Topological semantics of polymodal provability logic

Lev Beklemishev

Steklov Mathematical Institute, Moscow

In memoriam Leo Esakia

TACL, Marseille, July 26–30, 2011
Lindenbaum algebras

Lindenbaum algebra of a theory $T$:
$L_T = \{\text{sentences of } T\}/\sim_T$, where

$$\varphi \sim_T \psi \iff T \vdash (\varphi \leftrightarrow \psi)$$

$L_T$ is a boolean algebra with operations $\wedge$, $\vee$, $\neg$.
$1 = \text{the set of provable sentences of } T$
$0 = \text{the set of refutable sentences of } T$

For consistent gödelian $T$ all such algebras are countable atomless, hence pairwise isomorphic.

Kripke, Pour-El: even computably isomorphic
Magari algebras

Emerged in 1970s: Macintyre/Simmons, Magari, Smoryński, . . .

Let $T$ be a gödelian theory (formalizing its own syntax), $\text{Con}(T) = \langle T \text{ is consistent} \rangle$

Consistency operator $\diamond : \varphi \mapsto \text{Con}(T + \varphi)$ acting on $\mathcal{L}_T$.

$(\mathcal{L}_T, \diamond) = \text{Magari algebra of } T$

$\square \varphi = \neg \diamond \neg \varphi = \langle \varphi \text{ is provable in } T \rangle$

Characteristic of $(M, \diamond)$:
$ch(M) = \min \{ k : \diamond^k 1 = 0 \}$;
$ch(M) = \infty$, if no such $k$ exists.

Remark. If $\mathbb{N} \models T$, then $ch(\mathcal{L}_T) = \infty$. 
Magari algebras

Emerged in 1970s: Macintyre/Simmons, Magari, Smoryński, ... 

Let $T$ be a gödelian theory (formalizing its own syntax), $\text{Con}(T) = \text{«}T \text{ is consistent}\text{»}$

Consistency operator $\Diamond : \varphi \mapsto \text{Con}(T + \varphi)$ acting on $\mathcal{L}_T$.

$(\mathcal{L}_T, \Diamond) = \text{Magari algebra of } T$

$\square \varphi = \neg \Diamond \neg \varphi = \text{«} \varphi \text{ is provable in } T \text{»}$

Characteristic of $(M, \Diamond)$:

$ch(M) = \min\{k : \Diamond^k 1 = 0\}$

$ch(M) = \infty$, if no such $k$ exists.

Remark. If $\mathbb{N} \not\models T$, then $ch(\mathcal{L}_T) = \infty$. 
K. Gödel (33), M.H. Löb (55): Algebra \((\mathcal{L}_T, \lozenge)\) satisfies the following set of identities \(GL\):

- boolean identities
- \(\lozenge 0 = 0\)
- \(\lozenge (\varphi \lor \psi) = (\lozenge \varphi \lor \lozenge \psi)\)
- \(\lozenge \varphi = \lozenge (\varphi \land \neg \lozenge \varphi)\) (Löb’s identity)

\(GL\)-algebras = Magari algebras = diagonalizable algebras
Identities of Magari algebras

K. Gödel (33), M.H. Löb (55): Algebra \((\mathcal{L}_T, \lozenge)\) satisfies the following set of identities \(GL\):

- boolean identities
- \(\lozenge 0 = 0\)
- \(\lozenge (\varphi \lor \psi) = (\lozenge \varphi \lor \lozenge \psi)\)
- \(\lozenge \varphi = \lozenge (\varphi \land \neg \lozenge \varphi)\) (Löb’s identity)

\(GL\)-algebras = Magari algebras = diagonalizable algebras
Provability logic

Let $\mathcal{A} = (A, \diamond)$ be a boolean algebra with an operator $\diamond$, and $\varphi(\bar{x})$ a term.

**Def.** Denote $\mathcal{A} \models \varphi$ if $\mathcal{A} \models \forall \bar{x} (\varphi(\bar{x}) = 1)$;

- The logic of $\mathcal{A}$ is $\text{Log}(\mathcal{A}) = \{ \varphi : \mathcal{A} \models \varphi \}$.

R. Solovay (76): If $\text{ch}(\mathcal{L}_T) = \infty$, then $\text{Log}(\mathcal{L}_T, \boxdot) = \text{GL}$.

$\text{GL}$ is nice as a modal logic (decidable, Kripke complete, fmp, Craig, cut-free calculus, ...).
Let $\mathcal{A} = (A, \Box)$ be a boolean algebra with an operator $\Box$, and $\varphi(\vec{x})$ a term.

**Def.** Denote $\mathcal{A} \models \varphi$ if $\mathcal{A} \models \forall \vec{x} (\varphi(\vec{x}) = 1)$;

- The logic of $\mathcal{A}$ is $\text{Log}(\mathcal{A}) = \{ \varphi : \mathcal{A} \models \varphi \}$.

