Topological semantics of polymodal provability logic

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In memoriam Leo Esakia

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Lindenbaum algebras

Lindenbaum algebra of a theory T:
$\mathcal{L}_T = \{\text{sentences of } T\}/\sim_T$, where

$$\varphi \sim_T \psi \iff T \vdash (\varphi \leftrightarrow \psi)$$

$\mathcal{L}_T$ is a boolean algebra with operations $\wedge$, $\vee$, $\neg$.
$\mathbf{1}$ = the set of provable sentences of T
$\mathbf{0}$ = the set of refutable sentences of T

For consistent gödelian T all such algebras are countable atomless, hence pairwise isomorphic.

Kripke, Pour-El: even computably isomorphic

Magari algebras

Emerged in 1970s: Macintyre/Simmons, Magari, Smoryński, ...

Let T be a gödelian theory (formalizing its own syntax),
$\mathsf{Con}(T) =$ «T is consistent»

Consistency operator $\Diamond : \varphi \longmapsto \mathsf{Con}(T+\varphi)$ acting on $\mathcal{L}_T$.

$(\mathcal{L}_T, \Diamond) =$ *Magari algebra of T*
$\Box\varphi = \neg\Diamond\neg\varphi =$ «φ is provable in T»

Characteristic of $(M, \Diamond)$:
$ch(M) = \min\{k : \Diamond^k \mathbf{1} = \mathbf{0}\}$;
$ch(M) = \infty$, if no such k exists.

Remark. If $\mathbb{N} \models T$, then $ch(\mathcal{L}_T) = \infty$.

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Identities of Magari algebras

K. Gödel (33), M.H. Löb (55): Algebra $(\mathcal{L}_T, \Diamond)$ satisfies the following set of identities *GL*:

- boolean identities
- $\Diamond\mathbf{0} = \mathbf{0}$
- $\Diamond(\varphi \vee \psi) = (\Diamond\varphi \vee \Diamond\psi)$
- $\Diamond\varphi = \Diamond(\varphi \wedge \neg\Diamond\varphi)$ (Löb's identity)

GL-algebras = Magari algebras = diagonalizable algebras

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GL-algebras = Magari algebras = diagonalizable algebras

Provability logic

Let $\mathcal{A} = (A, \Diamond)$ be a boolean algebra with an operator $\Diamond$, and $\varphi(\vec{x})$ a term.

Def. Denote

- $\mathcal{A} \vDash \varphi$ if $\mathcal{A} \vDash \forall \vec{x}\,(\varphi(\vec{x}) = \mathbf{1})$;
- The logic of $\mathcal{A}$ is $Log(\mathcal{A}) = \{\varphi : \mathcal{A} \vDash \varphi\}$.

R. Solovay (76): If $ch(\mathcal{L}_T) = \infty$, then $Log(\mathcal{L}_T, \Diamond) = GL$.

GL is nice as a modal logic (decidable, Kripke complete, fmp, Craig, cut-free calculus, . . .)

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n-consistency

Def. A gödelian theory T is n-consistent, if every provable Σ^0_n-sentence of T is true.

$n\text{-Con}(T)$ = « T is n-consistent»

n-consistency operator $\langle n \rangle : \mathcal{L}_T \to \mathcal{L}_T$

$$\varphi \longmapsto n\text{-Con}(T + \varphi).$$

$[n] = \neg \langle n \rangle \neg$ (n-provability)

The algebra of n-provability

$\mathcal{M}_T = (\mathcal{L}_T; \langle 0 \rangle, \langle 1 \rangle, \ldots)$.

The following identities *GLP* hold in $\mathcal{M}_T$:

- *GL*, for all $\langle n \rangle$;
- $\langle n+1 \rangle \varphi \to \langle n \rangle \varphi$;
- $\langle n \rangle \varphi \to [n+1] \langle n \rangle \varphi$.

G. Japaridze (86): If $\mathbb{N} \models T$, then $Log(\mathcal{M}_T) = GLP$.

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The significance of GLP

GLP is

- Useful for proof theory:
  - Ordinal notations and consistency proof for PA;
  - Independent combinatorial assertion;
  - Characterization of provably total computable functions of PA.
- Fairly complicated and not so nice modal-logically:
  - no Kripke completeness, no cut-free calculus;
  - though it is decidable and has Craig interpolation.

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Set-theoretic interpretation

Let X be a nonempty set, $\mathcal{P}(X)$ the b.a. of subsets of X.

Consider any operator $\delta : \mathcal{P}(X) \to \mathcal{P}(X)$ and the structure $(\mathcal{P}(X), \delta)$.

Question: Can $(\mathcal{P}(X), \delta)$ be a GL-algebra and, if yes, when?

Def. Write $(X, \delta) \vDash \varphi$ if $(\mathcal{P}(X), \delta) \vDash \varphi$. Also let $Log(X, \delta) := Log(\mathcal{P}(X), \delta)$.

