr{P}(X)$ given by

$$\phi_{\mathscr{E}}(u) = \{x \in X \mid \xi(x)^*(u) \cong 1_{\mathbf{Set}}\} .$$

We denote the topological space obtained by endowing the set X with the topology $\tau_{\xi}^{\mathscr{E}}$ by $X_{\tau_{\xi}^{\mathscr{E}}}$.

The interest of this notion lies in its level of generality, as well as in its formulation as a topos-theoretic invariant admitting a 'natural behaviour' with respect to sites. Moreover, the following fact will be crucial for us.

Fact

If P is a separating set of points for $\mathscr{E}$ (for example, the set of all the points of a localic topos having enough points) then the frame $\mathcal{O}(X_{\tau_{\xi}^{\mathscr{E}}})$ of open sets of the space $X_{\tau_{\xi}^{\mathscr{E}}}$ is isomorphic (via $\phi_{\mathscr{E}}$) to the frame $\mathrm{Sub}_{\mathscr{E}}(1)$ of subterminals of the topos $\mathscr{E}$.

Examples of subterminal topologies I

Definition

Let $(\mathscr{C}, \leq)$ be a preorder. A J-prime filter on $\mathscr{C}$ is a subset $F \subseteq \mathscr{C}$ such that F is non-empty, $a \in F$ implies $b \in F$ whenever $a \leq b$, for any $a, b \in F$ there exists $c \in F$ such that $c \leq a$ and $c \leq b$, and for any J-covering sieve $\{a_i \to a \mid i \in I\}$ in $\mathscr{C}$ if $a \in F$ then there exists $i \in I$ such that $a_i \in F$.

Theorem

Let $\mathscr{C}$ be a preorder and J be a Grothendieck topology on it. Then the space $X_{\tau_{\mathbf{Sh}(\mathscr{C},J)}}$ has as set of points the collection $\mathscr{F}^J_{\mathscr{C}}$ of the J-prime filters on $\mathscr{C}$ and as open sets the sets the form

$$\mathscr{F}_I = \{F \in \mathscr{F}^J_{\mathscr{C}} \mid F \cap I \neq \emptyset\},$$

where I ranges among the J-ideals on $\mathscr{C}$. In particular, a sub-basis for this topology is given by the sets

$$\mathscr{F}_c = \{F \in \mathscr{F}^J_{\mathscr{C}} \mid c \in F\},$$

where c varies among the elements of $\mathscr{C}$.

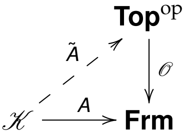
Examples of subterminal topologies II

- The Alexandrov topology ($\mathscr{E} = [\mathscr{P}, \mathbf{Set}]$, where $\mathscr{P}$ is a preorder and ξ is the indexing of the set of points of $\mathscr{E}$ corresponding to the elements of $\mathscr{P}$)
- The Stone topology for distributive lattices ($\mathscr{E} = \mathbf{Sh}(\mathscr{D}, J_{coh})$ and ξ is an indexing of the set of all the points of $\mathscr{E}$, where $\mathscr{D}$ is a distributive lattice and J_{coh} is the coherent topology on it)
- A topology for meet-semilattices ($\mathscr{E} = [\mathscr{M}^{\mathrm{op}}, \mathbf{Set}]$ and ξ is an indexing of the set of all the points of $\mathscr{E}$, where $\mathscr{M}$ is a meet-semilattice)
- The space of points of a locale ($\mathscr{E} = \mathbf{Sh}(L)$ for a locale L and ξ is an indexing of the set of all the points of $\mathscr{E}$)
- A logical topology ($\mathscr{E} = \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is the classiying topos of a geometric theory $\mathbb{T}$ and ξ is any indexing of the set of all the points of $\mathscr{E}$ i.e. models of $\mathbb{T}$)
- The Zariski topology

...