R. Solovay (76): If $\text{ch}(\mathcal{L}_T) = \infty$, then $\text{Log}(\mathcal{L}_T, \Box) = \text{GL}$.

$\text{GL}$ is nice as a modal logic (decidable, Kripke complete, fmp, Craig, cut-free calculus, ...).
Let $\mathcal{A} = (A, \Diamond)$ be a boolean algebra with an operator $\Diamond$, and $\varphi(\vec{x})$ a term.

**Def.** Denote $\mathcal{A} \models \varphi$ if $\mathcal{A} \models \forall \vec{x} \ (\varphi(\vec{x}) = 1)$;

- The logic of $\mathcal{A}$ is $\text{Log}(\mathcal{A}) = \{ \varphi : \mathcal{A} \models \varphi \}$.

R. Solovay (76): If $\text{ch}(\mathcal{L}_T) = \infty$, then $\text{Log}(\mathcal{L}_T, \Diamond) = \text{GL}$.

$\text{GL}$ is nice as a modal logic (decidable, Kripke complete, fmp, Craig, cut-free calculus, ...)
Def. A gödelian theory $T$ is $n$-consistent, if every provable $\Sigma^0_n$-sentence of $T$ is true.

$n$-Consistency operator $\langle n \rangle : \mathcal{L}_T \to \mathcal{L}_T$

$$\varphi \mapsto n\text{-Con}(T + \varphi).$$

$$[n] = \neg \langle n \rangle \neg \quad (n\text{-provability})$$
The algebra of $n$-provability

$$
\mathcal{M}_T = (\mathcal{L}_T; \langle 0 \rangle, \langle 1 \rangle, \ldots).
$$

The following identities $GLP$ hold in $\mathcal{M}_T$:

- $GL$, for all $\langle n \rangle$;
- $\langle n + 1 \rangle \varphi \rightarrow \langle n \rangle \varphi$;
- $\langle n \rangle \varphi \rightarrow [n + 1] \langle n \rangle \varphi$.

G. Japaridze (86): If $\mathbb{N} \models T$, then $\text{Log}(\mathcal{M}_T) = GLP$. 
The algebra of $n$-provability

$\mathcal{M}_T = (\mathcal{L}_T; \langle 0 \rangle, \langle 1 \rangle, \ldots )$.

The following identities $GLP$ hold in $\mathcal{M}_T$:

- $GL$, for all $\langle n \rangle$;
- $\langle n + 1 \rangle \varphi \rightarrow \langle n \rangle \varphi$;
- $\langle n \rangle \varphi \rightarrow [n + 1] \langle n \rangle \varphi$.

G. Japaridze (86): If $\mathbb{N} \models T$, then $\text{Log}(\mathcal{M}_T) = GLP$. 
The significance of GLP

GLP is

- Useful for proof theory:
  - Ordinal notations and consistency proof for PA;
  - Independent combinatorial assertion;
  - Characterization of provably total computable functions of PA.

- Fairly complicated and not so nice modal-logically:
  - no Kripke completeness, no cut-free calculus;
  - though it is decidable and has Craig interpolation.

GLPₙ is GLP in the language with n operators. GLP₁ = GL.
The significance of GLP

GLP is

- Useful for proof theory:
  - Ordinal notations and consistency proof for PA;
  - Independent combinatorial assertion;
  - Characterization of provably total computable functions of PA.

- Fairly complicated and not so nice modal-logically:
  - no Kripke completeness, no cut-free calculus;
  - though it is decidable and has Craig interpolation.

GLP\(_n\) is GLP in the language with \(n\) operators. GLP\(_1 = GL\).
The significance of GLP

GLP is

- Useful for proof theory:
  - Ordinal notations and consistency proof for PA;
  - Independent combinatorial assertion;
  - Characterization of provably total computable functions of PA.

- Fairly complicated and not so nice modal-logically:
  - no Kripke completeness, no cut-free calculus;
  - though it is decidable and has Craig interpolation.

$GLP_n$ is GLP in the language with $n$ operators. $GLP_1 = GL$. 
Let $X$ be a nonempty set, $\mathcal{P}(X)$ the b.a. of subsets of $X$.

Consider any operator $\delta : \mathcal{P}(X) \to \mathcal{P}(X)$ and the structure $(\mathcal{P}(X), \delta)$.

Question: Can $(\mathcal{P}(X), \delta)$ be a $GL$-algebra and, if yes, when?

Def. Write $(X, \delta) \models \varphi$ if $(\mathcal{P}(X), \delta) \models \varphi$. Also let $\text{Log}(X, \delta) := \text{Log}(\mathcal{P}(X), \delta)$. 

Set-theoretic interpretation
Set-theoretic interpretation

Let $X$ be a nonempty set, $\mathcal{P}(X)$ the b.a. of subsets of $X$.

