Topological Semantics of Modal Logic

David Gabelaia

Overview

- Personal story
- Three gracious ladies
- Completeness in C-semantics
  - Quasiorders as topologies
  - Finite connected spaces are interior images of the real line
  - Connected logics
- Completeness in d-semantics
  - Incompleteness
  - Ordinal completeness of **GL**
  - Completeness techniques for **wK4** and **K4.Grz**

Motivations

- Gödel's translation
  - Bringing intuitionistic reasoning into theclassical setting.
- Tarski's impetus towards "algebraization"
  - Algebra of Topology, McKinsey and Tarski, 1944.
- Quine's criticism
  - Making Modal Logic meaningful in the rest of mathematics

Three Graces

Topological space (X,τ)

Three Graces

Topological space (X,τ)

Heyting Algebra
Op(X)

Closure Algebra
$(\wp(X), \mathbf{C})$

Derivative Algebra
$(\wp(X), \mathbf{d})$

Three Graces

Topological space (X,τ)

Hegemone

Heyting Algebra
Op(X)

Cleta

Closure Algebra
$(\wp(X), \mathbf{C})$

Delia

Derivative Algebra
$(\wp(X), \mathbf{d})$

The discourses of the Graces

- Hegemone talks about open subsets.

  $U \cap (U \Rightarrow V) \subseteq V$

The discourses of the Graces

- Hegemone talks about open subsets.

  $U \cap (U \Rightarrow V) \subseteq V$

- Cleta can talk about everything Hegemone can:

  $\mathbf{I}A \cap \mathbf{I}\,(-\mathbf{I}A \cup \mathbf{I}B) \subseteq \mathbf{I}B$

  – and more:
    - $A \subseteq \mathbf{C}B$ subset B is "dense over" A
    - $\mathbf{C}A \cap \mathbf{C}B = \varnothing$ subsets A and B are "apart"

The discourses of the Graces

- Hegemone talks about open subsets.
  $U \cap (U \Rightarrow V) \subseteq V$
- Cleta can talk about everything Hegemone can:
  $\mathbf{I}A \cap \mathbf{I}\,(-\mathbf{I}A \cup \mathbf{I}B) \subseteq \mathbf{I}B$
  - and more:
    - $A \subseteq \mathbf{C}B$ — subset B is "dense over" A
    - $\mathbf{C}A \cap \mathbf{C}B = \varnothing$ — subsets A and B are "apart"
- Delia can talk about everything Cleta can:
  $A \cup \mathbf{d}A = \mathbf{C}A$
  - and more:

    $A \subseteq \mathbf{d}A$ — A is dense-in-itself (dii)

Three Graces

Hegemone

Heyting Algebra
Op(X)

Cleta

Closure Algebra
$(\wp(X), \mathbf{C})$

Delia

Derivative Algebra
$(\wp(X), \mathbf{d})$

Three Graces

Hegemone

Heyting Algebra
Op(X)

Heyting identities

HC

Cleta

Closure Algebra
($\wp$(X), **C**)

Delia

Derivative Algebra
($\wp$(X), **d**)

Three Graces

Hegemone

Heyting Algebra
Op(X)

Heyting identities

HC

Cleta

Closure Algebra
$(\wp(X), \mathbf{C})$

Kuratowski Axioms
$\mathbf{C}\varnothing = \varnothing$
$\mathbf{C}(A \cup B) = \mathbf{C}A \cup \mathbf{C}B$
$A \subseteq \mathbf{C}A$
$\mathbf{CC}A = \mathbf{C}A$

Delia

Derivative Algebra
$(\wp(X), \mathbf{d})$

S4

Three Graces

Hegemone

Heyting Algebra
Op(X)

Heyting identities

HC

Cleta

Closure Algebra
$(\wp(X), \mathbf{C})$

Kuratowski Axioms
$\mathbf{C}\varnothing = \varnothing$
$\mathbf{C}(A \cup B) = \mathbf{C}A \cup \mathbf{C}B$
$A \subseteq \mathbf{C}A$
$\mathbf{CC}A = \mathbf{C}A$

S4

Delia

Derivative Algebra
$(\wp(X), \mathbf{d})$

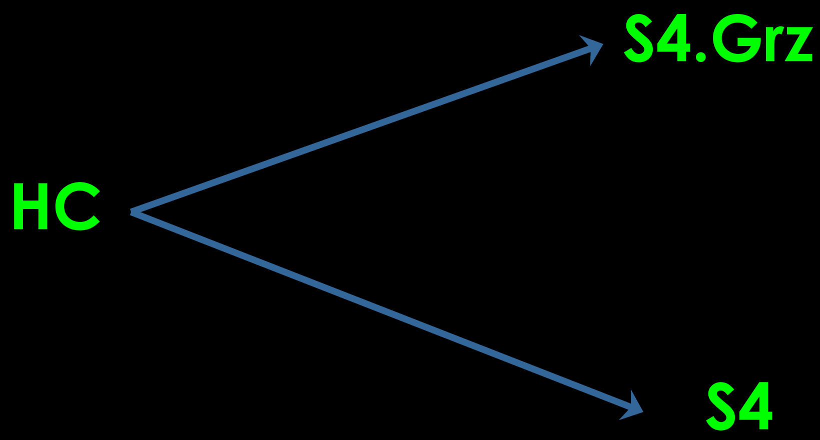
$\mathbf{d}\varnothing = \varnothing$
$\mathbf{d}(A \cup B) = \mathbf{d}A \cup \mathbf{d}B$
$\mathbf{dd}A \subseteq A \cup \mathbf{d}A$

wK4

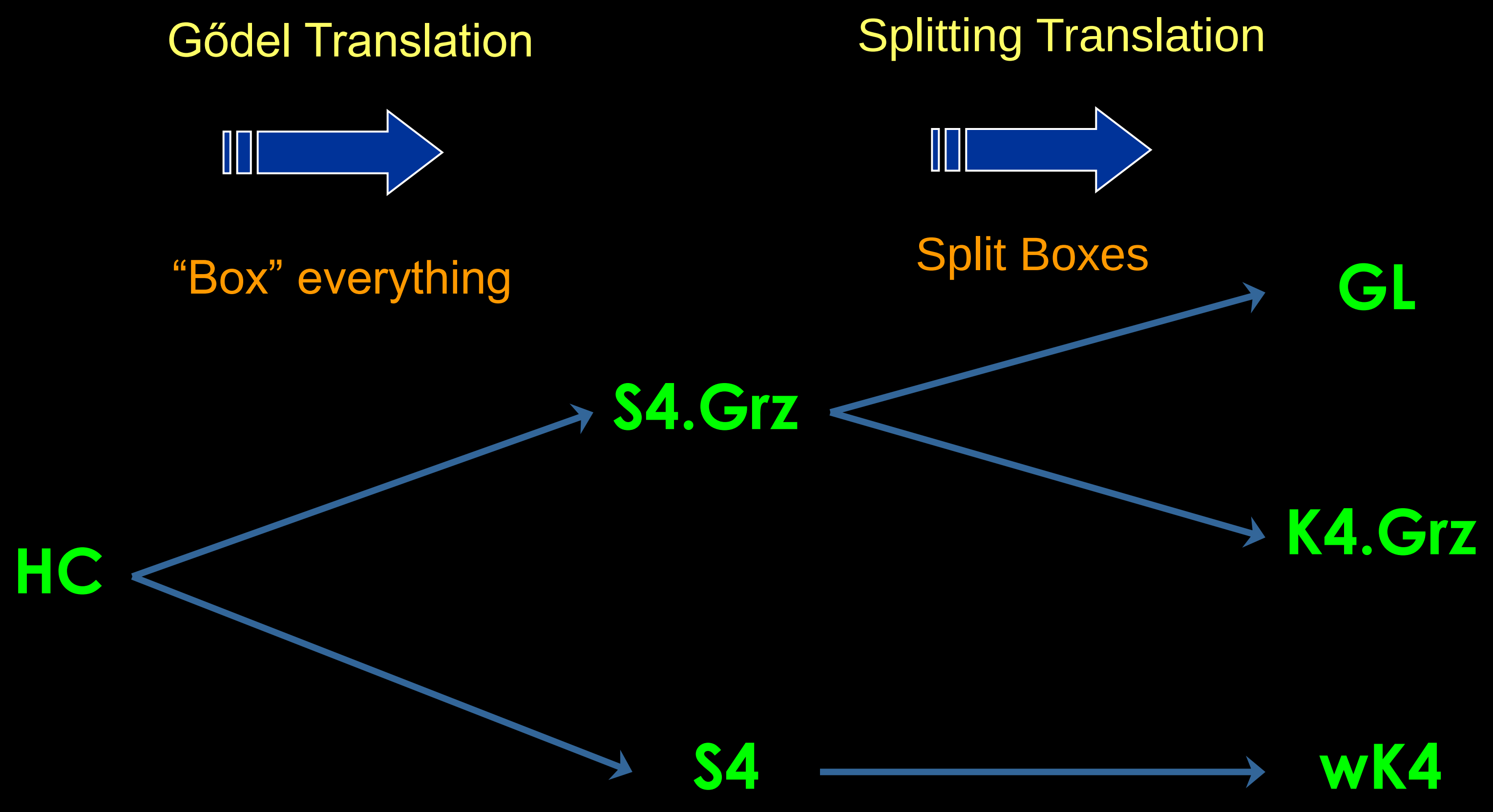
Graceful translations

Gődel Translation

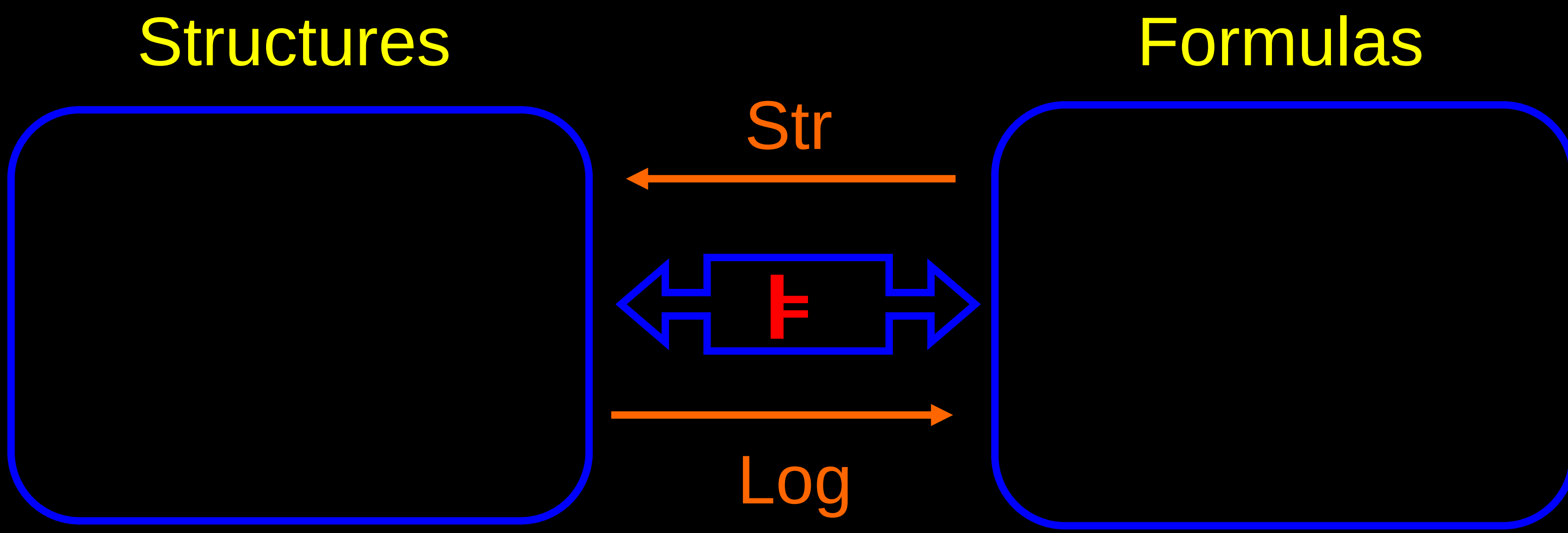
"Box" everything



Graceful translations



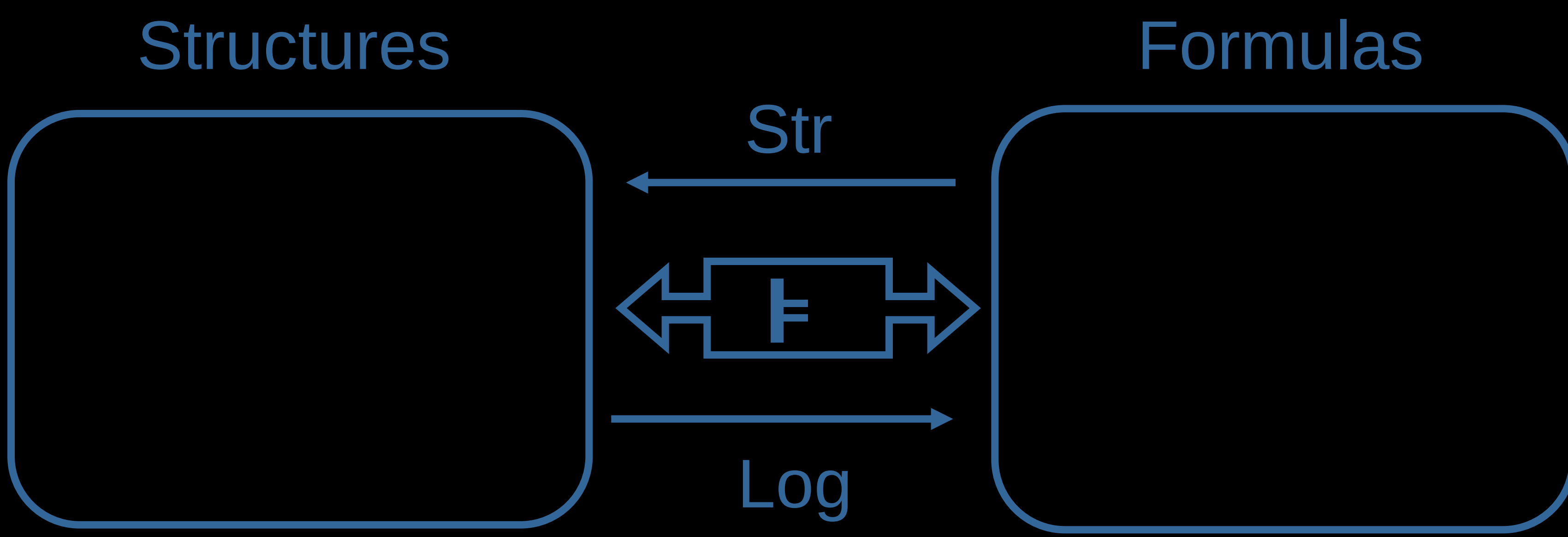
Syntax and Semantics



$\Gamma \subseteq \mathrm{Log}\ \mathrm{Str}\ (\Gamma)$

$K \subseteq \mathrm{Str}\ \mathrm{Log}\ (K)$

Syntax and Semantics



K is definable, if

K = Str Log (K)

Γ ⊆ Log Str (Γ)

K ⊆ Str Log (K)

Γ is complete, if

Γ = Log Str (Γ)

Completeness for Hegemone

- Heyting Calculus (**HC**) is complete wrt the class of all topological spaces

  [Tarski 1938]

- **HC** is also complete wrt the class of finite topological spaces
- **HC** is also complete wrt the class of finite partial orders
- Is there an intermediate logic that is topologically incomplete? (Kuznetsov's Problem).

