Topology, Algebra, and Categories in Logic
TACL 2011
Marseilles, France, July 26-30 2011

Formulas of Finite Number Propositional Variables in the Intuitionistic Logic With the Solovay Modality

Leo Esakia and Revaz Grigolia
Tbilisi State University, Institute of Cybernetics

In

["Provability interpretation of modal logics", *Israel Journal of Mathematics*, ***25(1976), 287-304.***]

Robert Solovay presented a set-theoretical translation of modal formulas by putting $\Box$*p to mean "p is true in every* transitive model of Zermelo-Fraenkel Set Theory ZF".

By defining an interpretation as a function *s* sending modal formulas to sentences of ZF which commutes with the Boolean connectives and putting *s*($\Box$ *p*) *to be equal to the* statement "*s(p) is true in every transitive model of ZF",* Solovay formulated a modal system, which we call here *SOL, and announced its ZF-completeness.*

SOL is the classical modal system which results from the Gödel-Löb system *GL (alias, the provability logic) by adding the formula*

$$\Box(\Box p \to \Box q) \vee \Box(\Box q \to \Box p \wedge p)$$

as a new axiom.

ZF-completeness: *For any modal formula p, SOL* $\vdash$ *p iff ZF* $\vdash$ *s(p) for any Solovay's interpretation s.*

We introduce a simple system *I.SOL*, which is an Intuitionistic "companion" of *SOL:*
the composition of the well-known Gödel's modal translation of Heyting Calculus and split-map
(= splitting a formula $\Box p$ into the formula $p \wedge \Box p$)
provides the needed embedding *of I.SOL* into *SOL.*

The proof-intuitionistic logic *KM (=Kuznetsov-Muravitsky* [*"On superintuitionistic logics as fragments of Proof Logic extensions", Studia logica,* ***45 (1986), 77-99.***]) is the Heyting propositional calculus *HC* enriched by □ as *Prov* modality satisfying the following conditions:

$p \to \Box p; \Box p \to (q \vee (q \to p)); (\Box p \to p) \to p.$

An intuitionistic modal system I.SOL is an extension of the proof-intuitionistic logic KM obtained by postulating the formula

$$(\Box p \to \Box q) \vee (\Box q \to p)$$

as a new axiom.

A Heyting algebra with an operator $\Box$ is called *Solovay algebra**, if the following conditions are satisfied:*

$$p \leq \Box p; \ \Box p \leq q \vee (q \rightarrow p); \ \Box p \rightarrow p = p;$$

$$(\Box p \rightarrow \Box q) \vee (\Box q \rightarrow p) = 1.$$

The class of all Solovay algebras forms a variety, which we denote by **SA**.

A finite Heyting algebra H *is* *a Boolean cascade*, *if there exist Boolean lattices* $B_1 , \dots , B_k$ such that

$$H = B_1 + \dots + B_k,$$

where each B_i *is a convex sublattice of* H and $B_i + B_{i+1}$ *denotes the ordinal sum of* B_i *and* B_{i+1} *in which the Smallest* element of B_i *and the largest element of* B_{i+1} *are identified.*

A topological space *X with binary relation R is said to be* GL-frame *if :*

1) *X is a Stone space (i. e. 0-dimensional, Hausdorf and compact topological* space);

2) $R(x)$ *and* $R^{-1}(x)$ *are closed sets for every* $x \in X$ *and* $R^{-1}(A)$ *is a clopen* for every clopen *A of X;*

3) for every clopen *A of X and every element* $x \in A$ *there is an element* $y \in A \setminus R^{-1}(A)$ *such that either* xRy *or* $x \in A \setminus R^{-1}(A)$.

A map $f : X_1 \to X_2$ *from a GL-frame* X_1 *to a GL-frame* X_2 *is said to* be *strongly isotone if*

$$f(x)R_2y \Leftrightarrow (\exists z \in X_1)(xR_1z \& f(z) = y).$$

Let ***G*** be the category of *GL*-frames and strongly isotone maps.

The category ***G*** is dually equivalent to the category **D** of diagonalizable algebras with diagonalizable algebra homomorphisms.

An algebra $(A; \vee, \wedge, \Diamond, -, 0, 1)$ *is said to be diagonalizable algebra if* $(A; \vee, \wedge, -, 0, 1)$ *is a Boolean algebra and* $\Diamond$ *satisfies the following conditions :*

(1) $\Diamond(a \vee b) = \Diamond(a) \vee \Diamond(b)$,

(2) $\Diamond(0) = 0$,

(3) $\Diamond(a) \leq \Diamond(a \vee -\Diamond(a))$.