Consider any operator $\delta : \mathcal{P}(X) \to \mathcal{P}(X)$ and the structure $(\mathcal{P}(X), \delta)$.

**Question:** Can $(\mathcal{P}(X), \delta)$ be a $GL$-algebra and, if yes, when?

**Def.** Write $(X, \delta) \models \varphi$ if $(\mathcal{P}(X), \delta) \models \varphi$. Also let $\text{Log}(X, \delta) := \text{Log}(\mathcal{P}(X), \delta)$. 
Set-theoretic interpretation

Let $X$ be a nonempty set, $\mathcal{P}(X)$ the b.a. of subsets of $X$.

Consider any operator $\delta : \mathcal{P}(X) \to \mathcal{P}(X)$ and the structure $(\mathcal{P}(X), \delta)$.

Question: Can $(\mathcal{P}(X), \delta)$ be a $GL$-algebra and, if yes, when?

Def. Write $(X, \delta) \models \varphi$ if $(\mathcal{P}(X), \delta) \models \varphi$. Also let $\text{Log}(X, \delta) := \text{Log}(\mathcal{P}(X), \delta)$. 
Let $X$ be a topological space, $A \subseteq X$. Derived set $d(A)$ of $A$ is the set of limit points of $A$:

$$x \in d(A) \iff \forall U_x \text{ open } \exists y \neq x \ y \in U_x \cap A.$$ 

Fact. If $(X, \delta) \models GL$ then $X$ naturally bears a topology $\tau$ for which $\delta = d_\tau$, that is, $\delta : A \mapsto d_\tau(A)$, for each $A \subseteq X$.

In fact, we can define: $A$ is $\tau$-closed iff $\delta(A) \subseteq A$. Equivalently, $c(A) = A \cup \delta(A)$ is the closure of $A$. 

\textit{Derived set operators}
**Derived set operators**

Let $X$ be a topological space, $A \subseteq X$.

**Derived set** $d(A)$ of $A$ is the set of limit points of $A$:

$$x \in d(A) \iff \forall U_x \text{ open } \exists y \neq x \ y \in U_x \cap A.$$ 

**Fact.** If $(X, \delta) \vDash GL$ then $X$ naturally bears a topology $\tau$ for which $\delta = d_\tau$, that is, $\delta : A \mapsto d_\tau(A)$, for each $A \subseteq X$.

In fact, we can define: $A$ is $\tau$-closed iff $\delta(A) \subseteq A$.

Equivalently, $c(A) = A \cup \delta(A)$ is the closure of $A$. 


Derived set operators

Let $X$ be a topological space, $A \subseteq X$. 

Derived set $d(A)$ of $A$ is the set of limit points of $A$:

$$x \in d(A) \iff \forall U_x \text{ open } \exists y \neq x \ y \in U_x \cap A.$$

Fact. If $(X, \delta) \models GL$ then $X$ naturally bears a topology $\tau$ for which $\delta = d_\tau$, that is, $\delta : A \mapsto d_\tau(A)$, for each $A \subseteq X$.

In fact, we can define: $A$ is $\tau$-closed iff $\delta(A) \subseteq A$. Equivalently, $c(A) = A \cup \delta(A)$ is the closure of $A$. 

Scattered spaces

Definition (Cantor): $X$ is scattered if every nonempty $A \subseteq X$ has an isolated point.

Cantor-Bendixson sequence:

\[ X_0 = X, \quad X_{\alpha+1} = d(X_\alpha), \quad X_\lambda = \bigcap_{\alpha < \lambda} X_\alpha, \text{if } \lambda \text{ is limit.} \]

Notice that all $X_\alpha$ are closed and $X_0 \supset X_1 \supset X_2 \supset \ldots$

Fact (Cantor): $X$ is scattered $\iff \exists \alpha : X_\alpha = \emptyset$. 
**Scattered spaces**

**Definition (Cantor):** $X$ is scattered if every nonempty $A \subseteq X$ has an isolated point.

Cantor-Bendixon sequence:

$$X_0 = X, \quad X_{\alpha + 1} = d(X_\alpha), \quad X_\lambda = \bigcap_{\alpha < \lambda} X_\alpha, \text{ if } \lambda \text{ is limit.}$$

Notice that all $X_\alpha$ are closed and $X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$

**Fact (Cantor):** $X$ is scattered $\iff \exists \alpha : X_\alpha = \emptyset.$
Scattered spaces

Definition (Cantor): \( X \) is scattered if every nonempty \( A \subseteq X \) has an isolated point.