Completeness for Cleta

Kuratowski Axioms

$\mathbf{C}\emptyset = \emptyset$

$\mathbf{C}(A \cup B) = \mathbf{C}A \cup \mathbf{C}B$

$A \subseteq \mathbf{C}A$

$\mathbf{CC}A = \mathbf{C}A$

Axioms of modal S4

$\Diamond 0 = 0$

$\Diamond(p \vee q) = \Diamond p \vee \Diamond q$

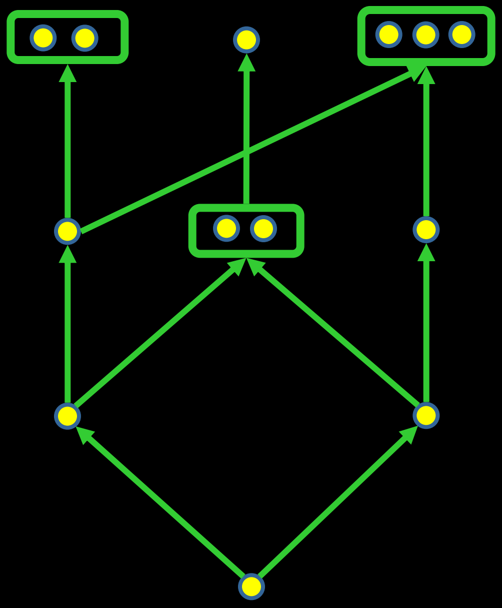
$p \to \Diamond p = 1$

$\Diamond\Diamond p = \Diamond p$

So **S4** is definitely valid on all topological spaces (soundness). How do we know that nothing extra goes through (completeness)?

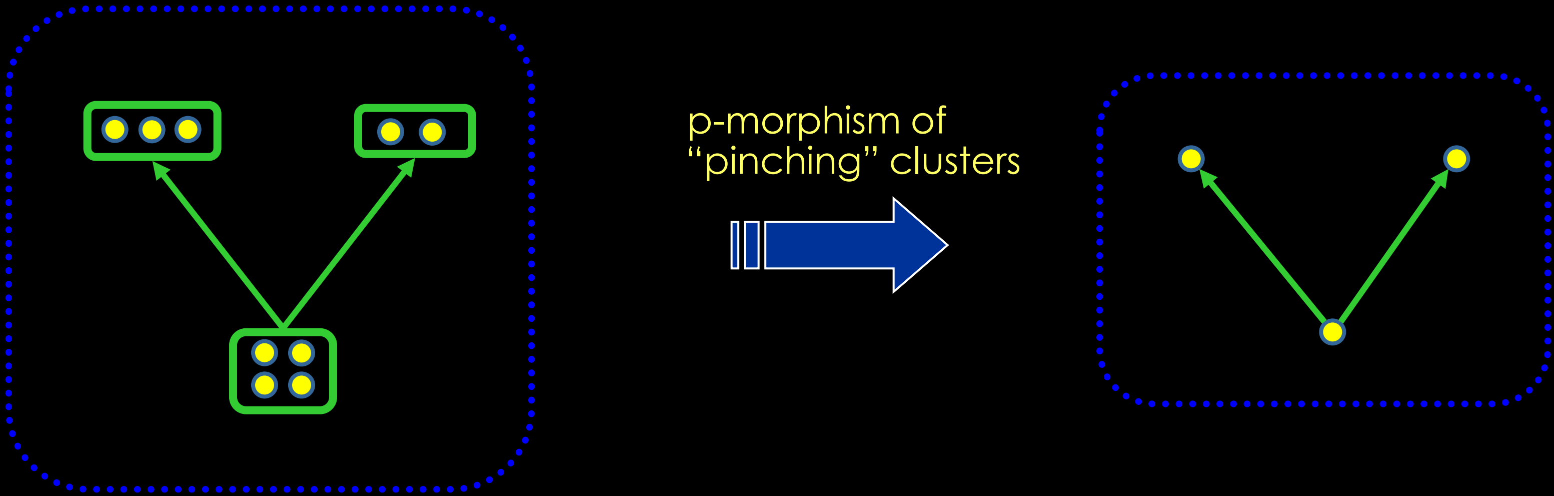
Kripke semantics for S4

- Quasiorders are **reflexive-transitive frames**.
- Just **partial orders** with **clusters**.
- **S4** is the logic of all quasiorders.
- Indeed, finite **tree-like** quasiorders suffice to generate **S4** (unravelling).

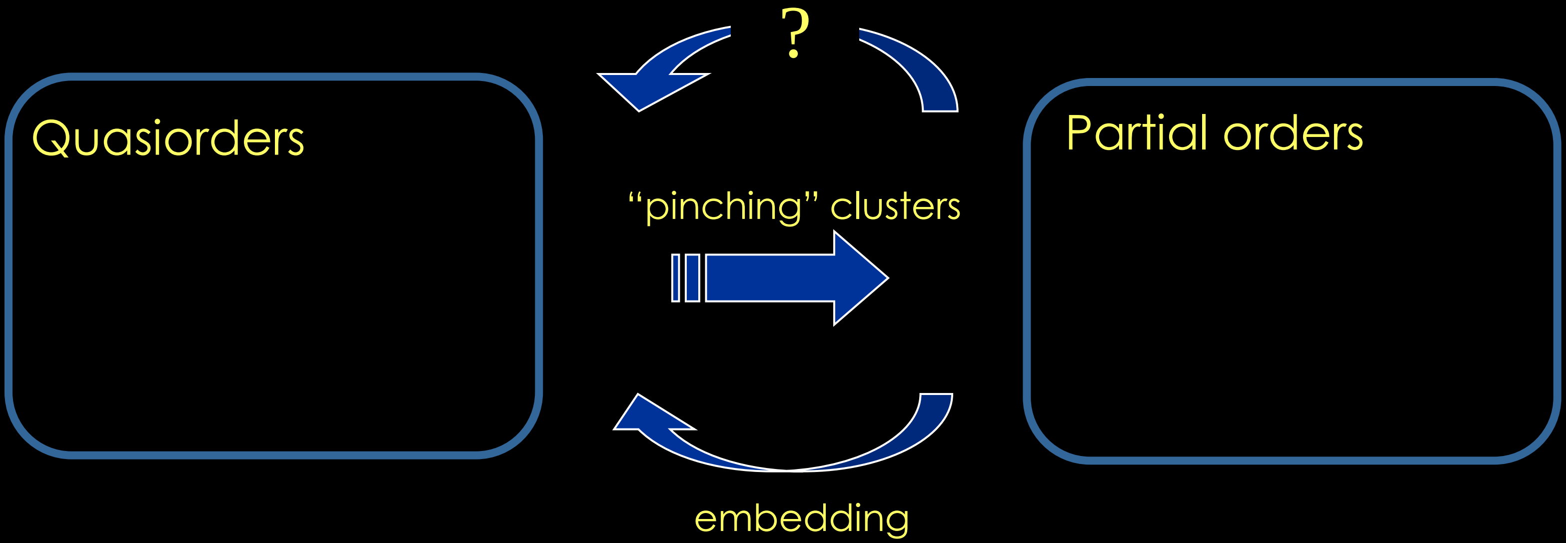


Intermezzo: Gödel Translation (quasi)orderly

HC |- φ iff **S4** |- Tr(φ)

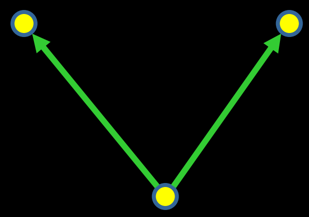


Intermezzo: Gödel Translation (quasi)orderly



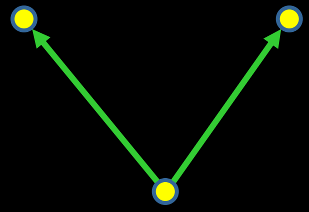
Quasiorder as a partially ordered sum of clusters

External skeleton
(partial order)

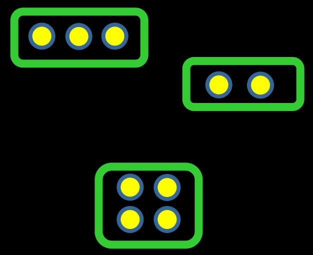


Quasiorder as a partially ordered sum of clusters

External skeleton
(partial order)

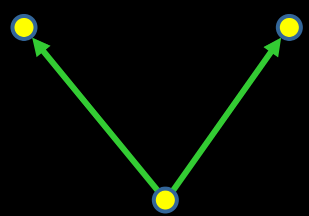


Internal worlds
(clusters)

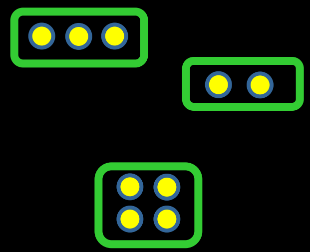


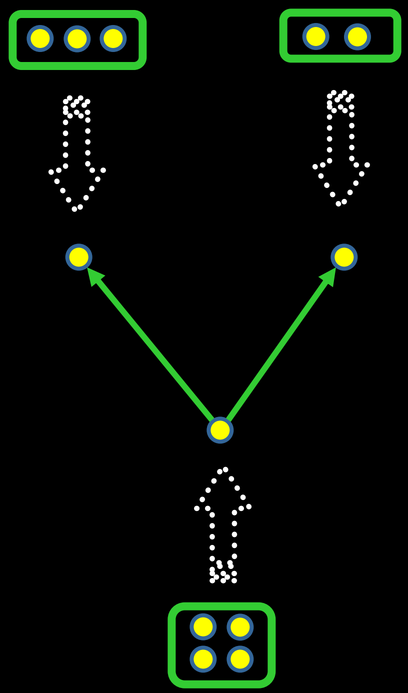
Quasiorder as a partially ordered sum of clusters

External skeleton
(partial order)



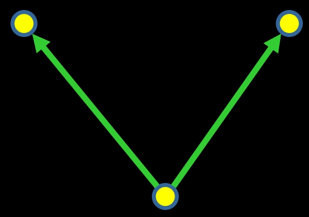
Internal worlds
(clusters)



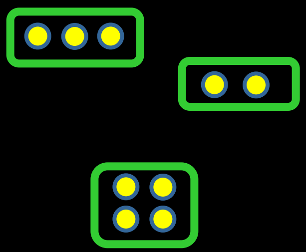


Quasiorder as a partially ordered sum of clusters

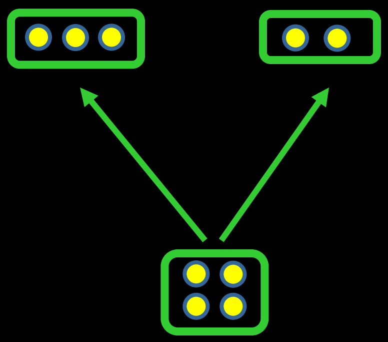
External skeleton
(partial order)



Internal worlds
(clusters)

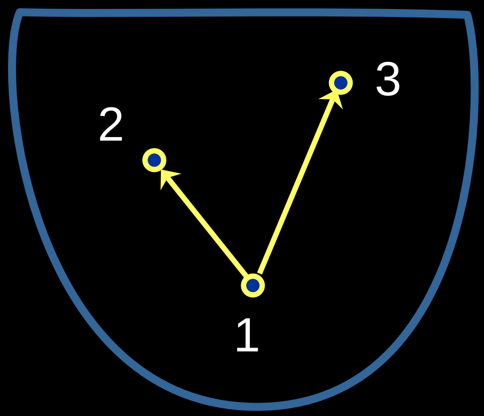


The sum



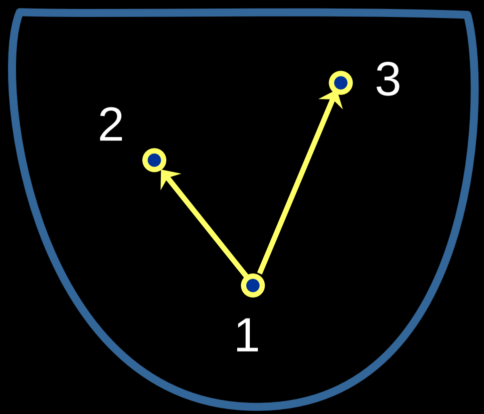
Partially ordered sums

A partial order P
(Skeleton)

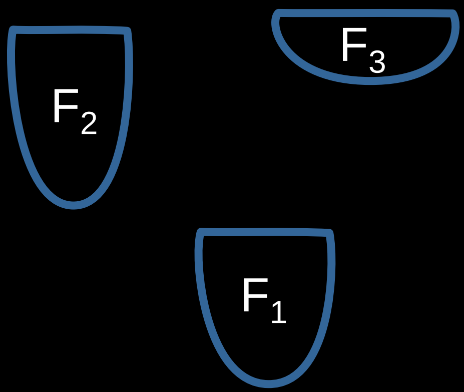


Partially ordered sums

A partial order P
(Skeleton)

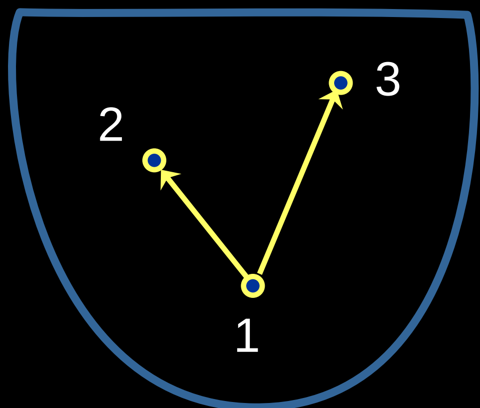


Family of frames $(F_i)_{i \in P}$
indexed by P
(Components)

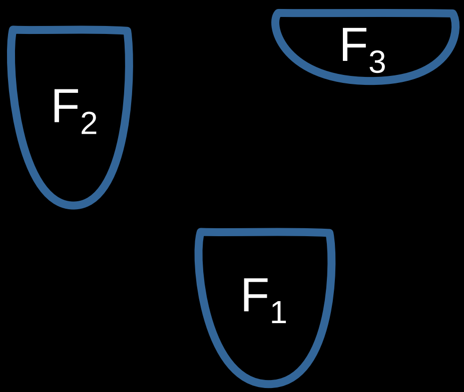


Partially ordered sums

A partial order P
(Skeleton)



Family of frames $(F_i)_{i \in P}$
indexed by P
(Components)

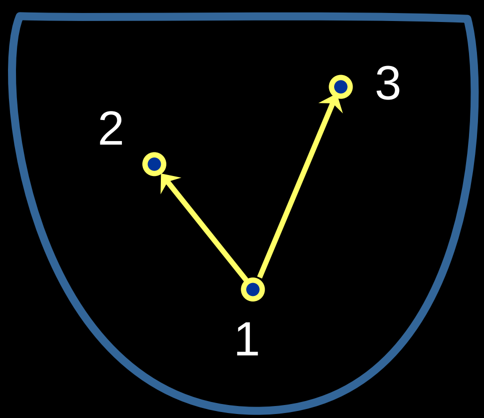


P-ordered sum of (F_i)

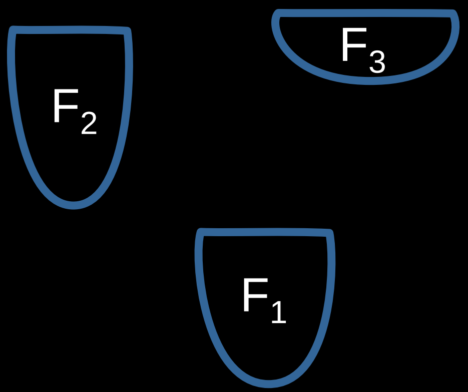
$\oplus_P F_i$

Partially ordered sums

A partial order P
(Skeleton)



Family of frames $(F_i)_{i \in P}$
indexed by P
(Components)