On every diagonalizable algebra A is defined a unary operator $\Box$ which is dual to $\Diamond$: $\Box(x) = -\Diamond-(x)$. The sublattice $H = \{\Box(a) \wedge a : a \in A\}$ *forms a* Heyting algebra, where $a \to b = \Box(-a \vee b) \wedge (-a \vee b)$.

The class of all algebras $(H; \vee, \wedge, \to, \Box, 0, 1)$ *,* where $(H; \vee, \wedge, \to, 0, 1)$ *is a Heyting, forms a variety which we denote by* $\mathbf{H}_\Box$*.*

The variety of Solovay algebras **SA** is a subvariety of $\mathbf{H}_\Box$

A pair $(X;R)$ *is said to be S-frame if :* 1) $(X;R)$ *is GL*-frame*;* 2) $(X;R_{\circ})$ is a poset; 3) for every *x; y; z; u* $\in X$ *if uRx,uRz,xRy and* $\neg$*(xRz), then zRy.*

Let **S** be the category of *S-frames and continuous strongly isotone maps.*

The duality between the category of Solovay algebras and the category of *S-frames is obtained by specialization of the duality between the categories* **D** and **G** on the case of Solovay algebras.

For any *S-frame* ($X;R$) *and* $U, V \in SA(X)$*(= the set of all clopen cones of X) define*:

$$U \to V = X \setminus (R^{-1}(U \setminus V) \cup (U \setminus V)),$$

$$\Box U = X \setminus R^{-1}(U \setminus V)$$

Then the algebra

$$SA((X;R)) = (SA(X); \cup, \cap, \to, \Box, 0, 1)$$

is a Solovay algebra.

As follows from a duality there is one-to-one correspondence between homomorphic images of a Solovay algebra *A* and closed cones of S(*A*), and between subalgebras of a *SA*-algebra *A* and correct partitions of S(A), where a correct partition of a (X;R) $\in$ **S** is a such equivalence relation E on X that

- *E is a closed equivalence relation, i. e. E-saturation of any closed* subset is closed;
- *E-saturation of any upper cone is an upper cone;*
- $(\forall x \in X)(E(x) \cap R^{-1}(E(x)) \neq \varnothing \Rightarrow E(x) \subset R^{-1}(E(x)));$
- *there is S-frame* $(Y;Q)$ *and a strongly isotone map* $f : X \to Y$ *such that* $Ker f = E$.

Suppose *(X;R)* is an *S-frame, A = SA*((*X;R*)) and $g_1,\ldots, g_n \in A$. *Now* we will present a criterion deciding whether *A* is generated by $g_1,\ldots, g_n$. Our criterion extends the analogous one for descriptive intuitionistic frames from to *S- frames.*

Denote by **n** the set {1, ... , *n*}. Let $G_p = g_1^{\varepsilon 1} \cap \ldots \cap g_n^{\varepsilon n}$, where $\varepsilon_i \in \{0, 1\}$, $p = \{i : \varepsilon_i = 1\}$ *and* $g_i^{\varepsilon i} = g_i$ *if* $\varepsilon_I = 1$, and $g_i^{\varepsilon i} = \neg g_i$ *if* $\varepsilon_I = 0$.

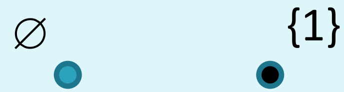
It is obvious that $\{Gp\}_{p \subset \mathbf{n}}$ *is a partition of X which we call a* colouring *of X. A point* $x \in G_p$ *is said to have the* colour *p, written as Col(x) = p. Let* us remark that $g_i = \bigcup_{i \in p} G_p$.