Cantor-Bendixon sequence:

\[
X_0 = X, \quad X_{\alpha+1} = d(X_{\alpha}), \quad X_\lambda = \bigcap_{\alpha < \lambda} X_\alpha, \text{ if } \lambda \text{ is limit.}
\]

Notice that all \( X_\alpha \) are closed and \( X_0 \supset X_1 \supset X_2 \supset \ldots \)

Fact (Cantor): \( X \) is scattered \( \iff \exists \alpha : X_\alpha = \emptyset. \)
Examples

1. **Left topology** $\tau_\prec$ on a strict partial ordering $(X, \prec)$. $A \subseteq X$ is open iff $\forall x, y (y \prec x \in A \Rightarrow y \in A)$.

   Fact: $(X, \prec)$ is well-founded iff $(X, \tau_\prec)$ is scattered.

2. Ordinal $\Omega$ with the usual order topology generated by intervals $(\alpha, \beta)$, $[0, \beta)$, $(\alpha, \Omega)$ such that $\alpha < \beta$. 
Examples

Left topology $\tau_<$ on a strict partial ordering $(X, \prec)$. $A \subseteq X$ is open iff $\forall x, y \,(y \prec x \in A \Rightarrow y \in A)$.

Fact: $(X, \prec)$ is well-founded iff $(X, \tau_\prec)$ is scattered.

Ordinal $\Omega$ with the usual order topology generated by intervals $(\alpha, \beta), [0, \beta), (\alpha, \Omega)$ such that $\alpha < \beta$. 
Löb’s identity = scatteredness

Simmons 74, Esakia 81

Löb’s identity: $\Diamond A = \Diamond(A \land \neg \Diamond A)$.

Topological reading:

$$d(A) = d(A \setminus d(A)) = d(\text{iso}(A)),$$

where $\text{iso}(A) = A \setminus d(A)$ is the set of isolated points of $A$.

Fact: The following are equivalent:

- $X$ is scattered;
- $d(A) = d(\text{iso}(A))$ for any $A \subseteq X$;
- $(X, d) \models GL$. 

Löb’s identity = scatteredness

Simmons 74, Esakia 81

Löb’s identity: $\Diamond A = \Diamond (A \land \neg \Diamond A)$.

Topological reading:

$$d(A) = d(A \setminus d(A)) = d(\text{iso}(A)),$$

where $\text{iso}(A) = A \setminus d(A)$ is the set of isolated points of $A$.

Fact: The following are equivalent:

- $X$ is scattered;
- $d(A) = d(\text{iso}(A))$ for any $A \subseteq X$;
- $(X, d) \models GL$. 
Completeness theorems

Theorem (Esakia 81): There is a scattered $X$ such that $\text{Log}(X, d) = GL$. In fact, $X$ is the left topology on a countable well-founded partial ordering.

Theorem (Abashidze/Blass 87/91): Consider $\Omega \geq \omega^\omega$ with the order topology. Then $\text{Log}(\Omega, d) = GL$. 
Completeness theorems

Theorem (Esakia 81): There is a scattered $X$ such that $\text{Log}(X, d) = \text{GL}$. In fact, $X$ is the left topology on a countable well-founded partial ordering.

Theorem (Abashidze/Blass 87/91): Consider $\Omega \geq \omega^\omega$ with the order topology. Then $\text{Log}(\Omega, d) = \text{GL}$. 
Topological models for GLP

We consider poly-topological spaces \((X; \tau_0, \tau_1, \ldots)\) where modality \(\langle n \rangle\) corresponds to the derived set operator \(d_n\) w.r.t. \(\tau_n\).

**Definition:** \(X\) is a GLP-space if

- \(\tau_0\) is scattered;
- For each \(A \subseteq X\), \(d_n(A)\) is \(\tau_{n+1}\)-open;
- \(\tau_n \subseteq \tau_{n+1}\).

**Remark:** In a GLP-space, all \(\tau_n\) are scattered.
Topological models for GLP

We consider poly-topological spaces \((X; \tau_0, \tau_1, \ldots)\) where modality \(\langle n \rangle\) corresponds to the derived set operator \(d_n\) w.r.t. \(\tau_n\).

**Definition:** \(X\) is a GLP-space if

- \(\tau_0\) is scattered;
- For each \(A \subseteq X\), \(d_n(A)\) is \(\tau_{n+1}\)-open;
- \(\tau_n \subseteq \tau_{n+1}\).

**Remark:** In a GLP-space, all \(\tau_n\) are scattered.
Basic example

Consider a bitopological space \((\Omega, \tau_0, \tau_1)\), where

- \(\Omega\) is an ordinal;
- \(\tau_0\) is the left topology on \(\Omega\);
- \(\tau_1\) is the interval topology on \(\Omega\).

Fact (Esakia): \((\Omega, \tau_0, \tau_1)\) is a model of \(GLP_2\), but not an exact one: linearity axiom holds for \(\langle 0 \rangle\), that is,

\[
[0](\varphi \rightarrow (\psi \lor \langle 0 \rangle \psi)) \lor [0](\psi \rightarrow (\varphi \lor \langle 0 \rangle \varphi)).
\]
Let \((X, \tau)\) be a scattered space.