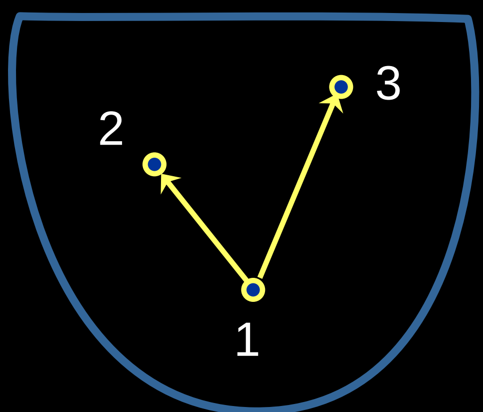


P-ordered sum of (F_i)

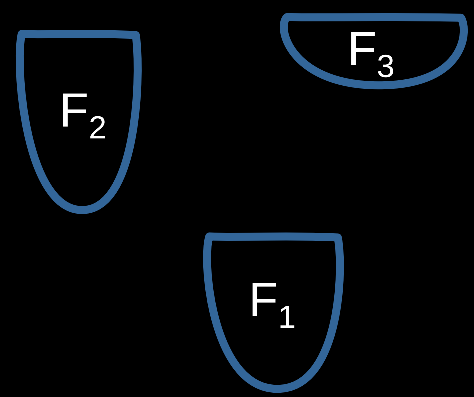
$\oplus_P F_i$

Partially ordered sums

A partial order P
(Skeleton)

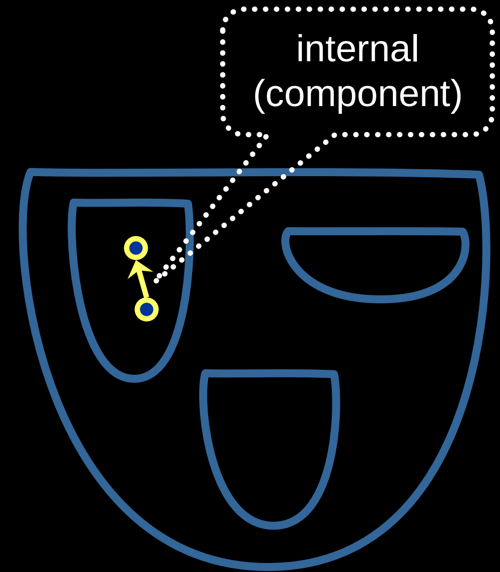


Family of frames $(F_i)_{i \in P}$
indexed by P
(Components)



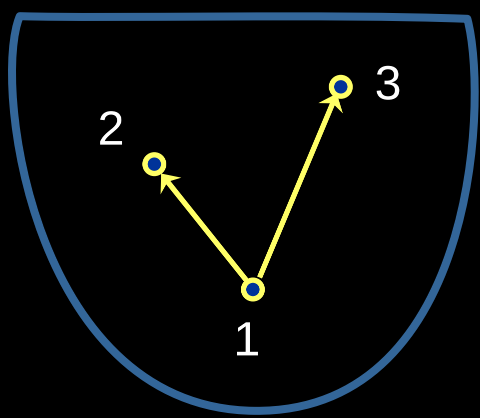
P-ordered sum of (F_i)

$\bigoplus_P F_i$

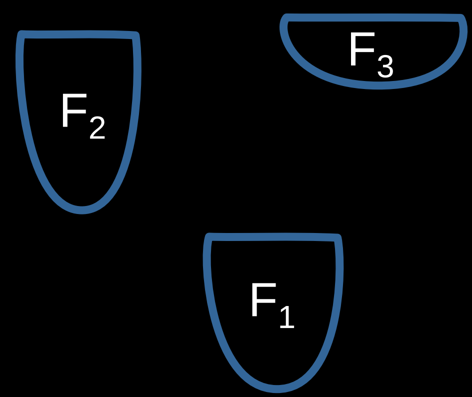


Partially ordered sums

A partial order P
(Skeleton)

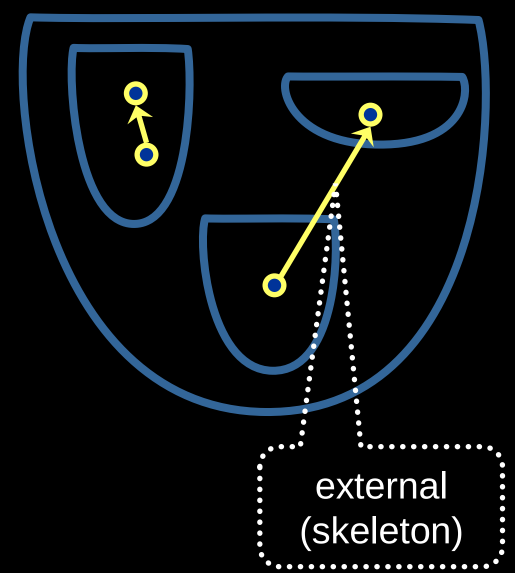


Family of frames $(F_i)_{i \in P}$
indexed by P
(Components)



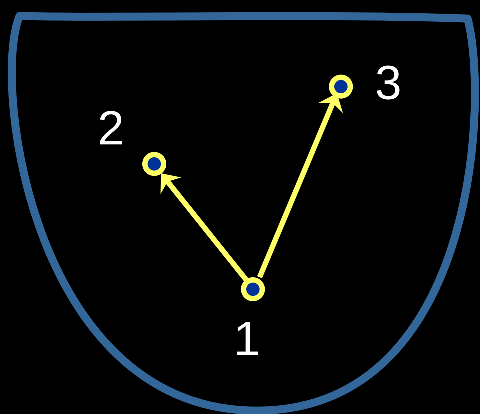
P-ordered sum of (F_i)

$\bigoplus_P F_i$

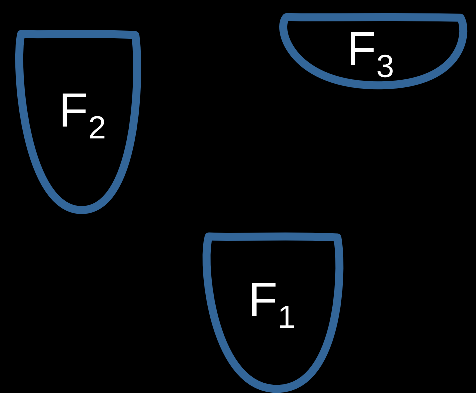


Partially ordered sums

A partial order P
(Skeleton)

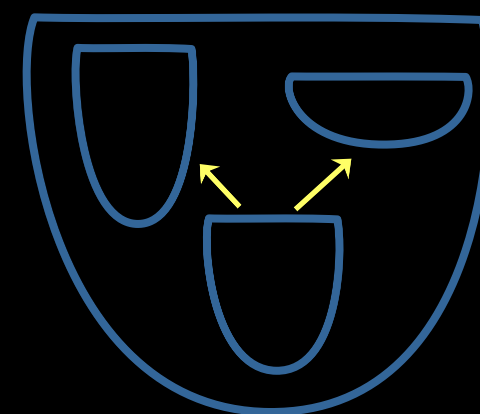


Family of frames $(F_i)_{i \in P}$
indexed by P
(Components)



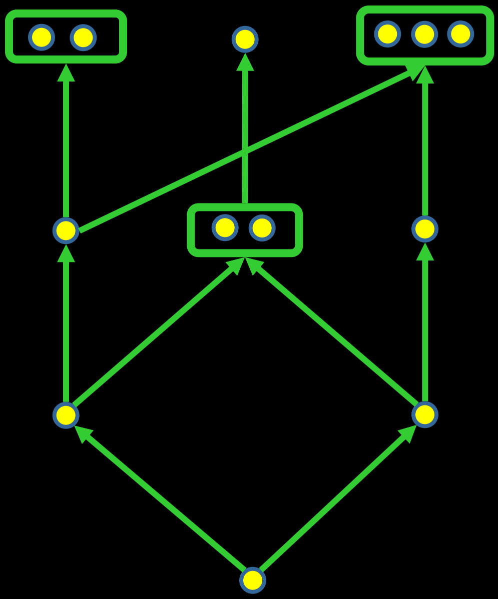
P-ordered sum of (F_i)

$\oplus_P F_i$



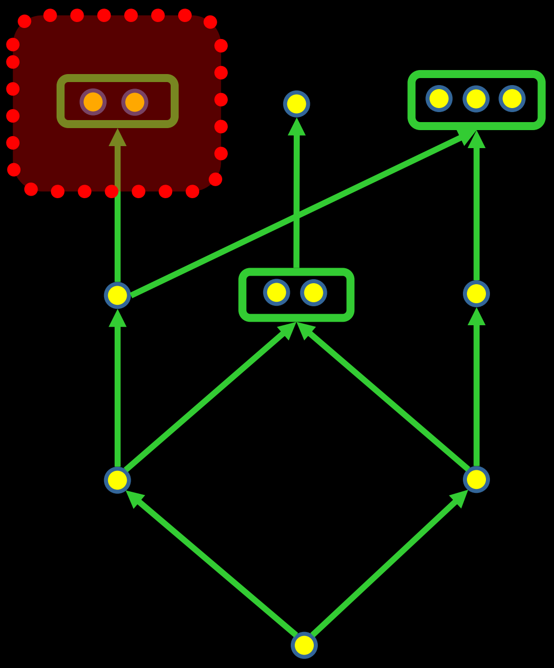
Quasiorders as topologies

- **Topology** is generated by **upwards closed sets**.



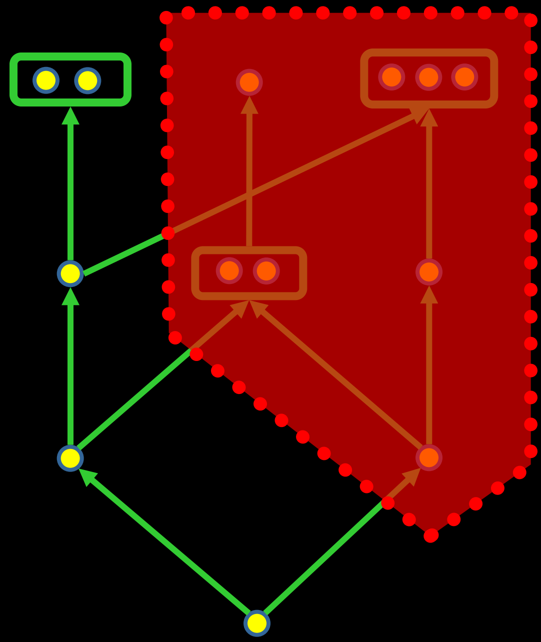
Quasiorders as topologies

- **Topology** is generated by **upwards closed sets**.



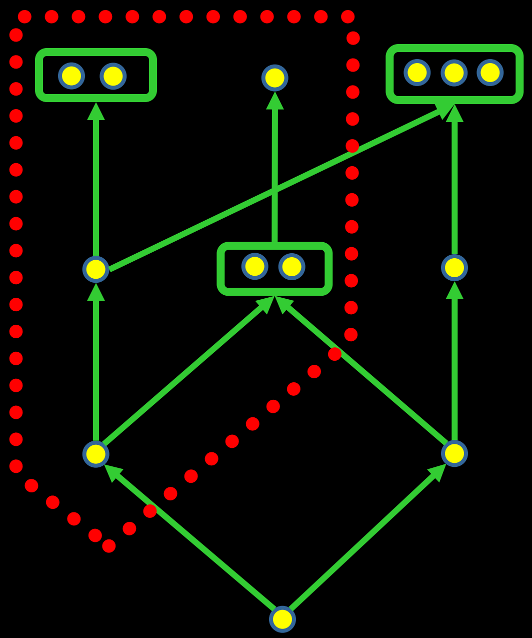
Quasiorders as topologies

- **Topology** is generated by **upwards closed sets**.

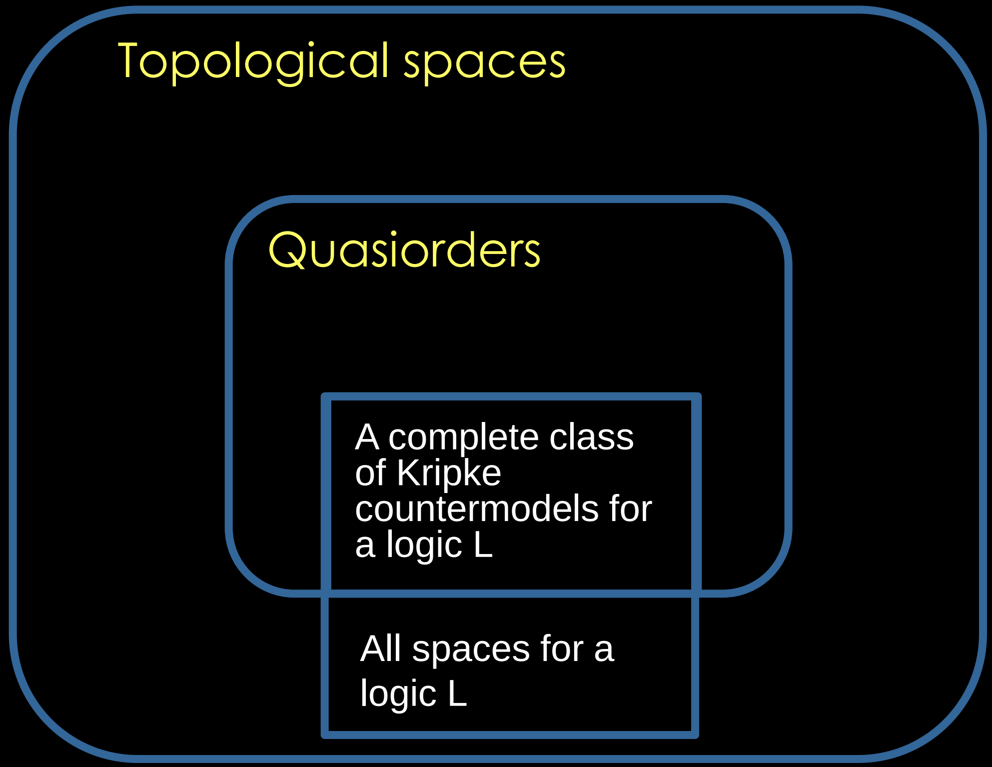


Quasiorders as topologies

- **Topology** is generated by **upwards closed sets**.