Dualities with categories of topological spaces

- By using the subterminal topology, we can 'lift' the equivalences with frames established above to dualities with topological spaces, provided that the toposes involved have enough points (notice that this condition is automatically satisfied, at least under some form of the axiom of choice, if the topologies $J_{\mathscr{C}}$ are finitary).
- Indeed, the construction of the subterminal topology can be naturally made functorial.
- Thus, by assigning sets of points of the toposes corresponding to the structures in a natural way, we obtain a functor $\tilde{A} : \mathscr{K} \to \mathbf{Top}^{\mathrm{op}}$ such that $\mathscr{O} \circ \tilde{A} \cong A$, where $\mathscr{O} : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Frm}$ the usual functor taking the frame of open sets of a topological space:



The power of the machinery

- Functorializing general equivalences $\mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Sh}(\mathscr{D}, K)$ (where $\mathscr{C}$ is a K-dense subcategory of $\mathscr{D}$ and J is induced by K on $\mathscr{C}$), we are able to recover all the dualities mentioned at the beginning of the talk as special cases generated through our machinery.
- At the same time, our framework allows enough flexibility to construct many new interesting dualities with particular properties.
- In fact, we essentially have four degrees of freedom:
  - (i) The choice of the structures $\mathscr{C}$;
  - (ii) The choice of the structures $\mathscr{D}$;
  - (iii) The choice of the topologies J and K;
  - (iv) The choice of points of the toposes $\mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Sh}(\mathscr{D}, K)$.

New dualities I

Among the new dualities that we obtain though our machinery, we have:

- A duality between the category of meet-semilattices and meet-semilattices homomorphisms betweeen them and the category of locales whose objects are the locales with a basis of supercompact elements which is closed under finite intersections and whose arrows are the locale maps whose associated frame homomorphisms send supercompact elements to supercompact elements.
- A duality between the category of frames with a basis of supercompact elements and complete homomorphisms between them and the category of posets (endowed with the Alexandrov topology), which restricts to the Lindenbaum-Tarski duality.
- A duality between the category of disjunctively distributive lattices and the category whose objects are the sober topological spaces which have a basis of disjunctively compact open sets which is closed under finite intersection and satisfies the property that any covering of a basic open set has a disjunctively compact refinement by basic open sets and whose arrows are the continuous maps between such spaces such that the inverse image of any disjunctively compact open set is a disjunctively compact open set.

New dualities II

- For any regular cardinal k, a duality between the category of k-frames and the category whose objects are the frames which have a basis of k-compact elements which is closed under finite meets and whose arrows are the frame homomorphisms between them which send k-compact elements to k-compact elements.
- A duality between the category of disjunctive frames and the category $\mathbf{Pos}_{dis}$ which has as objects the posets $\mathscr{P}$ such that for any $a, b \in \mathscr{P}$ there exists a family $\{c_i \mid i \in I\}$ of elements of $\mathscr{P}$ such that for any $p \in \mathscr{P}$, $p \leq a$ and $p \leq b$ if and only if $p \leq c_i$ for a unique $i \in I$ and as arrows $\mathscr{P} \to \mathscr{P}'$ the monotone maps $g : \mathscr{P} \to \mathscr{P}'$ such that for any $b \in \mathscr{P}'$ there exists a family $\{c_i \mid i \in I\}$ of elements of $\mathscr{P}$ such that for any $p \in \mathscr{P}$, $g(p) \leq b$ if and only if $p \leq c_i$ for a unique $i \in I$.