**Fact:** There is the coarsest topology \(\tau^+\) on \(X\) such that \((X; \tau, \tau^+)\) is a \(GLP_2\)-space.

The next topology \(\tau^+\) is generated by \(\tau\) and \(\{d(A) : A \subseteq X\}\) (as a subbase).

Thus, any \((X, \tau)\) generates a \(GLP\)-space \((X; \tau_0, \tau_1, \ldots)\) with \(\tau_0 = \tau\) and \(\tau_{n+1} = \tau_n^+\), for each \(n\).
Next topology and generated *GLP*-space

Let \((X, \tau)\) be a scattered space.

**Fact:** There is the coarsest topology \(\tau^+\) on \(X\) such that \((X; \tau, \tau^+)\) is a \(GLP_2\)-space.

The next topology \(\tau^+\) is generated by \(\tau\) and \(\{d(A) : A \subseteq X\}\) (as a subbase).

Thus, any \((X, \tau)\) generates a \(GLP\)-space \((X; \tau_0, \tau_1, \ldots)\) with \(\tau_0 = \tau\) and \(\tau_{n+1} = \tau_n^+\), for each \(n\).
Completeness for $GLP_2$

$GLP_2$ is complete w.r.t. $GLP_2$-spaces generated from the left topology on a well-founded partial ordering (with Guram Bezhanishvili and Thomas Icard).

**Theorem:** There is a countable $GLP_2$-space $X$ such that $\text{Log}(X, d_0, d_1) = GLP_2$.

In fact, $X$ has the form $(X; \tau\prec, \tau^\prec)$ where $(X, \prec)$ is a well-founded partial ordering.

**Aside:** This seems to be the first naturally occurring example of a logic that is topologically complete but not Kripke complete.
Completeness for $GLP_2$

$GLP_2$ is complete w.r.t. $GLP_2$-spaces generated from the left topology on a well-founded partial ordering (with Guram Bezhanishvili and Thomas Icard).

**Theorem:** There is a countable $GLP_2$-space $X$ such that $\text{Log}(X, d_0, d_1) = GLP_2$.

In fact, $X$ has the form $(X; \tau_\prec, \tau^\bot_\prec)$ where $(X, \prec)$ is a well-founded partial ordering.

**Aside:** This seems to be the first naturally occurring example of a logic that is topologically complete but not Kripke complete.
Difficulties for three or more operators.

**Fact.** If \((X, \tau)\) is Hausdorff and first-countable (i.e. if each point has a countable neighborhood base), then \((X, \tau^+)\) is discrete.

**Proof:** Each \(a \in X\) is a unique limit of a countable sequence \(A = \{a_n\}\). Hence, \(\{a\} = d(A)\) is open.
Let $\tau_0$ be the left topology on an ordinal $\Omega$. It generates a $GLP$-space $(\Omega; \tau_0, \tau_1, \ldots)$. What are these topologies?

**Fact:** $\tau_1$ is the order topology on $\Omega$. 
Def. Let $\alpha$ be a limit ordinal.

- $C \subseteq \alpha$ is a club in $\alpha$ if $C$ is $\tau_1$-closed and unbounded below $\alpha$.
- The filter generated by clubs in $\alpha$ is called the club filter. It is improper iff $\alpha$ has countable cofinality.

Fact. $\tau_2$ is the club filter topology:

- $\tau_2$-isolated points are ordinals of countable cofinality;
- if $\text{cf}(\alpha) > \omega$ then clubs in $\alpha$ form a neighborhood base of $\alpha$;
- the least non-isolated point is $\omega_1$. 
Def. Let $\alpha$ be a limit ordinal.

- $C \subseteq \alpha$ is a club in $\alpha$ if $C$ is $\tau_1$-closed and unbounded below $\alpha$.
- The filter generated by clubs in $\alpha$ is called the club filter. It is improper iff $\alpha$ has countable cofinality.

Fact. $\tau_2$ is the club filter topology:

- $\tau_2$-isolated points are ordinals of countable cofinality;
- if $\text{cf}(\alpha) > \omega$ then clubs in $\alpha$ form a neighborhood base of $\alpha$;
- the least non-isolated point is $\omega_1$. 
Def. $A \subseteq \alpha$ is stationary in $\alpha$ if $A$ intersects every club in $\alpha$.

We have: $d_2(A) = \{ \alpha : \text{cf}(\alpha) > \omega \text{ and } A \cap \alpha \text{ is stationary} \}$

Remark: Set theorists call $d_2$ Mahlo operation. Ordinals in $d_2(\text{Reg})$, where $\text{Reg}$ is the class of regular cardinals, are called weakly Mahlo cardinals. Their existence implies consistency of $\text{ZFC}$.
Stationary reflection

Studied by: Solovay, Harrington, Jech, Shelah, Magidor, and many more.