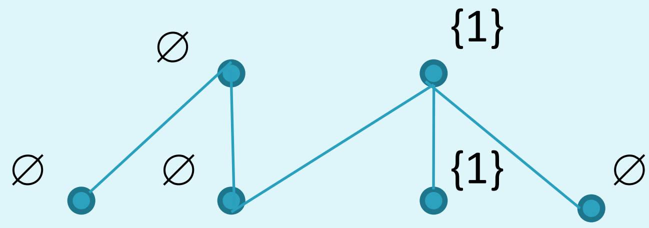
Lemma 1. *Suppose E is a correct partition of X. The following two conditions are mutually equivalent:*

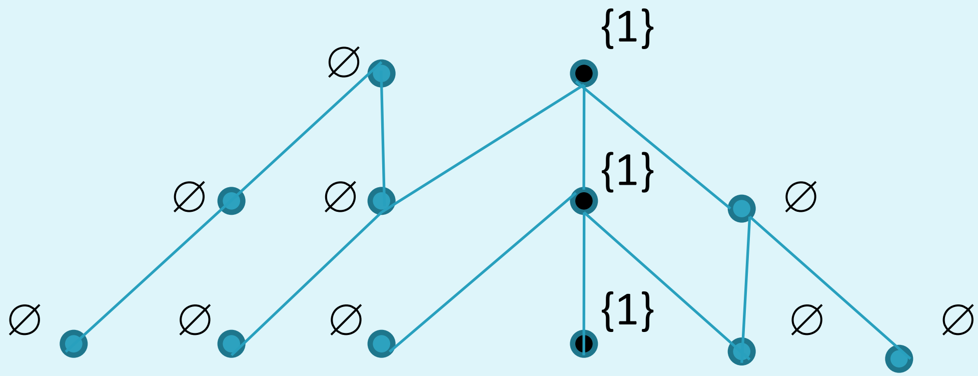
1) *Every* g_i *is E-saturated, that is* $E(g_i) = g_i$ $(1 \leq i \leq n)$;

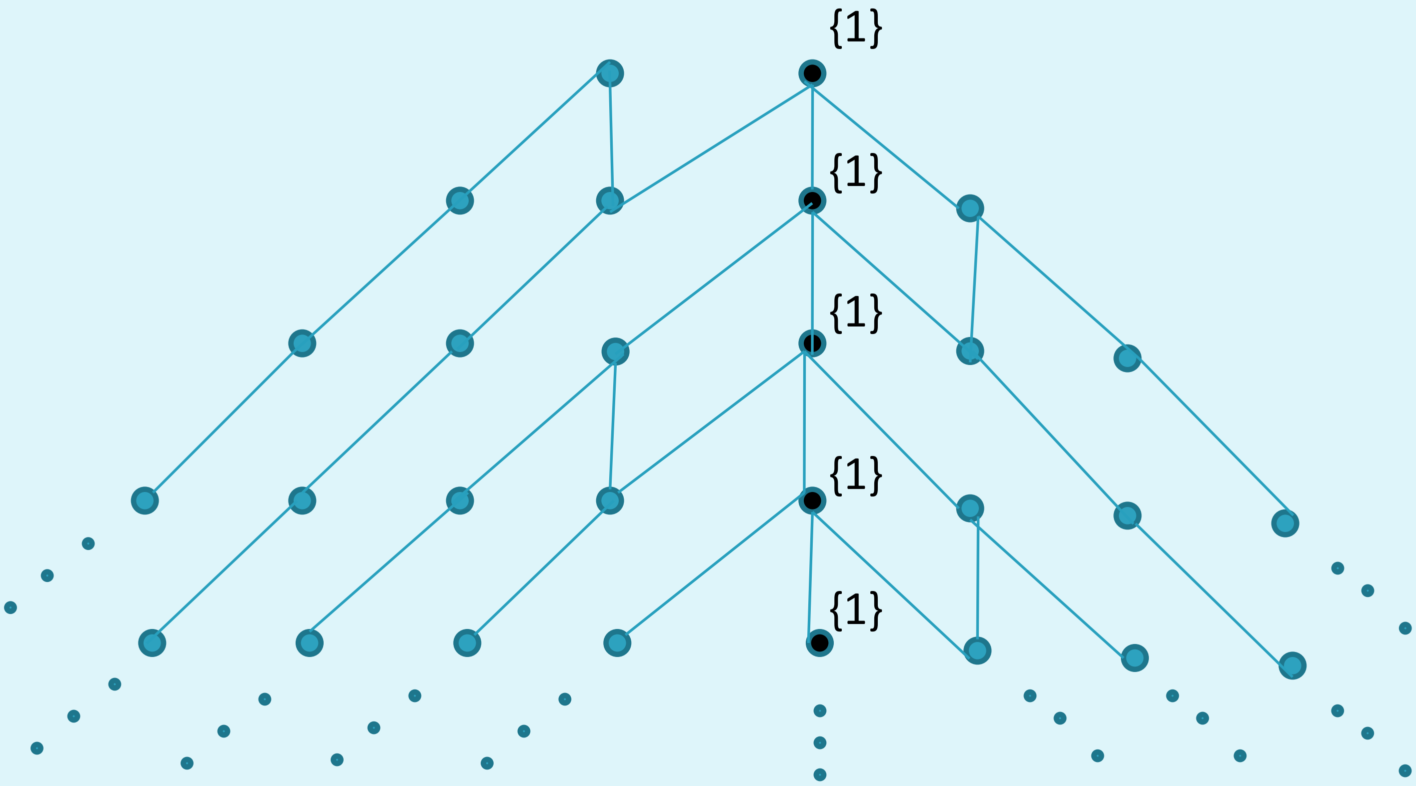
2) *Every class* G_p *is E-saturated, that is* $E(G_p) = G_p$ $(p \subset n)$.