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Derived set operators

Let X be a topological space, $A \subseteq X$.
Derived set $d(A)$ of A is the set of limit points of A:

$$x \in d(A) \iff \forall U_x \text{ open } \exists y \neq x \; y \in U_x \cap A.$$

Fact. If $(X, \delta) \vDash GL$ then X naturally bears a topology τ for which $\delta = d_\tau$, that is, $\delta : A \longmapsto d_\tau(A)$, for each $A \subseteq X$.

In fact, we can define: A is τ-closed iff $\delta(A) \subseteq A$.
Equivalently, $c(A) = A \cup \delta(A)$ is the closure of A.

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Scattered spaces

Definition (Cantor): X is scattered if every nonempty $A \subseteq X$ has an isolated point.

Cantor-Bendixon sequence:

$$X_0 = X, \quad X_{\alpha+1} = d(X_\alpha), \quad X_\lambda = \bigcap_{\alpha<\lambda} X_\alpha, \text{if } \lambda \text{ is limit.}$$

Notice that all X_α are closed and $X_0 \supset X_1 \supset X_2 \supset \dots$

Fact (Cantor): X is scattered $\iff \exists \alpha : X_\alpha = \varnothing$.

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Examples

- Left topology $\tau_{\prec}$ on a strict partial ordering $(X, \prec)$.
  $A \subseteq X$ is open iff $\forall x, y\ (y \prec x \in A \Rightarrow y \in A)$.

  Fact: $(X, \prec)$ is well-founded iff $(X, \tau_{\prec})$ is scattered.

- Ordinal Ω with the usual order topology generated by intervals (α, β), $[0, \beta)$, (α, Ω) such that $\alpha < \beta$.

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Löb's identity = scatteredness

Simmons 74, Esakia 81

Löb's identity: $\Diamond A = \Diamond(A \wedge \neg\Diamond A)$.

Topological reading:

$$d(A) = d(A \setminus d(A)) = d(\mathrm{iso}(A)),$$

where $\mathrm{iso}(A) = A \setminus d(A)$ is the set of isolated points of A.

Fact: The following are equivalent:

- X is scattered;
- $d(A) = d(\mathrm{iso}(A))$ for any $A \subseteq X$;
- $(X, d) \vDash GL$.

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- X is scattered;
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Completeness theorems

Theorem (Esakia 81): There is a scattered X such that $Log(X, d) = GL$. In fact, X is the left topology on a countable well-founded partial ordering.

Theorem (Abashidze/Blass 87/91): Consider $\Omega \geq \omega^{\omega}$ with the order topology. Then $Log(\Omega, d) = GL$.

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Topological models for GLP

We consider poly-topological spaces $(X; \tau_0, \tau_1, \ldots)$ where modality $\langle n \rangle$ corresponds to the derived set operator d_n w.r.t. τ_n.

Definition: X is a GLP-space if

- τ_0 is scattered;
- For each $A \subseteq X$, $d_n(A)$ is τ_{n+1}-open;
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Remark: In a GLP-space, all τ_n are scattered.

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Basic example

Consider a bitopological space (Ω, τ_0, τ_1), where

- Ω is an ordinal;
- τ_0 is the left topology on Ω;
- τ_1 is the interval topology on Ω.

Fact (Esakia): (Ω, τ_0, τ_1) is a model of GLP_2, but not an exact one: linearity axiom holds for $\langle 0 \rangle$, that is,

$$[0](\varphi \to (\psi \vee \langle 0 \rangle \psi)) \vee [0](\psi \to (\varphi \vee \langle 0 \rangle \varphi)).$$

Next topology and generated GLP-space

Let (X, τ) be a scattered space.

Fact: There is the coarsest topology τ^+ on X such that $(X; \tau, \tau^+)$ is a GLP_2-space.

The next topology τ^+ is generated by τ and $\{d(A) : A \subseteq X\}$ (as a subbase).

Thus, any (X, τ) generates a GLP-space $(X; \tau_0, \tau_1, \dots)$ with $\tau_0 = \tau$ and $\tau_{n+1} = \tau_n^+$, for each n.

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Completeness for GLP_2

GLP_2 is complete w.r.t. GLP_2-spaces generated from the left topology on a well-founded partial ordering (with Guram Bezhanishvili and Thomas Icard).

Theorem: There is a countable GLP_2-space X such that $Log(X, d_0, d_1) = GLP_2$.

In fact, X has the form $(X; \tau_{\prec}, \tau_{\prec}^{+})$ where $(X, \prec)$ is a well-founded partial ordering.

Aside: This seems to be the first naturally occurring example of a logic that is topologically complete but not Kripke complete.

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Difficulties

Difficulties for three or more operators.

Fact. If (X, τ) is hausdorff and first-countable (i.e. if each point has a countable neighborhood base), then (X, τ^+) is discrete.

Proof: Each $a \in X$ is a unique limit of a countable sequence $A = \{a_n\}$. Hence, $\{a\} = d(A)$ is open.

Ordinal GLP-spaces

Let τ_0 be the left topology on an ordinal Ω. It generates a *GLP*-space $(\Omega; \tau_0, \tau_1, \dots)$. What are these topologies?

Fact: τ_1 is the order topology on Ω.