**Def.** Ordinal $\kappa$ is *reflecting* if whenever $A$ is stationary in $\kappa$ there is an $\alpha < \kappa$ such that $A \cap \alpha$ is stationary in $\alpha$.

**Def.** Ordinal $\kappa$ is *doubly reflecting* if whenever $A, B$ are stationary in $\kappa$ there is an $\alpha < \kappa$ such that both $A \cap \alpha$ and $B \cap \alpha$ are stationary in $\alpha$.

**Theorem.** $\kappa$ is $\tau_3$-nonisolated iff $\kappa$ is doubly reflecting.
Stationary reflection

Studied by: Solovay, Harrington, Jech, Shelah, Magidor, and many more.

Def. Ordinal $\kappa$ is reflecting if whenever $A$ is stationary in $\kappa$ there is an $\alpha < \kappa$ such that $A \cap \alpha$ is stationary in $\alpha$.

Def. Ordinal $\kappa$ is doubly reflecting if whenever $A, B$ are stationary in $\kappa$ there is an $\alpha < \kappa$ such that both $A \cap \alpha$ and $B \cap \alpha$ are stationary in $\alpha$.

Theorem. $\kappa$ is $\tau_3$-nonisolated iff $\kappa$ is doubly reflecting.
Stationary reflection

Studied by: Solovay, Harrington, Jech, Shelah, Magidor, and many more.

**Def.** Ordinal $\kappa$ is *reflecting* if whenever $A$ is stationary in $\kappa$ there is an $\alpha < \kappa$ such that $A \cap \alpha$ is stationary in $\alpha$.

**Def.** Ordinal $\kappa$ is *doubly reflecting* if whenever $A, B$ are stationary in $\kappa$ there is an $\alpha < \kappa$ such that both $A \cap \alpha$ and $B \cap \alpha$ are stationary in $\alpha$.

**Theorem.** $\kappa$ is $\tau_3$-nonisolated iff $\kappa$ is doubly reflecting.
Fact (characterizing $\tau_3$):

- If $\kappa$ is not doubly reflecting, then $\kappa$ is $\tau_3$-isolated;
- If $\kappa$ is doubly reflecting, then the sets $d_2(A) \cap \kappa$, i.e.,
  \[
  \{ \alpha < \kappa : \text{cf}(\alpha) > \omega \text{ and } A \cap \alpha \text{ is stationary in } \alpha \},
  \]
  where $A$ is stationary in $\kappa$, form a base of $\tau_3$-open punctured neighborhoods of $\kappa$. 
Corollaries

Fact.

- If $\kappa$ is weakly compact then $\kappa$ is doubly reflecting.
- (Magidor) If $\kappa$ is doubly reflecting then $\kappa$ is weakly compact in $L$.

Cor. Assertion “$\tau_3$ is non-discrete” is equiconsistent with the existence of a weakly compact cardinal.

Cor. It is consistent with ZFC that $\tau_3$ is discrete and hence that $GLP_3$ is incomplete w.r.t. any ordinal space.
Corollaries

Fact.

\begin{itemize}
    \item If $\kappa$ is weakly compact then $\kappa$ is doubly reflecting.
    \item (Magidor) If $\kappa$ is doubly reflecting then $\kappa$ is weakly compact in $L$.
\end{itemize}

Cor. Assertion \texttt{“$\tau_3$ is non-discrete”} is equiconsistent with the existence of a weakly compact cardinal.

Cor. It is consistent with ZFC that $\tau_3$ is discrete and hence that $GLP_3$ is incomplete w.r.t. any ordinal space.
Corollaries

Fact.

- If $\kappa$ is weakly compact then $\kappa$ is doubly reflecting.
- (Magidor) If $\kappa$ is doubly reflecting then $\kappa$ is weakly compact in $L$.

Cor. Assertion “$\tau_3$ is non-discrete” is equiconsistent with the existence of a weakly compact cardinal.

Cor. It is consistent with $ZFC$ that $\tau_3$ is discrete and hence that $GLP_3$ is incomplete w.r.t.
any ordinal space.
Let $\theta_n$ denote the first limit point of $\tau_n$.