Theorem 2. (*Coloring Theorem*) *A Solovay algebra A is generated by* $g_1,\ldots, g_n$ *iff for every non-trivial correct partition E of X* (= *S*(*A*))*, there exists an equivalence class of E containing points of different colors.*

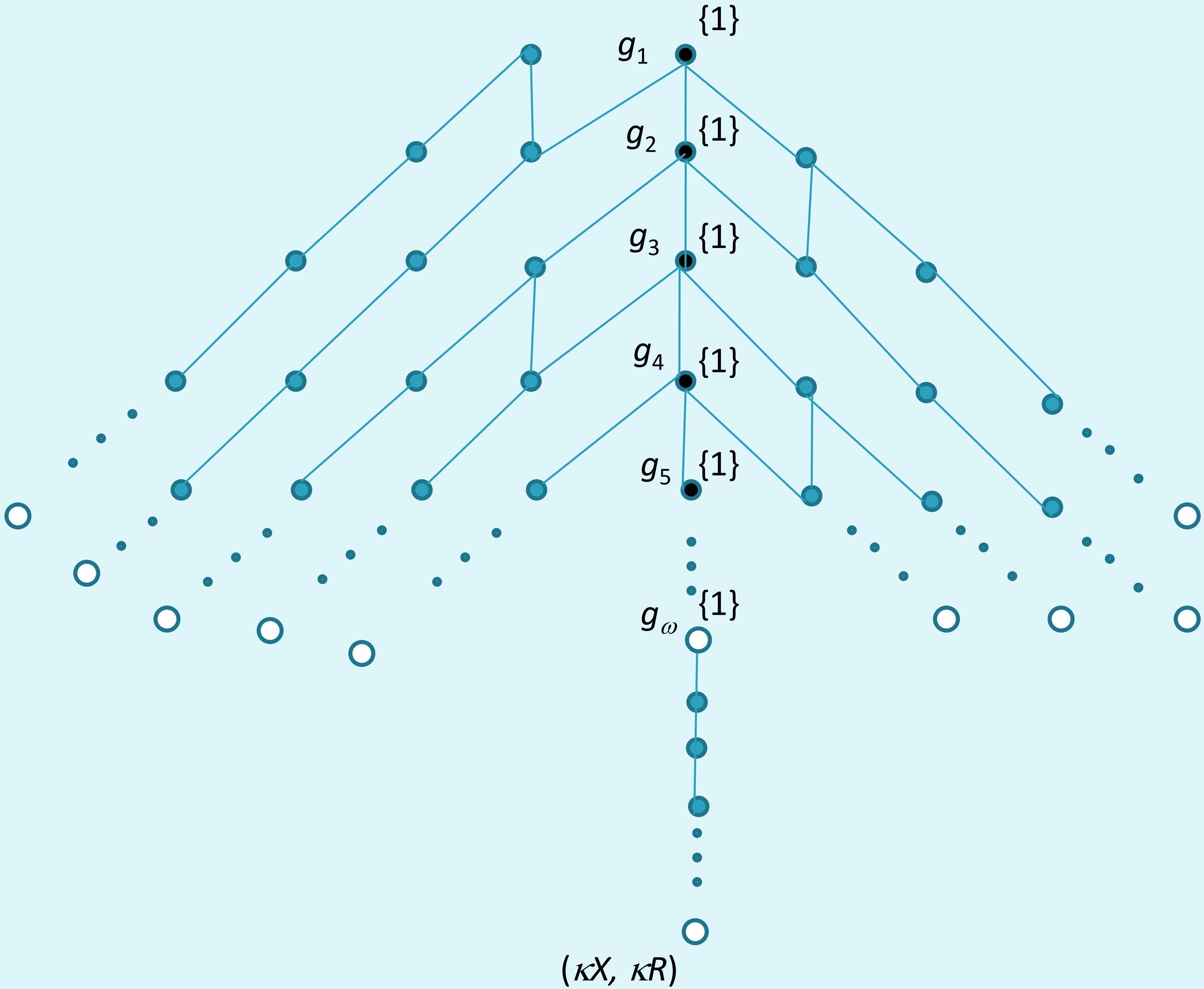

∅
{1}


∅
{1}
∅
∅
{1}
∅


{1}
∅
{1}
∅
∅
∅
{1}
∅
∅
∅
∅
∅



(X, R)


{1}
g_1
g_2
{1}
g_3
{1}
g_4
{1}
g_5
{1}
{1}
g_ω
$(\kappa X,\ \kappa R)$

Observation 3. *The algebra* (*Con(X),*□) *is a Solovay algebra.*

Let $K_0(X)$ be a ring of cones *of X obtained from the finite sets* $\Box^k\varnothing$ (*k-storied pyramid* of the figure) *and the sets* $X - R_\circ^{-1}(x)$ ($x \in X$) *by applying the* operations of union and intersection.

Theorem 4. *The ring of cones $K_0(X)$ is closed under the implication $\rightarrow$ and the box-operation $\Box$ of the Solovay algebra (Con(X), $\Box$). Thus the ring $K_0(X)$ is itself a Solovay algebra.*