New dualities III

- A duality between the category **DirIrrPFrm** of directedly generated preframes whose objects are the directedly generated preframes and whose arrows $\mathscr{D} \to \mathscr{D}'$ are the preframe homomorphisms $f : \mathscr{D} \to \mathscr{D}'$ between them such that the frame homomorphism $A(f) : Id_{J_{\mathscr{D}}}(\mathscr{D}) \to Id_{J'_{\mathscr{D}}}(\mathscr{D}')$ which sends an ideal I of $\mathscr{D}$ to the ideal of $\mathscr{D}'$ generated by $f(I)$ preserves arbitrary infima, and the category $\mathbf{Pos}_{dir}$ having as objects the posets $\mathscr{P}$ such that for any $a, b \in \mathscr{P}$ there is $c \in \mathscr{P}$ such that $c \leq a$ and $c \leq b$ and for any elements $d, e \in \mathscr{P}$ such that $d, e \leq a$ and $d, e \leq b$ there exists $z \in \mathscr{P}$ such that $z \leq a$, $z \leq b$, $d, e \leq z$, and as arrows $\mathscr{P} \to \mathscr{P}'$ the monotone maps $g : \mathscr{P} \to \mathscr{P}'$ with the property that for any $b \in \mathscr{P}'$ there exists $a \in \mathscr{P}$ such that $g(a) \leq p$ and for any two $u, v \in \mathscr{P}$ such that $g(u) \leq b$ and $g(v) \leq b$ there exists $z \in \mathscr{P}$ such that $u, v \leq z$ and $g(z) \leq b$.
  This duality restricts to the duality between algebraic lattices and sup-semilattices.
- An equivalence between the category of meet-semilattices and the category whose objects are the the meet-semilattices F with a bottom element 0_F which have the property that for any $a, b \in F$ with $a, b \neq 0$, $a \wedge b \neq 0$ and whose arrows are the meet-semilattice homomorphisms $F \to F'$ which send 0_F to $0_{F'}$ and any non-zero element of F to a non-zero element of F'.

New dualities IV

- A duality between the category **IrrDLat** whose objects are the irreducibly generated distributive lattices and whose arrows $\mathscr{D} \to \mathscr{D}'$ are the distributive lattices homomorphisms $f : \mathscr{D} \to \mathscr{D}'$ between them such that the frame homomorphism $A(f) : Id_{J_{\mathscr{D}}}(\mathscr{D}) \to Id_{J'_{\mathscr{D}}}(\mathscr{D}')$ which sends an ideal I of $\mathscr{D}$ to the ideal of $\mathscr{D}'$ generated by $f(I)$ preserves arbitrary infima, and the category $\mathbf{Pos}_{comp}$ whose objects are the posets and whose arrows $\mathscr{P} \to \mathscr{P}'$ are the monotone maps $g : \mathscr{P} \to \mathscr{P}'$ such that for any $q \in \mathscr{P}'$, there exists a finite family $\{a_k \mid k \in K\}$ of elements of $\mathscr{P}$ such that for any $p \in \mathscr{P}$, $g(p) \leq q$ if and only if $p \leq a_k$ for some $k \in K$.
  This duality restricts to Birkhoff duality.
- A duality between the category **AtDLat** whose objects are the atomic distributive lattices and whose arrows $\mathscr{D} \to \mathscr{D}'$ are the distributive lattices homomorphisms $f : \mathscr{D} \to \mathscr{D}'$ between them such that the frame homomorphism $A(f) : Id_{J_{\mathscr{D}}}(\mathscr{D}) \to Id_{J'_{\mathscr{D}}}(\mathscr{D}')$ which sends an ideal I of $\mathscr{D}$ to the ideal of $\mathscr{D}'$ generated by $f(I)$ preserves arbitrary infima, and the category $\mathbf{Set}_f$ whose objects are the sets and whose arrows $A \to B$ are the functions $f : A \to B$ such that the inverse image under f of any finite subset of B is a finite subset of A.
- ...

Other applications

A great amount of applications can be established, besides the construction of new dualities, by applying the technique 'toposes as bridges' to the equivalences of toposes considered above. Examples include:

- Adjunctions between categories of preorders and categories of frames or locales; for example, between meet-semilattices (resp. distributive lattices, preframes, Boolean algebras) and frames
- Translations of properties of preordered structures into properties of the corresponding locales or topological spaces (for example, characterizations of the Stone-type spaces associated to the structures which are trivial, almost discrete, extremally disconnected etc.)
- Representation theorems for preordered structures
- Priestley-type dualities for various kinds of preordered structures
- Completeness theorems for propositional logics

For further reading

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A topos-theoretic approach to Stone-type dualities,
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