<table>
<thead>
<tr>
<th>Name</th>
<th>$\theta_n$</th>
<th>$d_n(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_0$</td>
<td>left</td>
<td>1</td>
</tr>
<tr>
<td>$\tau_1$</td>
<td>order</td>
<td>$\omega$</td>
</tr>
<tr>
<td>$\tau_2$</td>
<td>club</td>
<td>$\omega_1$</td>
</tr>
<tr>
<td>$\tau_3$</td>
<td>Mahlo</td>
<td>$\theta_3$</td>
</tr>
</tbody>
</table>

$\theta_3$ is the first doubly reflecting cardinal.
On the location of the least non-isolated point

**Definition.** Let $\theta_n$ denote the first non-isolated point of $\tau_n$ (in the space of all ordinals).

We have: $\theta_0 = 1, \theta_1 = \omega, \theta_2 = \omega_1, \theta_3 = ?$

**ZFC** does not know much about the location of $\theta_3$:

- $\theta_3$ is regular, but not a successor of a regular cardinal;
- While weakly compact cardinals are non-isolated, $\theta_3$ need not be weakly compact: If infinitely many supercompact cardinals exist, then there is a model where $\aleph_{\omega+1}$ is doubly reflecting (Magidor);
- If $\theta_3$ is a successor of a singular cardinal, then some very strong large cardinal hypothesis must be consistent (Woodin cardinals).
**On the location of the least non-isolated point**

**Definition.** Let $\theta_n$ denote the first non-isolated point of $\tau_n$ (in the space of all ordinals).

We have: $\theta_0 = 1$, $\theta_1 = \omega$, $\theta_2 = \omega_1$, $\theta_3 =$?

**ZFC** does not know much about the location of $\theta_3$:

- $\theta_3$ is regular, but not a successor of a regular cardinal;
- While weakly compact cardinals are non-isolated, $\theta_3$ need not be weakly compact: If infinitely many supercompact cardinals exist, then there is a model where $\aleph_{\omega+1}$ is doubly reflecting (Magidor);
- If $\theta_3$ is a successor of a singular cardinal, then some very strong large cardinal hypothesis must be consistent (Woodin cardinals).
On the location of the least non-isolated point

Definition. Let $\theta_n$ denote the first non-isolated point of $\tau_n$ (in the space of all ordinals).

We have: $\theta_0 = 1$, $\theta_1 = \omega$, $\theta_2 = \omega_1$, $\theta_3 =$?

ZFC does not know much about the location of $\theta_3$:

- $\theta_3$ is regular, but not a successor of a regular cardinal;
- While weakly compact cardinals are non-isolated, $\theta_3$ need not be weakly compact: If infinitely many supercompact cardinals exist, then there is a model where $\aleph_{\omega+1}$ is doubly reflecting (Magidor);
- If $\theta_3$ is a successor of a singular cardinal, then some very strong large cardinal hypothesis must be consistent (Woodin cardinals).
On the location of the least non-isolated point

Definition. Let $\theta_n$ denote the first non-isolated point of $\tau_n$ (in the space of all ordinals).

We have: $\theta_0 = 1$, $\theta_1 = \omega$, $\theta_2 = \omega_1$, $\theta_3 = ?$

$\text{ZFC}$ does not know much about the location of $\theta_3$:

- $\theta_3$ is regular, but not a successor of a regular cardinal;
- While weakly compact cardinals are non-isolated, $\theta_3$ need not be weakly compact: If infinitely many supercompact cardinals exist, then there is a model where $\aleph_{\omega+1}$ is doubly reflecting (Magidor);
- If $\theta_3$ is a successor of a singular cardinal, then some very strong large cardinal hypothesis must be consistent (Woodin cardinals).
Completeness of $\text{GLP}_2$ for $\Omega$

A. Blass (91): 1) If $V = L$ and $\Omega \geq \aleph_\omega$, then $\text{GL}$ is complete w.r.t. $(\Omega, \tau_2)$. (Hence, "$\text{GL}$ is complete" is consistent with $\text{ZFC}$.)

2) On the other hand, if there is a weakly Mahlo cardinal, there is a model of $\text{ZFC}$ in which $\text{GL}$ is incomplete w.r.t. $(\Omega, \tau_2)$ (for any $\Omega$).

(This is based on a model of Harrington and Shelah in which $\aleph_2$ is reflecting for stationary sets of ordinals of countable cofinality.)

Theorem (B., 2009): If $V = L$ and $\Omega \geq \aleph_\omega$, then $\text{GLP}_2$ is complete w.r.t. $(\Omega; \tau_1, \tau_2)$. 
Completeness of \( GLP_2 \) for \( \Omega \)

A. Blass (91): 1) If \( V = L \) and \( \Omega \geq \aleph_\omega \), then \( GL \) is complete w.r.t. \( (\Omega, \tau_2) \). (Hence, «\( GL \) is complete» is consistent with \( ZFC \).)

2) On the other hand, if there is a weakly Mahlo cardinal, there is a model of \( ZFC \) in which \( GL \) is incomplete w.r.t. \( (\Omega, \tau_2) \) (for any \( \Omega \)).

(This is based on a model of Harrington and Shelah in which \( \aleph_2 \) is reflecting for stationary sets of ordinals of countable cofinality.)