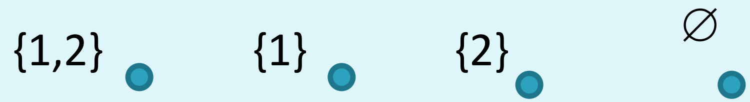
Denote by G *the cone* $\{g_i \in X : i \in \omega\}$(= *Trunk*) *and by* G_k *the cone* $\{g_i \in X : i \leq k \in \omega\}$*. Denote by K(X) the smallest subring of Con(X)* which contains the ring $K_0(X)$ *and the cones* $\square^k G$ $(k \in \omega)$.

Theorem 5. *The ring K(X) is also closed under operations* $\rightarrow$ *and* $\square$ *of the algebra Con(X) and hence is a Solovay algebra, which we denote by H(G).*

Theorem 6. *The Solovay algebra H(G) is the free cyclic algebra with generator G over the variety* ***SA****.*

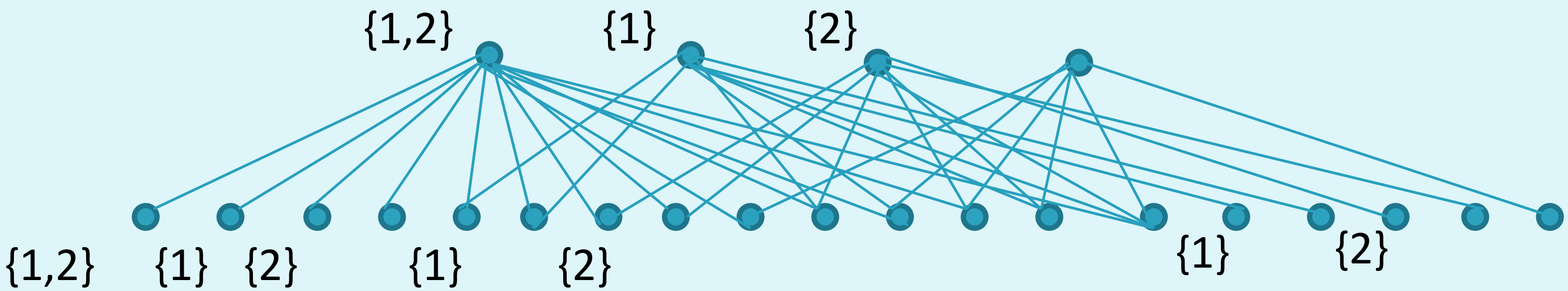
Finitely generated free Solovay Algebras

We describe finitely generated free Solovay algebras by means of a description of corres-ponding frames using the coloring technique. We describe a frame $X(n)$ *for* $n \geq 1$, *corresponding to n-generated free Solovay algebra* $F_{SA}(n)$, *by levels, i. e. by the elements of fixed depth.*