C-completeness via Kripke completeness

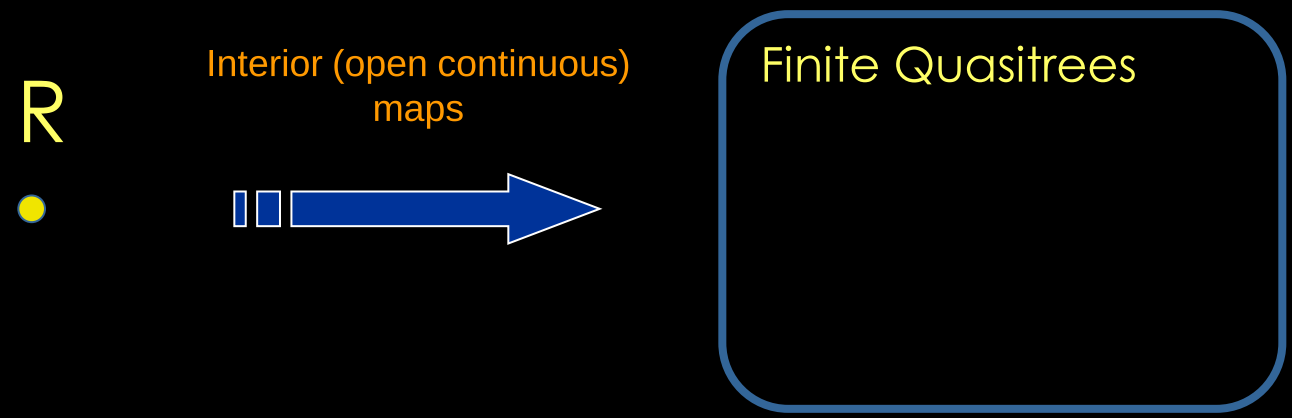


C-completeness via Kripke completeness

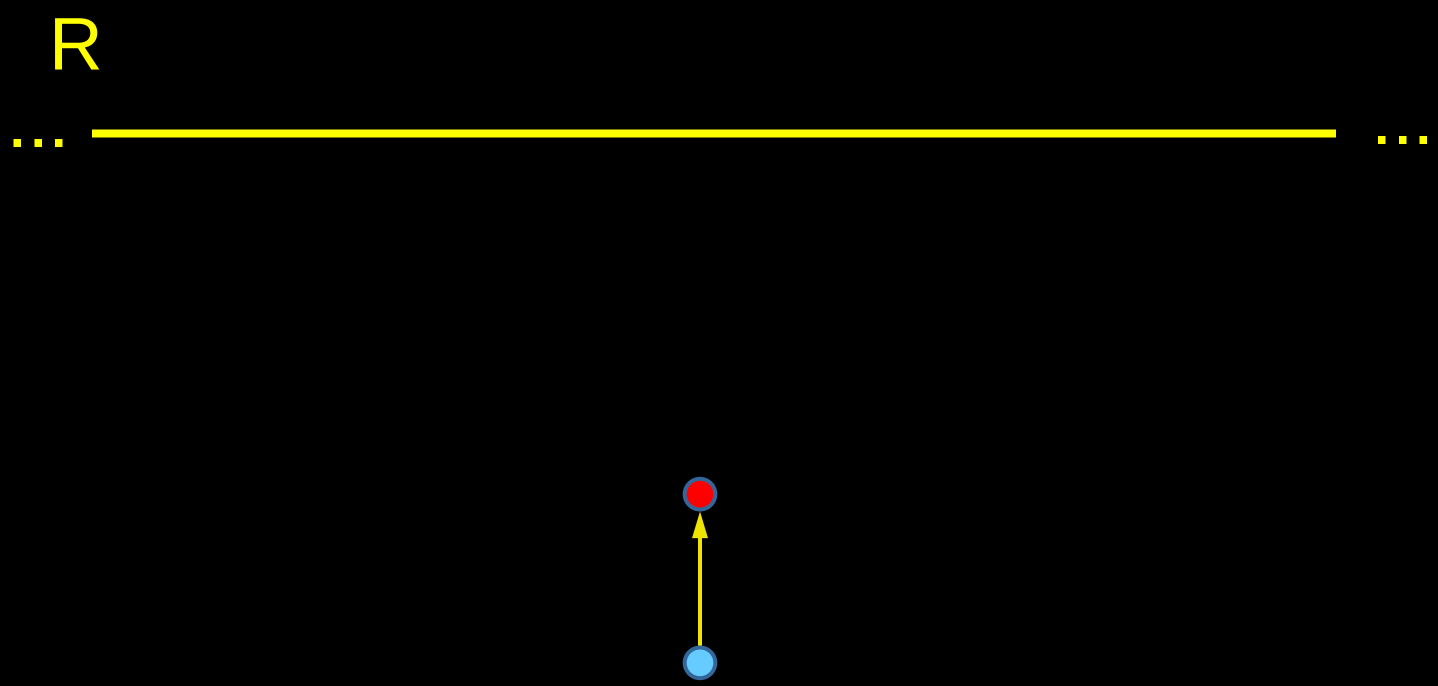
- Any Kripke complete logic above **S4** is topologically complete.
- There exist topologically complete logics that are not Kripke complete [Gerson 1975]
  - Even above **S4.Grz** [Shehtman 1998]
- Stronger completeness result by McKinsey and Tarski (1944):
  - **S4** is complete wrt any metric separable dense-in-itself space.
  - In particular, $\mathbf{Log_C(R) = S4}$.

$\mathrm{Log}_C(\mathbb{R})$ = S4: Insights

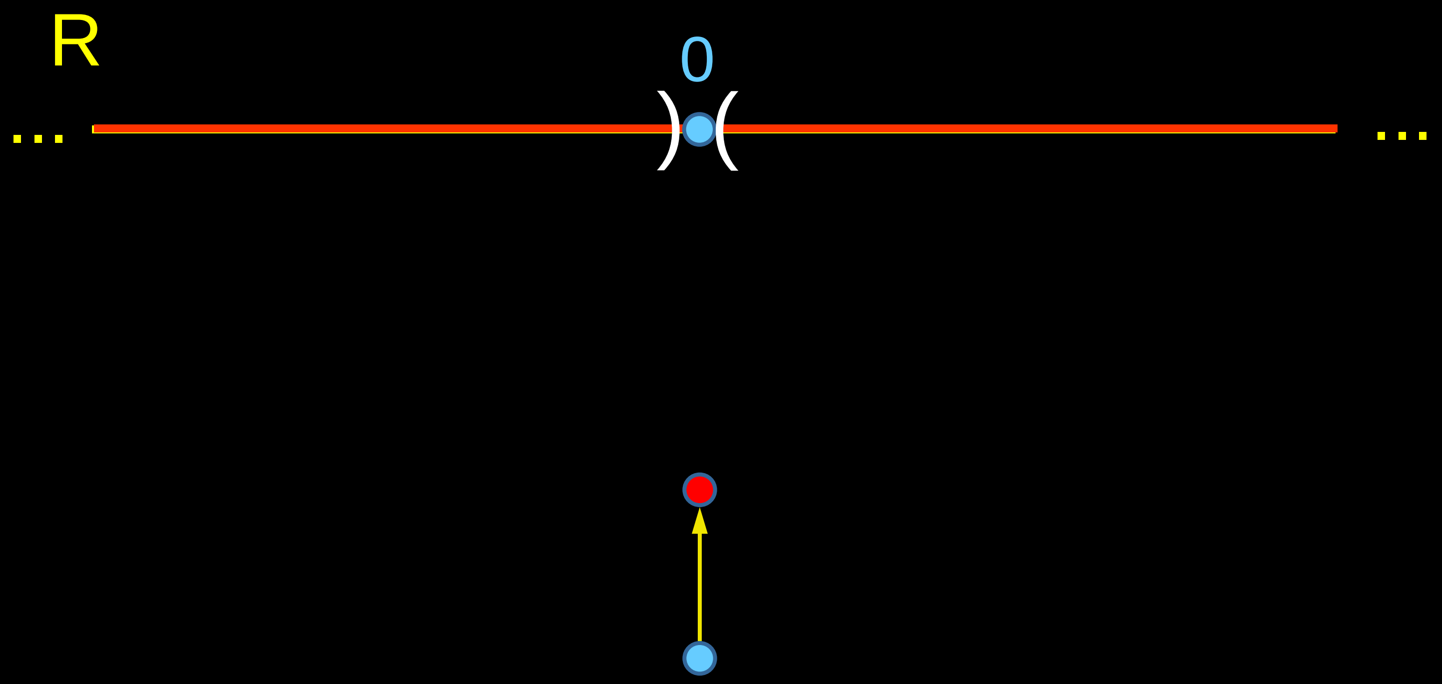
Following: G. Bezhanishvili, M. Gehrke. *Completeness of S4 with respect to the real line: revisited*, Annals of Pure and Applied Logic, 131 (2005), pp. 287—301.



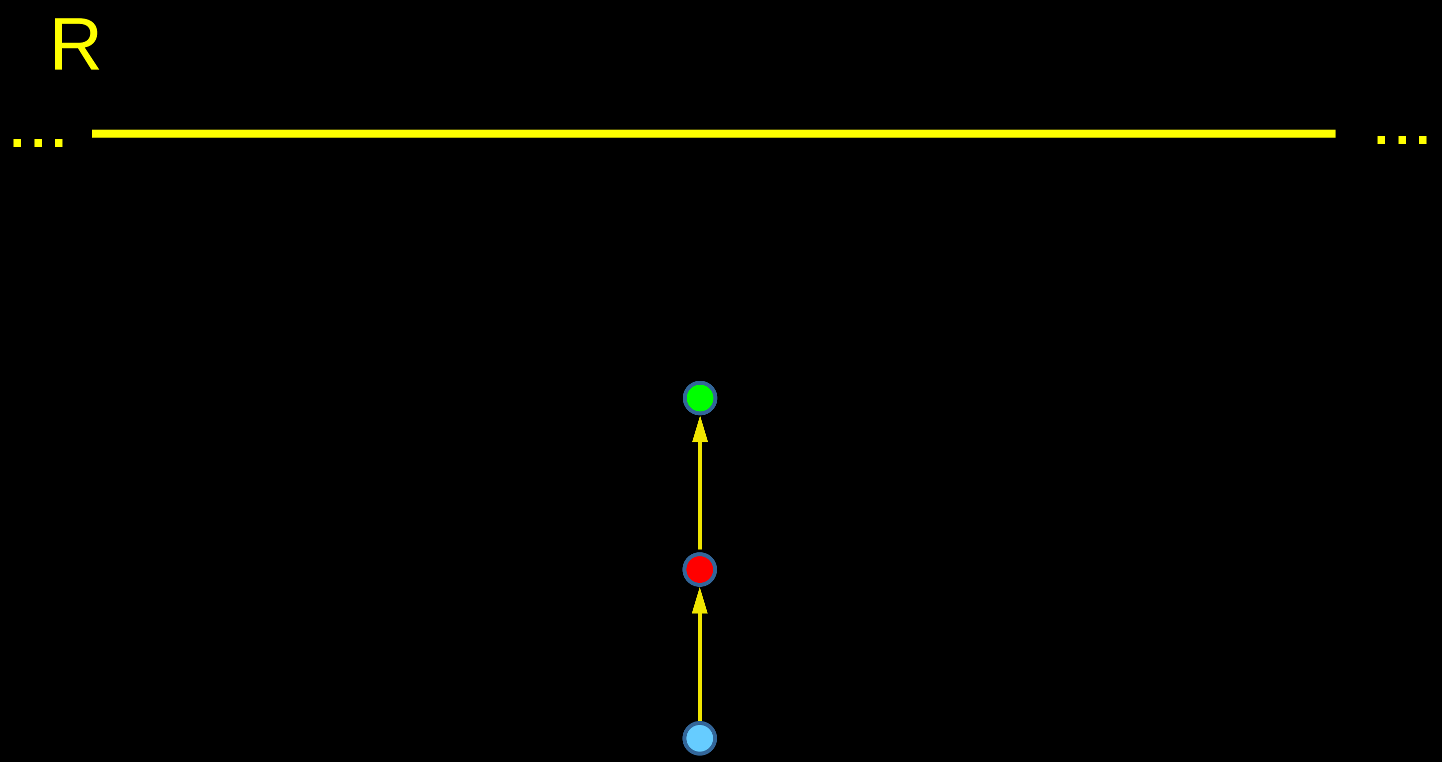
Mapping R onto finite connected quasiorders



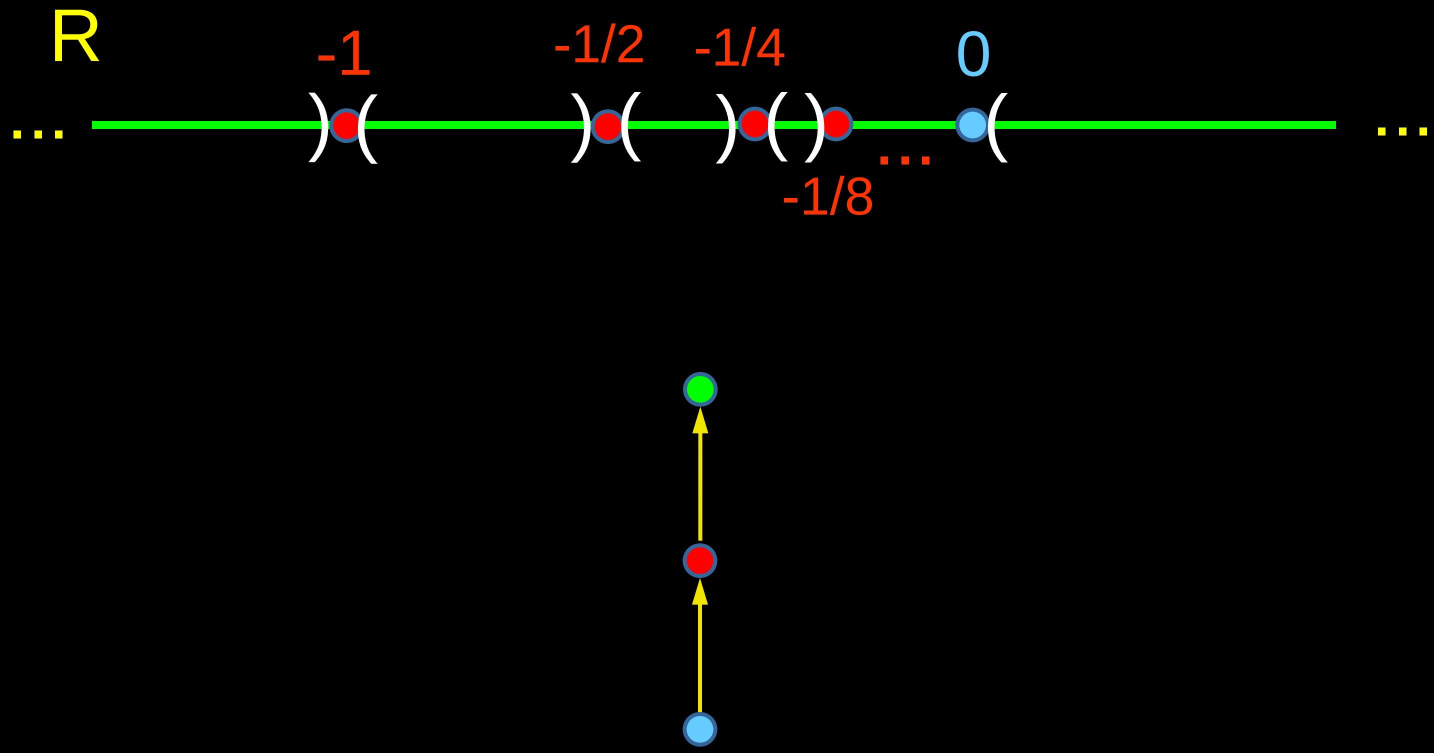
Mapping R onto finite connected quasiorders