Theorem (B., 2009): If \( V = L \) and \( \Omega \geq \aleph_\omega \), then \( GLP_2 \) is complete w.r.t. \( (\Omega; \tau_1, \tau_2) \).
Completeness of \( GLP_2 \) for \( \Omega \)

A. Blass (91): 1) If \( V = L \) and \( \Omega \geq \aleph_\omega \), then \( GL \) is complete w.r.t. \( (\Omega, \tau_2) \). (Hence, «\( GL \) is complete» is consistent with \( ZFC \).)

2) On the other hand, if there is a weakly Mahlo cardinal, there is a model of \( ZFC \) in which \( GL \) is incomplete w.r.t. \( (\Omega, \tau_2) \) (for any \( \Omega \)).

(This is based on a model of Harrington and Shelah in which \( \aleph_2 \) is reflecting for stationary sets of ordinals of countable cofinality.)

Theorem (B., 2009): If \( V = L \) and \( \Omega \geq \aleph_\omega \), then \( GLP_2 \) is complete w.r.t. \( (\Omega; \tau_1, \tau_2) \).
Further topologies: a conjecture
(for set-theorists)

Theorem (B., Philipp Schlicht): If $\kappa$ is $\Pi^1_n$-indescribable, then $\kappa$ is non-isolated w.r.t. $\tau_{n+2}$. Hence, if $\Pi^1_n$-indescribable cardinals below $\Omega$ exist for each $n$, then all topologies $\tau_n$ are non-discrete.

Conjecture: If $V = L$ and $\Pi^1_n$-indescribable cardinals below $\Omega$ exist for each $n$, then $GLP$ is complete w.r.t. $\Omega$. 
Further topologies: a conjecture
(for set-theorists)

Theorem (B., Philipp Schlicht): If $\kappa$ is $\Pi^1_n$-indescribable, then $\kappa$ is non-isolated w.r.t. $\tau_{n+2}$. Hence, if $\Pi^1_n$-indescribable cardinals below $\Omega$ exist for each $n$, then all topologies $\tau_n$ are non-discrete.

Conjecture: If $V = L$ and $\Pi^1_n$-indescribable cardinals below $\Omega$ exist for each $n$, then GLP is complete w.r.t. $\Omega$. 
**Topological completeness**

*GLP* is complete w.r.t. (countable, hausdorff) *GLP*-spaces.

**Theorem (B., Gabelaia 10):** There is a countable hausdorff *GLP*-space $X$ such that $\text{Log}(X) = \text{GLP}$.

In fact, $X$ is $\varepsilon_0$ equipped with topologies refining the order topology, where $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$.

If *GLP* complete w.r.t. a *GLP*-space $X$, then all topologies of $X$ have Cantor-Bendixon rank $\geq \varepsilon_0$. 
Topological completeness

GLP is complete w.r.t. (countable, hausdorff) GLP-spaces.

Theorem (B., Gabelaia 10): There is a countable hausdorff GLP-space $X$ such that $\text{Log}(X) = \text{GLP}$.

In fact, $X$ is $\varepsilon_0$ equipped with topologies refining the order topology, where $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots \}$.

If GLP complete w.r.t. a GLP-space $X$, then all topologies of $X$ have Cantor-Bendixon rank $\geq \varepsilon_0$. 
Topological completeness

\textit{GLP} is complete w.r.t. (countable, hausdorff) \textit{GLP}-spaces.

Theorem (B., Gabelaia 10): There is a countable hausdorff \textit{GLP}-space $X$ such that $\log(X) = \textit{GLP}$.

In fact, $X$ is $\varepsilon_0$ equipped with topologies refining the order topology, where $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots \}$.

If \textit{GLP} complete w.r.t. a \textit{GLP}-space $X$, then all topologies of $X$ have Cantor-Bendixon rank $\geq \varepsilon_0$. 
Conclusions

1. The notion of $GLP$-space seems to fit very naturally in the theory of scattered topological spaces.

2. Connections between provability logic and infinitary combinatorics (stationary reflection etc.) are fairly unexpected and would need further study.

3. From the point of view of applications to the study of modal logics such as $GLP$, the models obtained are still ‘too big’ and not very handy.
Conclusions

1. The notion of $GLP$-space seems to fit very naturally in the theory of scattered topological spaces.

2. Connections between provability logic and infinitary combinatorics (stationary reflection etc.) are fairly unexpected and would need further study.

3. From the point of view of applications to the study of modal logics such as $GLP$, the models obtained are still ‘too big’ and not very handy.
Conclusions

1. The notion of \textit{GLP}-space seems to fit very naturally in the theory of scattered topological spaces.

2. Connections between provability logic and infinitary combinatorics (stationary reflection etc.) are fairly unexpected and would need further study.

3. From the point of view of applications to the study of modal logics such as \textit{GLP}, the models obtained are still ‘too big’ and not very handy.


Thank you!