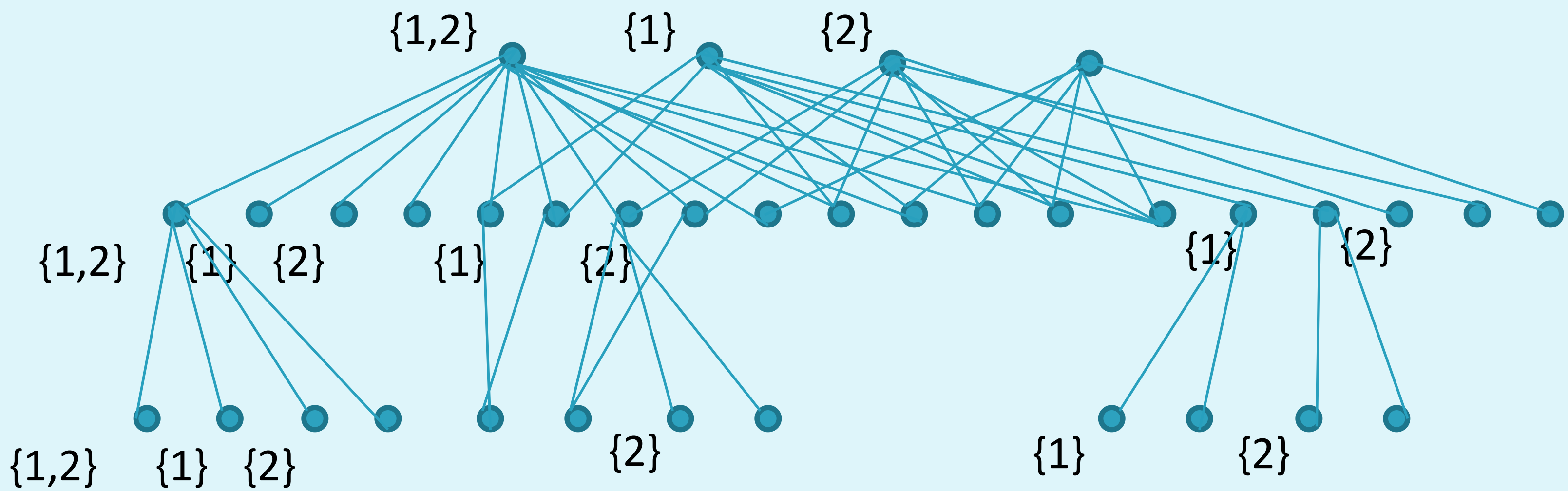
The set of elements of the first level (i. e. the set of elements with depth 1) $X_1(n)$ contains 2^n *elements, every of which has a color* $p \subset \{1, \dots , n\}$ in that way that different elements have different colors.

On $X_1(n)$ *define the binary relation* $R_1 \subset X_1^2(n)$: xR_1y is false for every $x, y \in X_1(n)$. *It is clear that the Solovay* algebra $F_{\mathbf{SA}1}(n)$ of all subsets of $X_1(n)$ is the free n-generated algebra in $\mathbf{SA}_1$. Observe, that the algebra $F_{\mathbf{SA}1}(n)$ is a diagonalizable algebra.



$X_2(n)$

For every element $a \in X_1(n)$ there are $a_1, \dots, a_k \in X_2(n)$ (= the set of all elements of the second level) with $Col(a_i) \subset Col(a)$, $i = 1, \dots, k$, such that ai is covered by only the element *a*. Further, for every set $\{u_1, \dots, u_k\}$ of incomparable elements of $X_1(n)$ there exists an element u_p such that $p = Col(u_p) \subset \bigcap_1^k Col(u_i)$ and u_p is covered by only the elements $u_1, \dots, u_k$.



Let $G_i = \{x \in X(n): i \in Col(x)\}$, $i = 1, \dots , n$. *Observe, that* G_i is an upper cone of *X(n). Let* $F_{\mathbf{SA}}(n)$ be an algebra generated by the set $\{G_1, \dots , G_n\}$ by operations $\cup, \cap, \rightarrow, \Box$, where $\Box Y = -R^{-1} - Y$.

Theorem 7. *The algebra* $F_{\mathbf{SA}}(n)$ *is n-generated free Solovay algebra for any positive integer n.*

THANK YOU