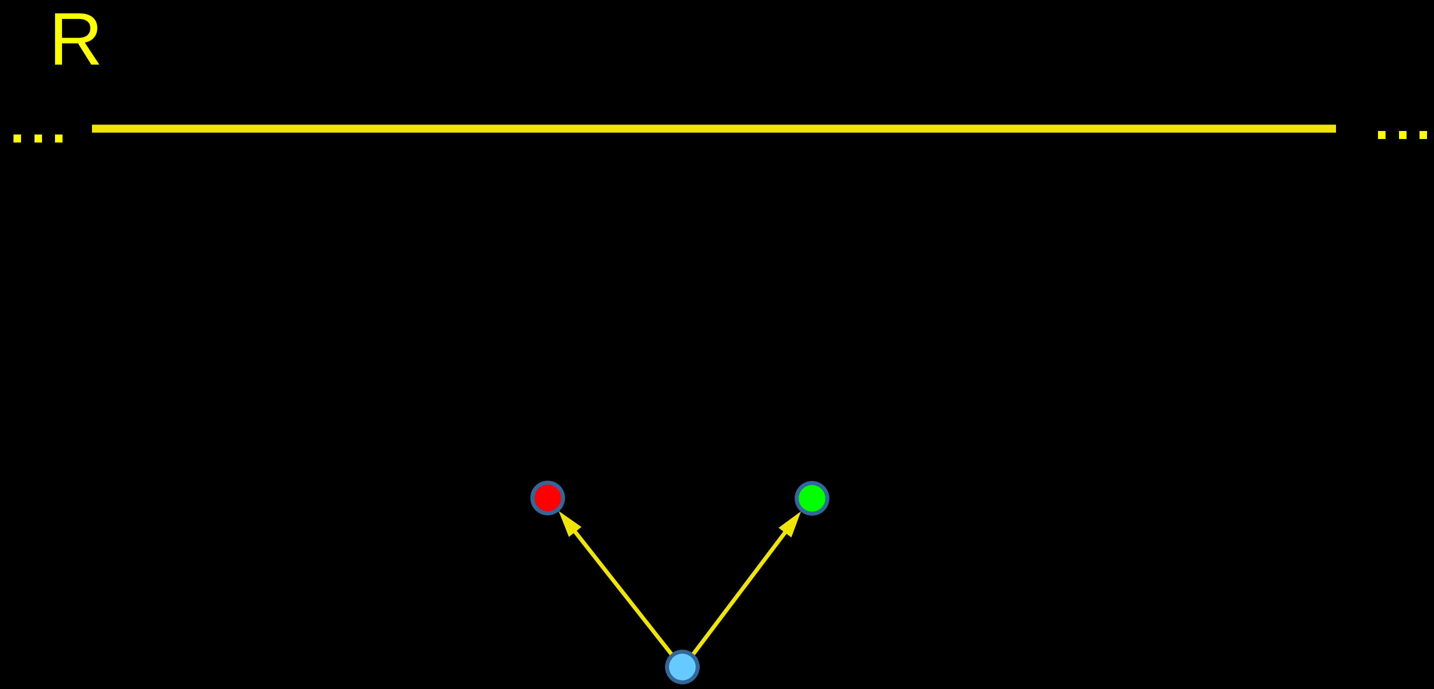
Mapping R onto finite connected quasiorders



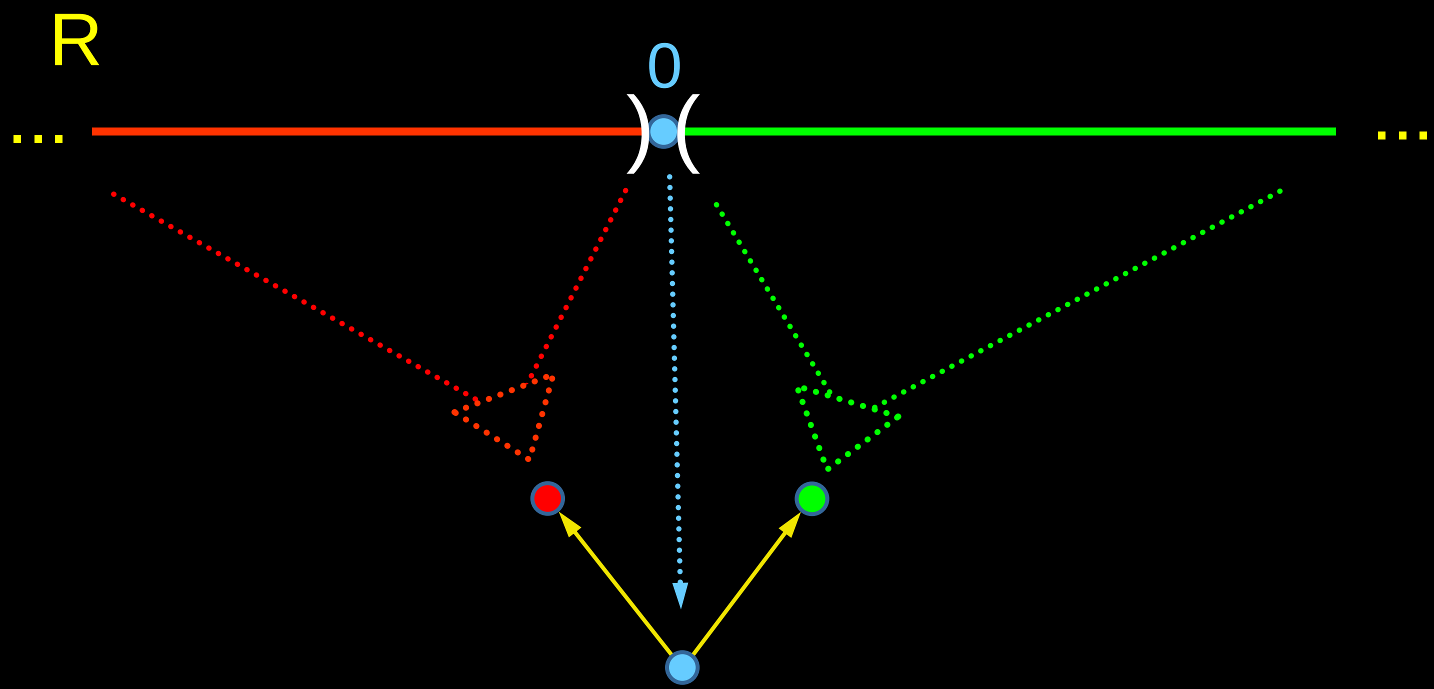
Mapping R onto finite connected quasiorders



Mapping R onto finite connected quasiorders



Mapping R onto finite connected quasiorders



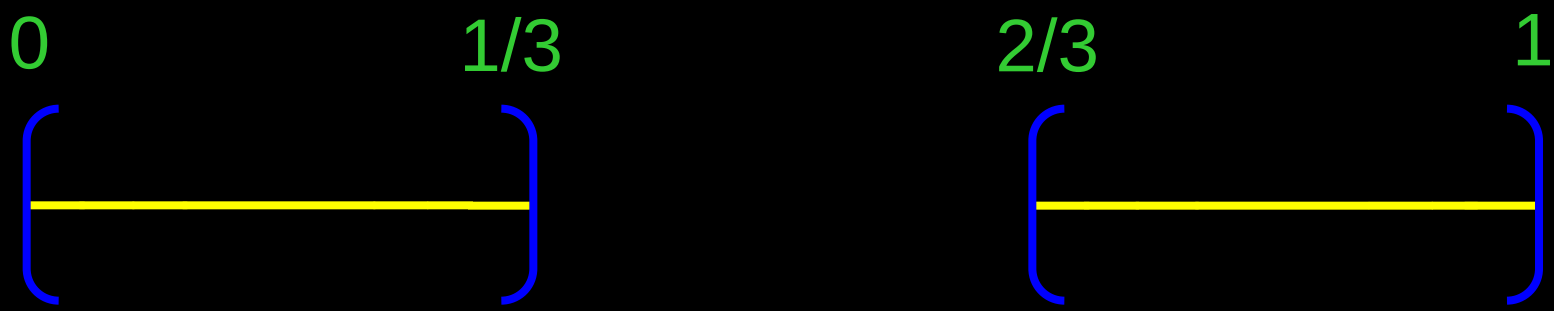
Problems: What if clusters are present?

What if the 3-fork is taken instead of the 2-fork??