Club filter topology

Def. Let α be a limit ordinal.

- $C \subseteq \alpha$ is a club in α if C is τ_1-closed and unbounded below α.
- The filter generated by clubs in α is called the club filter. It is improper iff α has countable cofinality.

Fact. τ_2 is the club filter topology:

- τ_2-isolated points are ordinals of countable cofinality;
- if $cf(\alpha) > \omega$ then clubs in α form a neighborhood base of α;
- the least non-isolated point is ω_1.

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Stationary sets

Def. $A \subseteq \alpha$ is stationary in α if A intersects every club in α.

We have: $d_2(A) = \{\alpha : \ cf(\alpha) > \omega$ and $A \cap \alpha$ is stationary$\}$

Remark: Set theorists call d_2 Mahlo operation.
Ordinals in $d_2(Reg)$, where Reg is the class of regular cardinals, are called weakly Mahlo cardinals. Their existence implies consistency of ZFC.

Stationary reflection

Studied by: Solovay, Harrington, Jech, Shelah, Magidor, and many more.

Def. Ordinal κ is *reflecting* if whenever A is stationary in κ there is an $\alpha < \kappa$ such that $A \cap \alpha$ is stationary in α.

Def. Ordinal κ is *doubly reflecting* if whenever A, B are stationary in κ there is an $\alpha < \kappa$ such that both $A \cap \alpha$ and $B \cap \alpha$ are stationary in α.

Theorem. κ is τ_3-nonisolated iff κ is doubly reflecting.

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Mahlo topology τ_3

Fact (characterizing τ_3):

- If κ is not doubly reflecting, then κ is τ_3-isolated;
- If κ is doubly reflecting, then the sets $d_2(A) \cap \kappa$, i.e.,

$$\{\alpha < \kappa : cf(\alpha) > \omega \text{ and } A \cap \alpha \text{ is stationary in } \alpha\},$$

  where A is stationary in κ, form a base of τ_3-open punctured neighborhoods of κ.

Corollaries

Fact.

- If κ is weakly compact then κ is doubly reflecting.
- (Magidor) If κ is doubly reflecting then κ is weakly compact in L.

Cor. Assertion "τ_3 is non-discrete" is equiconsistent with the existence of a weakly compact cardinal.

Cor. It is consistent with ZFC that τ_3 is discrete and hence that GLP_3 is incomplete w.r.t. any ordinal space.

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Summary

Let θ_n denote the first limit point of τ_n.

	name	θ_n	$d_n(A)$
τ_0	left	1	$\{\alpha : A \cap \alpha \neq \emptyset\}$
τ_1	order	ω	$\{\alpha \in Lim : A \cap \alpha$ is unbounded in $\alpha\}$
τ_2	club	ω_1	$\{\alpha : cf(\alpha) > \omega$ and $A \cap \alpha$ is stationary in $\alpha\}$
τ_3	Mahlo	θ_3	... ...

θ_3 is the first doubly reflecting cardinal.

On the location of the least non-isolated point

Definition. Let θ_n denote the first non-isolated point of τ_n (in the space of all ordinals).

We have: $\theta_0 = 1$, $\theta_1 = \omega$, $\theta_2 = \omega_1$, $\theta_3 =?$

ZFC does not know much about the location of θ_3:

- θ_3 is regular, but not a successor of a regular cardinal;
- While weakly compact cardinals are non-isolated, θ_3 need not be weakly compact: If infinitely many supercompact cardinals exist, then there is a model where $\aleph_{\omega+1}$ is doubly reflecting (Magidor);
- If θ_3 is a successor of a singular cardinal, then some very strong large cardinal hypothesis must be consistent (Woodin cardinals).

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Completeness of GLP_2 for Ω

A. Blass (91): 1) If $V = L$ and $\Omega \geq \aleph_\omega$, then GL is complete w.r.t. (Ω, τ_2). (Hence, «GL is complete» is consistent with ZFC.)

2) On the other hand, if there is a weakly Mahlo cardinal, there is a model of ZFC in which GL is incomplete w.r.t. (Ω, τ_2) (for any Ω).

(This is based on a model of Harrington and Shelah in which $\aleph_2$ is reflecting for stationary sets of ordinals of countable cofinality.)

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Further topologies: a conjecture (for set-theorists)

Theorem (B., Philipp Schlicht): If κ is Π^1_n-indescribable, then κ is non-isolated w.r.t. τ_{n+2}. Hence, if Π^1_n-indescribable cardinals below Ω exist for each n, then all topologies τ_n are non-discrete.

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Topological completeness

GLP is complete w.r.t. (countable, hausdorff) GLP-spaces.

Theorem (B., Gabelaia 10): There is a countable hausdorff GLP-space X such that $Log(X) = GLP$.

In fact, X is ε_0 equipped with topologies refining the order topology, where $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$.

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Conclusions

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2. Connections between provability logic and infinitary combinatorics (stationary reflection etc.) are fairly unexpected and would need further study.

3. From the point of view of applications to the study of modal logics such as *GLP*, the models obtained are still ‘too big’ and not very handy.

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Thank you!