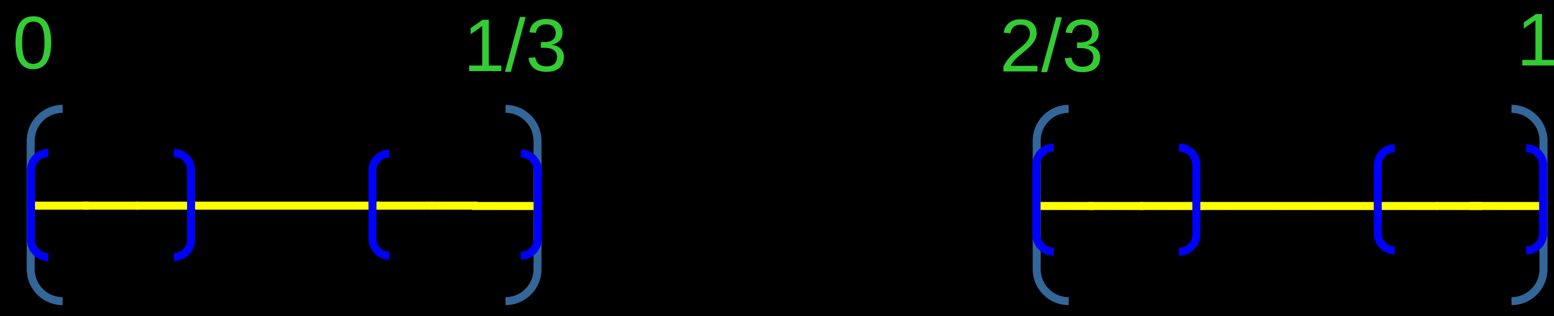
Cantor Space

Cantor Space

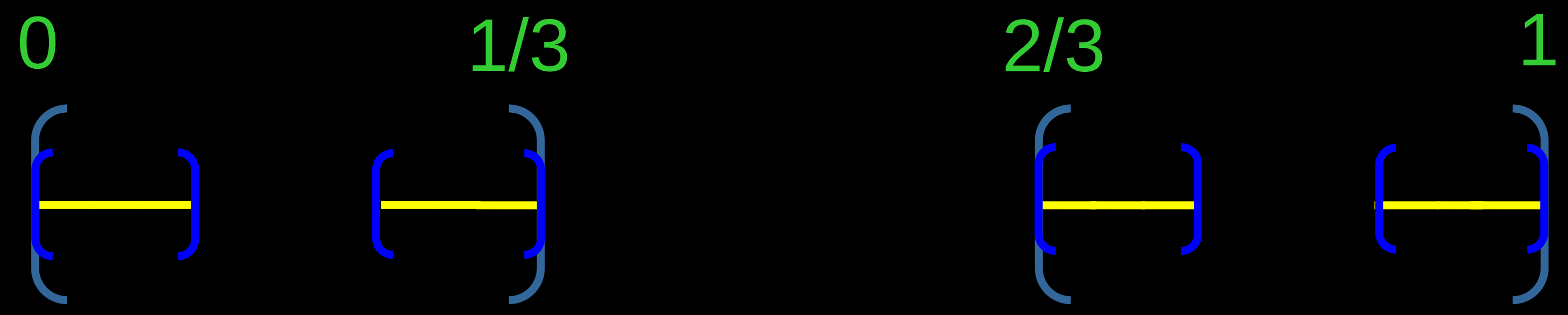
Cantor Space



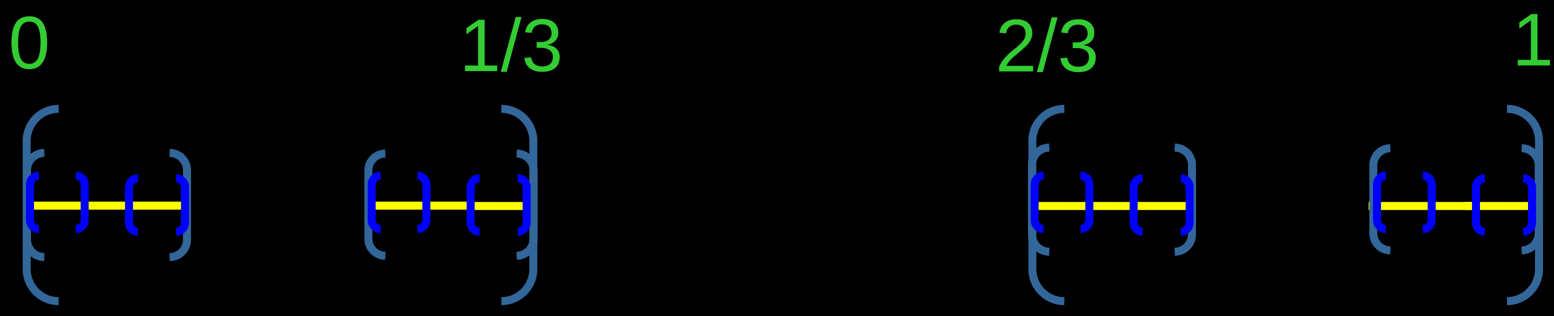
Cantor Space



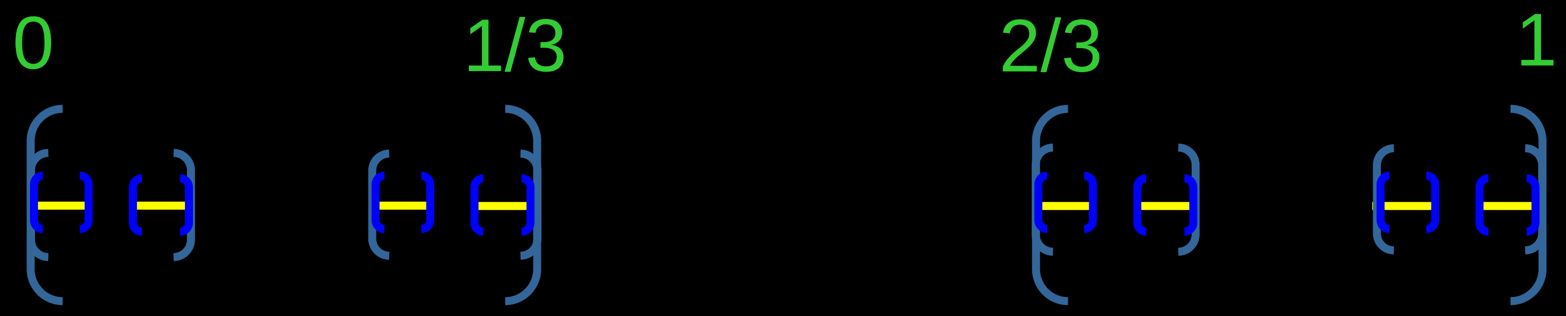
Cantor Space



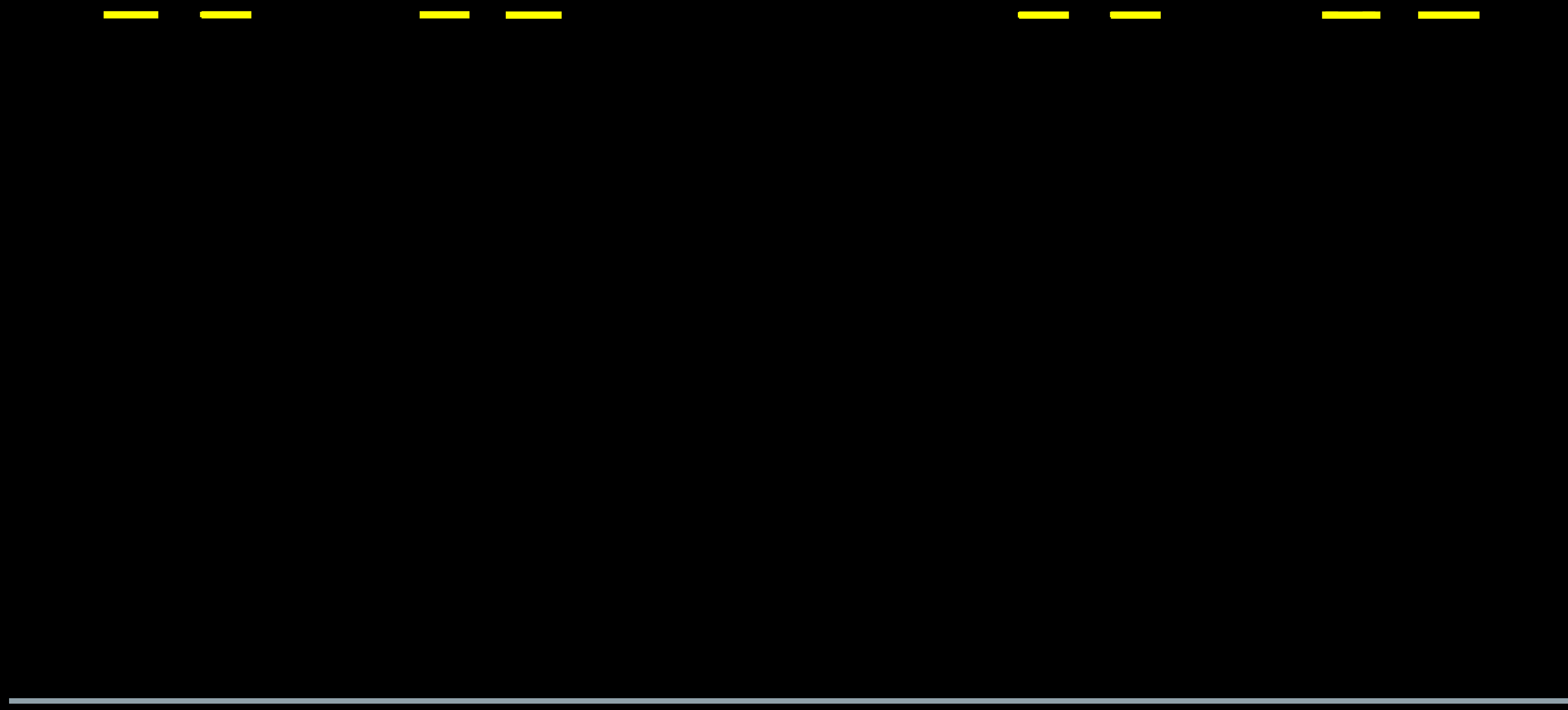
Cantor Space



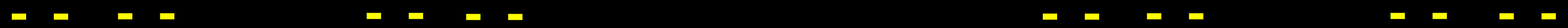
Cantor Space



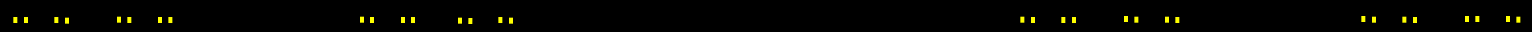
Cantor Space



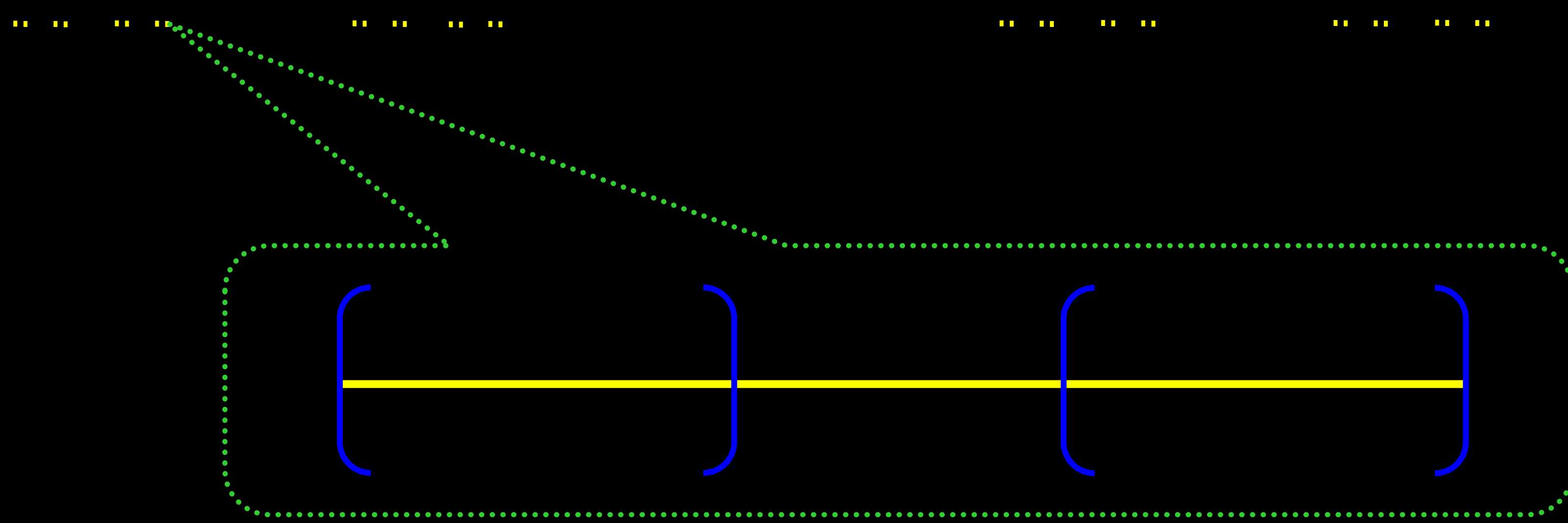
Cantor Space



Cantor Space

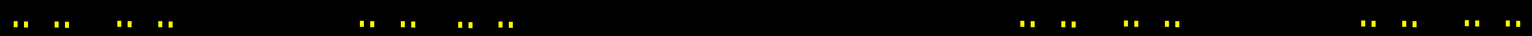


Cantor Space



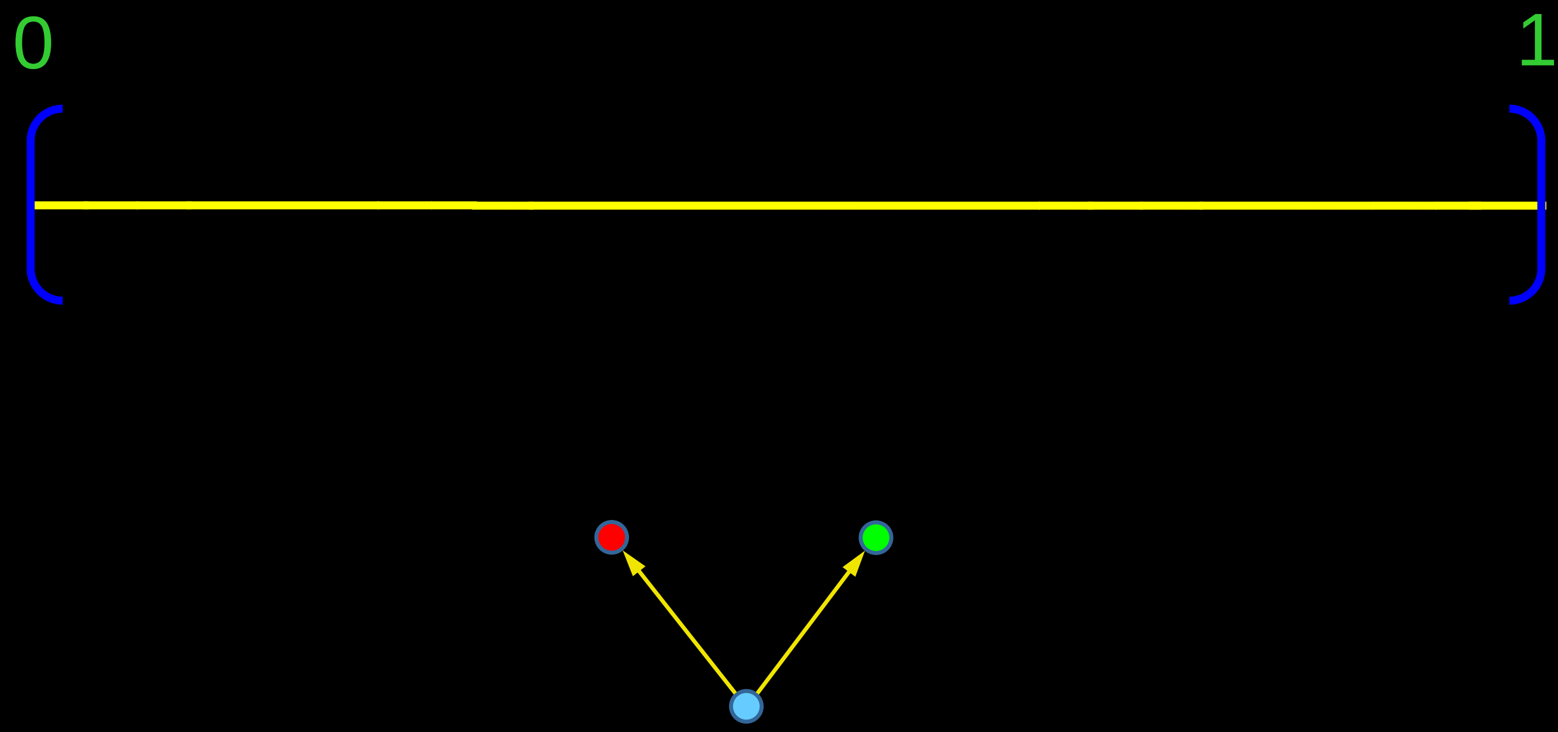
It’s fractal-like

Cantor Space

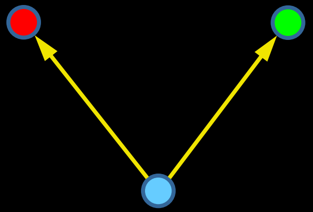


In the limit – Cantor set.

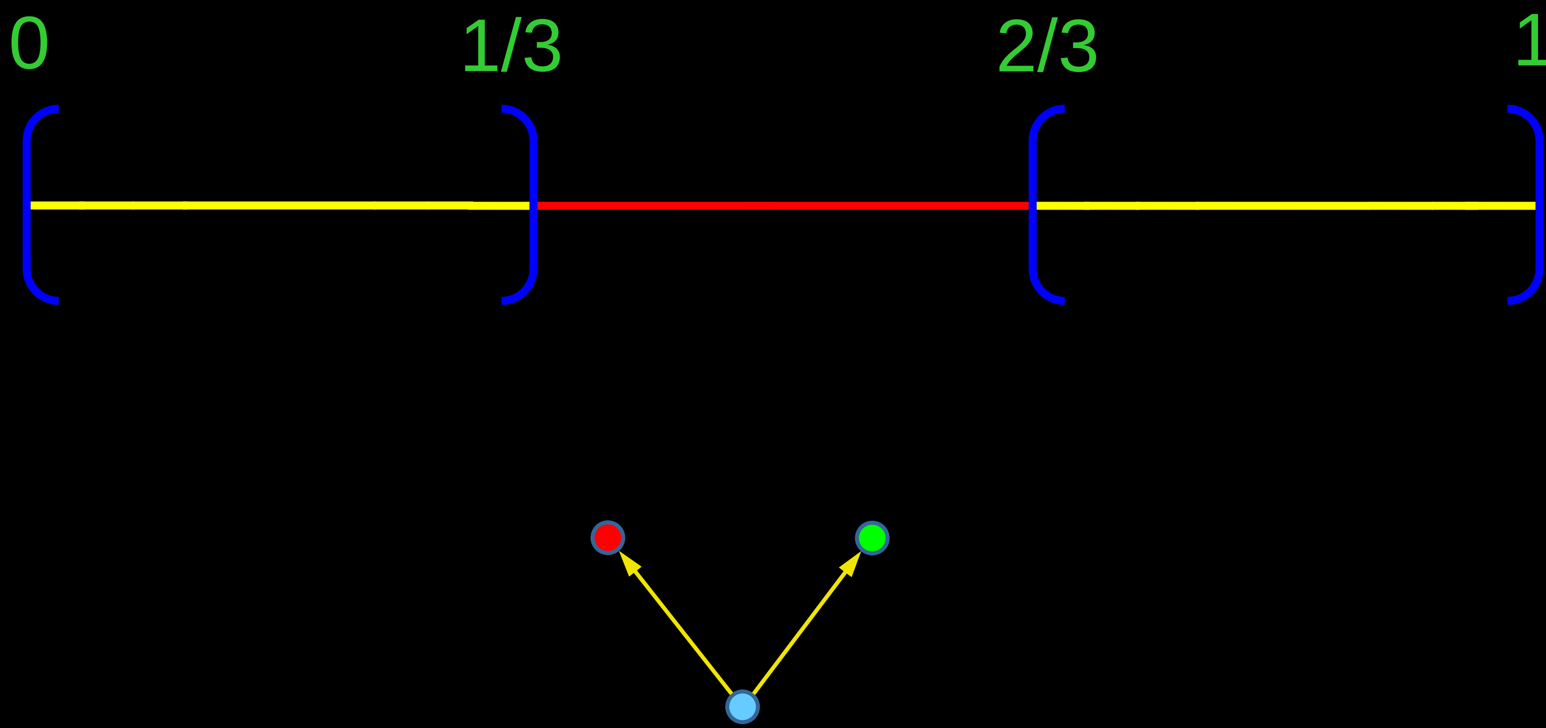
(0,1) mapped onto the fork



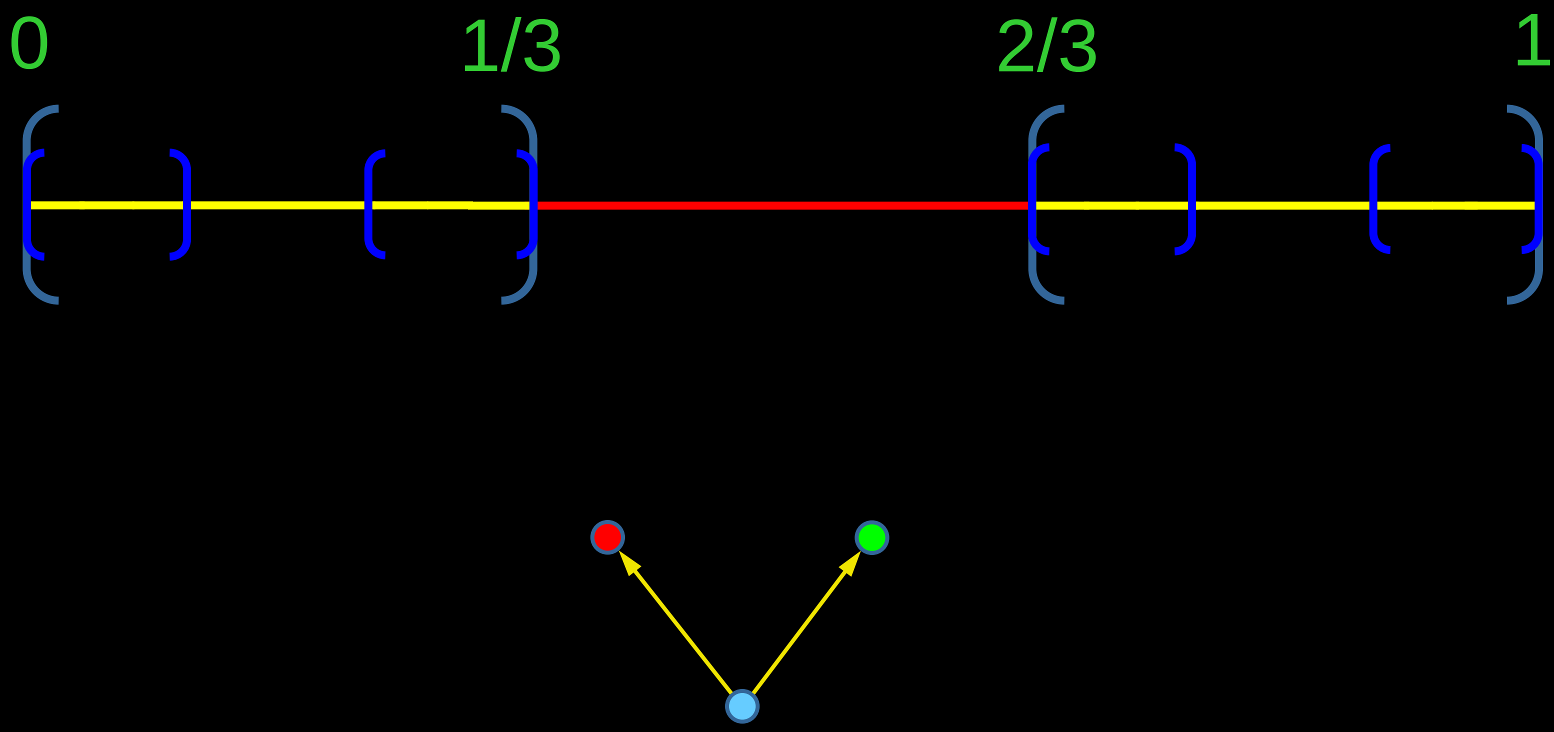
(0,1) mapped onto the fork



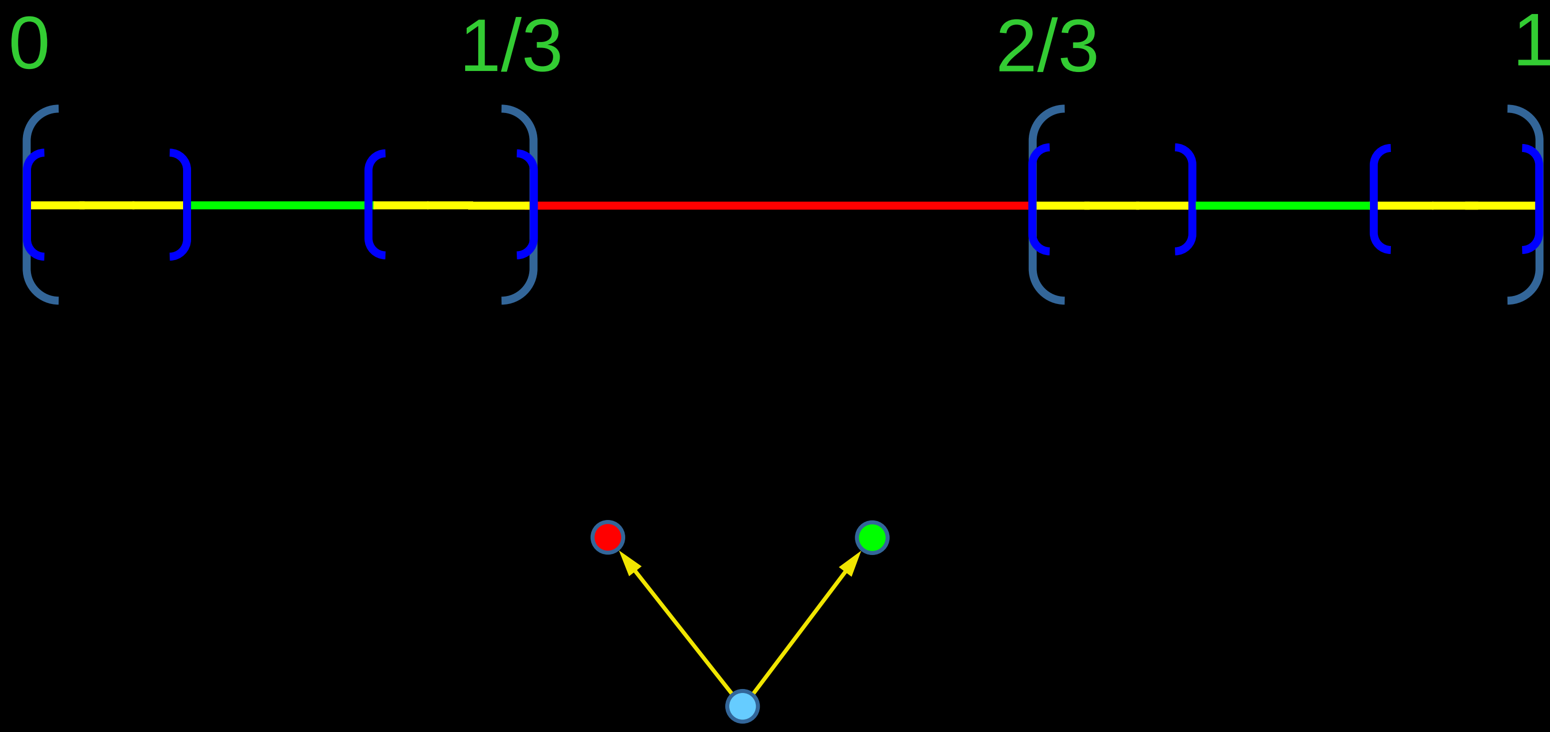
(0,1) mapped onto the fork



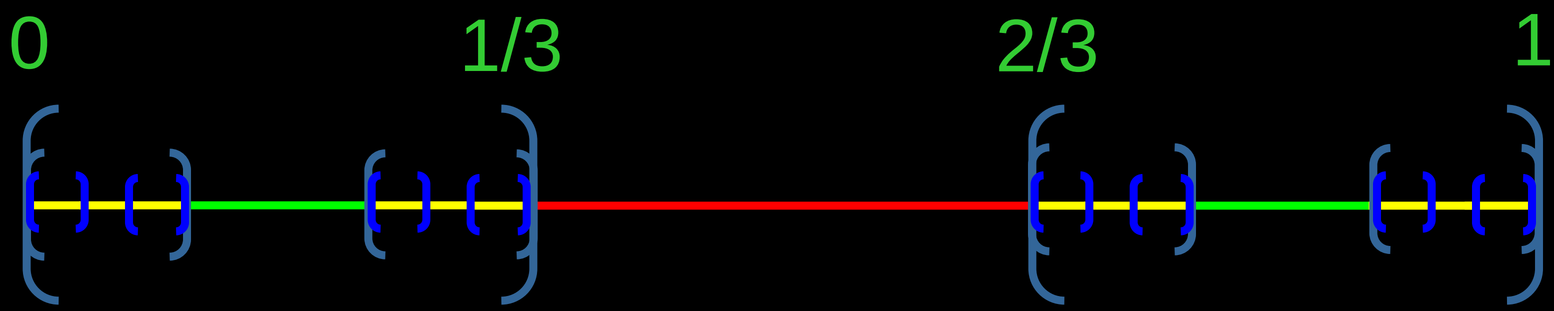
(0,1) mapped onto the fork

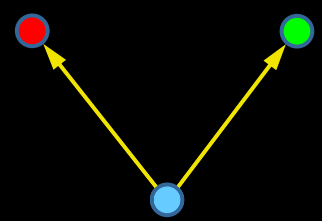


(0,1) mapped onto the fork

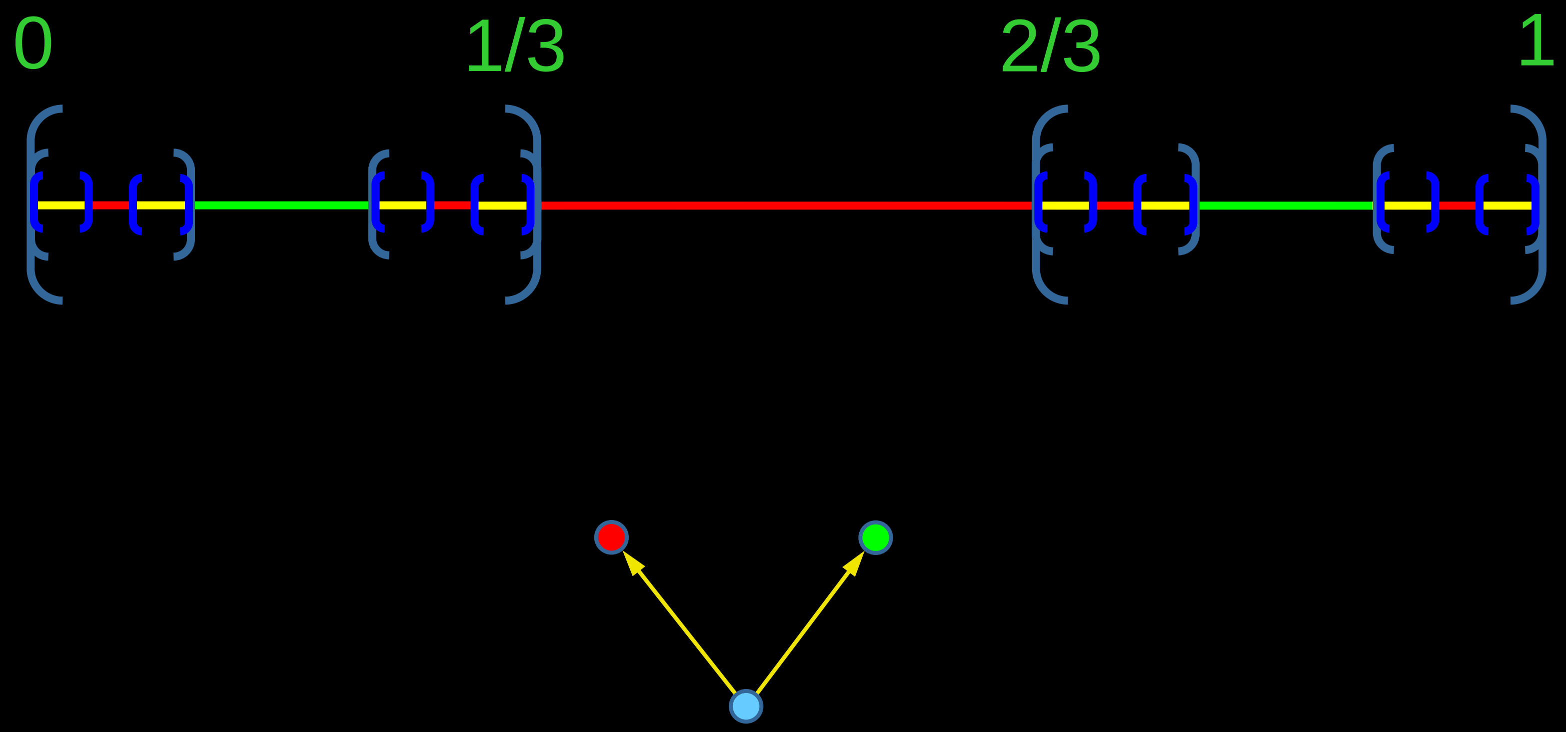


(0,1) mapped onto the fork

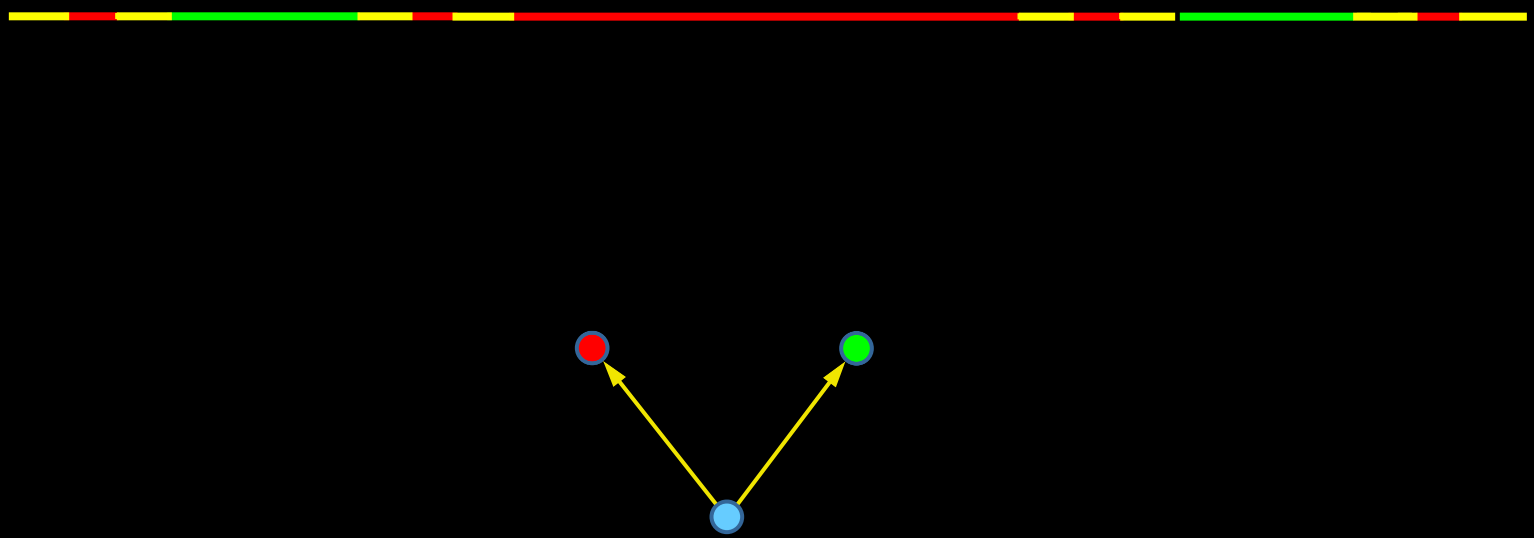




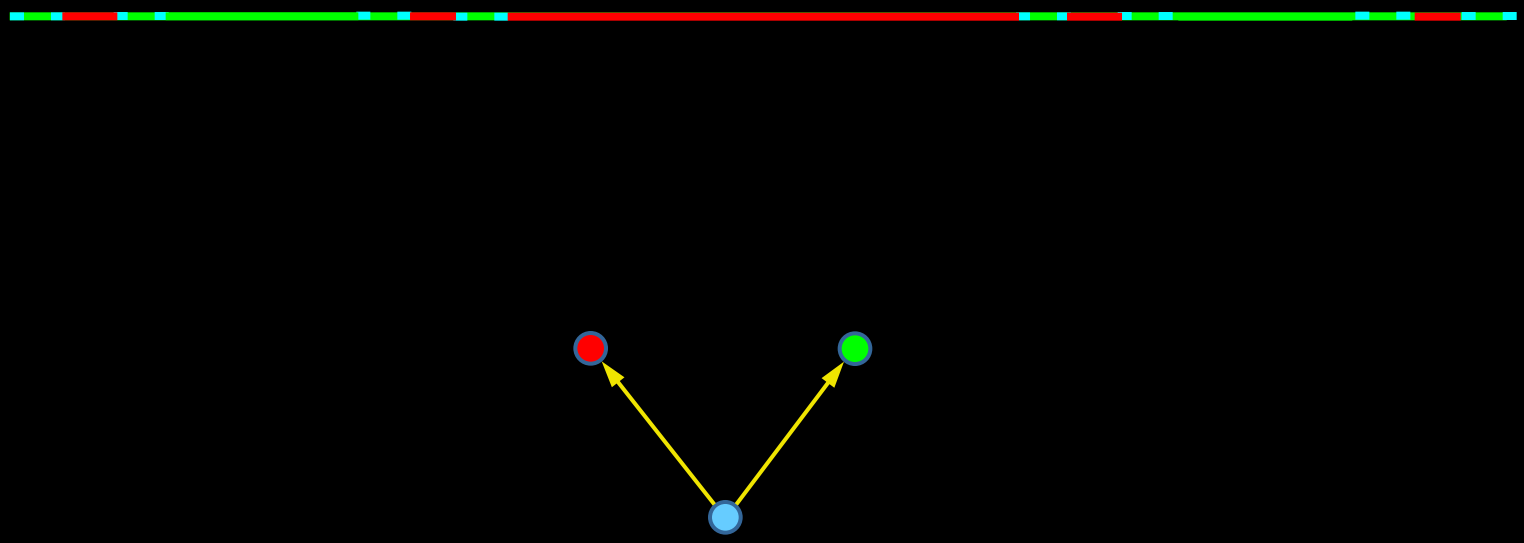
(0,1) mapped onto the fork



(0,1) mapped onto the fork



(0,1) mapped onto the fork



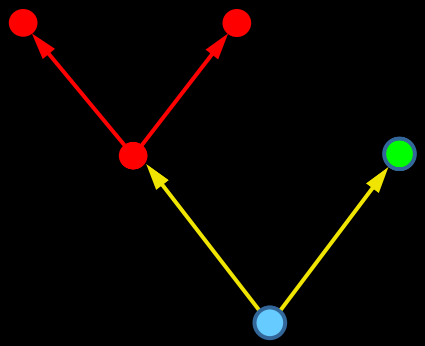
Problems solved

- It is straightforward to generalize this procedure to a 3-fork and, indeed, to any n-fork.

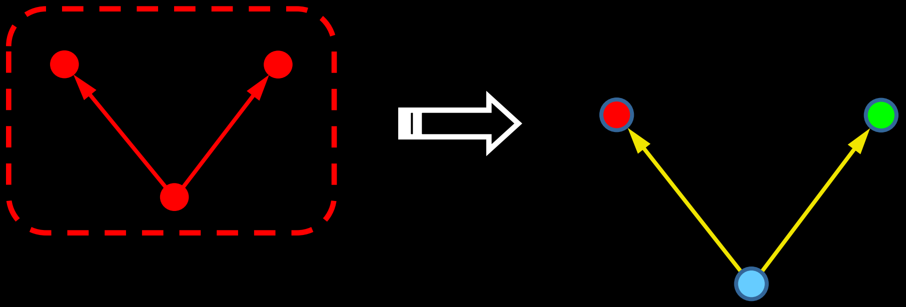
- Clusters are no problem:
  - the Cantor set can be decomposed into infinitely many disjoint subsets which are dense in it.
  - Similarly, an open interval (and thus, any open subset of the reals) can be decomposed into infinitely many disjoint, dense in it subsets.

- How about increasing the depth?

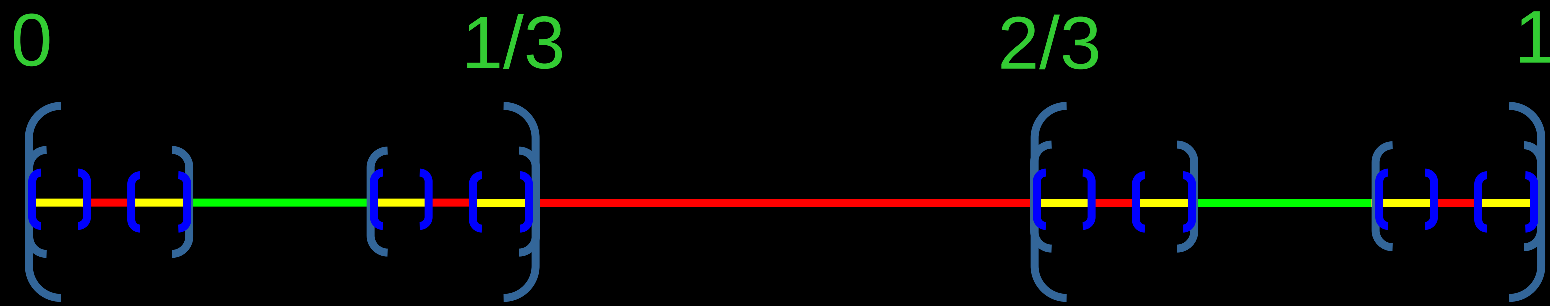
Iterating the procedure

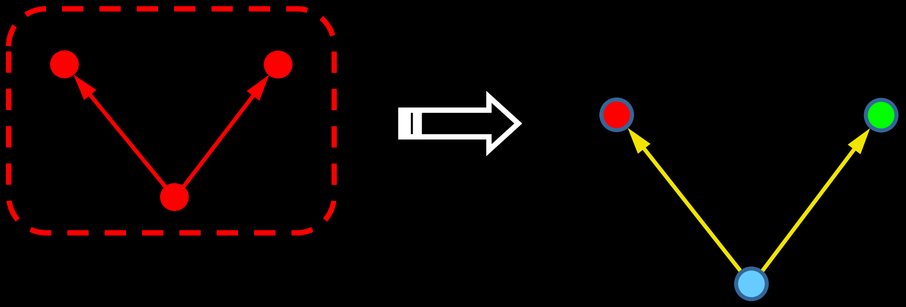


Iterating the procedure

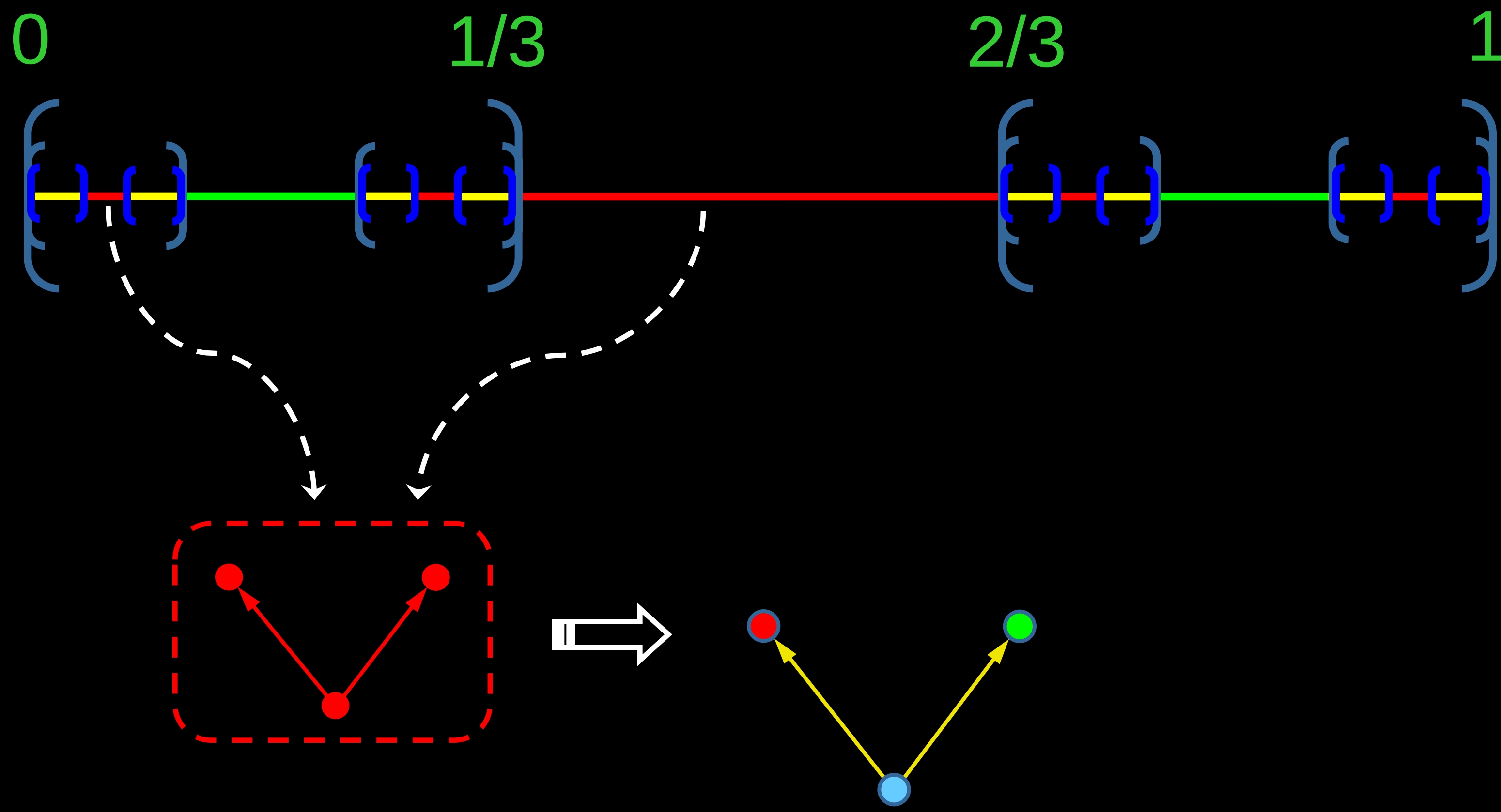


Iterating the procedure

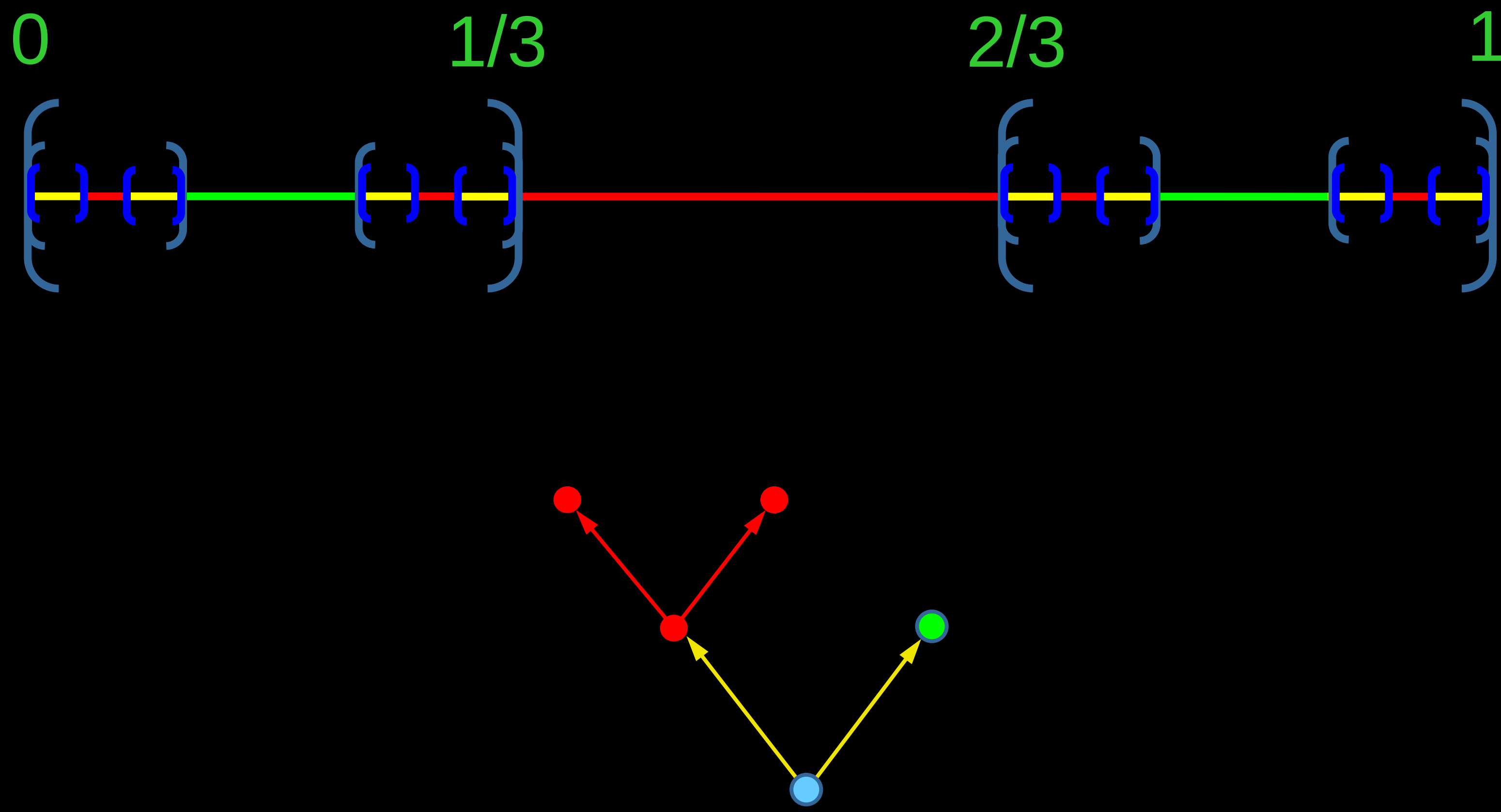




Iterating the procedure



Iterating the procedure



Connected logics

- What more can a modal logic say about the topology of **R** in C-semantics?

- Consider the closure algebra $\mathbf{R}^+ = (\wp(\mathbf{R}), \mathbf{C})$. Which modal logics can be generated by subalgebras of $\mathbf{R}^+$?

Answer: Any connected modal logic above S4 with fmp.

[G. Bezhanishvili, Gabelaia 2010]

- More questions like this – e.g. what about homomorphic images? What about logics without fmp?

- Recently Philip Kremer has shown strong completeness of **S4** wrt the real line!

Story of Delia

- d-completeness doesn’t straightforwardly follow from Kripke completeness.
- Incompleteness theorems.
- Extensions allow automatic transfer of d-completeness of **GL**.
- Completeness of **GL** wrt ordinals.
- Completeness of **wK4**
- Completeness of **K4.Grz**
- Some other recent results.

Story of Delia (d-semantics)

Axioms for derivation

$\mathbf{d}\varnothing = \varnothing$

$\mathbf{d}(A \cup B) = \mathbf{d}A \cup \mathbf{d}B$

$\mathbf{dd}A \subseteq A \cup \mathbf{d}A$

Axioms of wK4

$\Diamond 0 = 0$

$\Diamond(p \vee q) = \Diamond p \vee \Diamond q$

$\Diamond\Diamond p \leq p \vee \Diamond p$

wK4 – weak **K4**

wK4-frames are weakly transitive.

Tbilisi-Munich-Marseille is a transit flight,

Tbilisi-Munich-Tbilisi is not really a transit flight.

Story of Delia (d-semantics)

Axioms for derivation

$\mathbf{d}\varnothing = \varnothing$

$\mathbf{d}(A \cup B) = \mathbf{d}A \cup \mathbf{d}B$

$\mathbf{dd}A \subseteq A \cup \mathbf{d}A$

Axioms of **wK4**

$\Diamond 0 = 0$

$\Diamond(p \vee q) = \Diamond p \vee \Diamond q$

$\Diamond\Diamond p \leq p \vee \Diamond p$

wK4 – weak **K4**

wK4-frames are weakly transitive.

Story of Delia (d-semantics)

Axioms for derivation

$\mathbf{d}\varnothing = \varnothing$

$\mathbf{d}(A \cup B) = \mathbf{d}A \cup \mathbf{d}B$

$\mathbf{dd}A \subseteq A \cup \mathbf{d}A$

Axioms of **wK4**

$\Diamond 0 = 0$

$\Diamond(p \vee q) = \Diamond p \vee \Diamond q$

$\Diamond\Diamond p \leq p \vee \Diamond p$

wK4 – weak **K4**

wK4-frames are weakly transitive.

Tbilisi-Munich-Marseille is a transit flight,

Story of Delia (d-semantics)

Axioms for derivation

$\mathbf{d}\varnothing = \varnothing$

$\mathbf{d}(A \cup B) = \mathbf{d}A \cup \mathbf{d}B$

$\mathbf{dd}A \subseteq A \cup \mathbf{d}A$

Axioms of wK4

$\Diamond 0 = 0$

$\Diamond(p \vee q) = \Diamond p \vee \Diamond q$

$\Diamond\Diamond p \leq p \vee \Diamond p$

wK4 – weak **K4**

wK4-frames are weakly transitive.

Tbilisi-Munich-Marseille is a transit flight,

Tbilisi-Munich-Tbilisi is not really a transit flight.

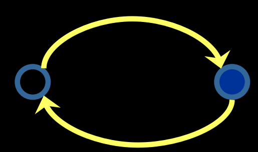
wK4-frames

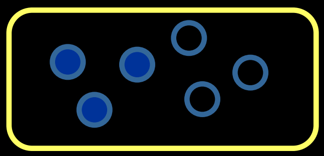
$$\forall xyz(xRy \wedge yRz \wedge x \neq z \rightarrow xRz)$$

- Weak quasiorders (delete any reflexive arrows in a quasiorder).

- Partially ordered sums of weak clusters

clusters with irreflexive points:





Delia is capricious (d-incompleteness)

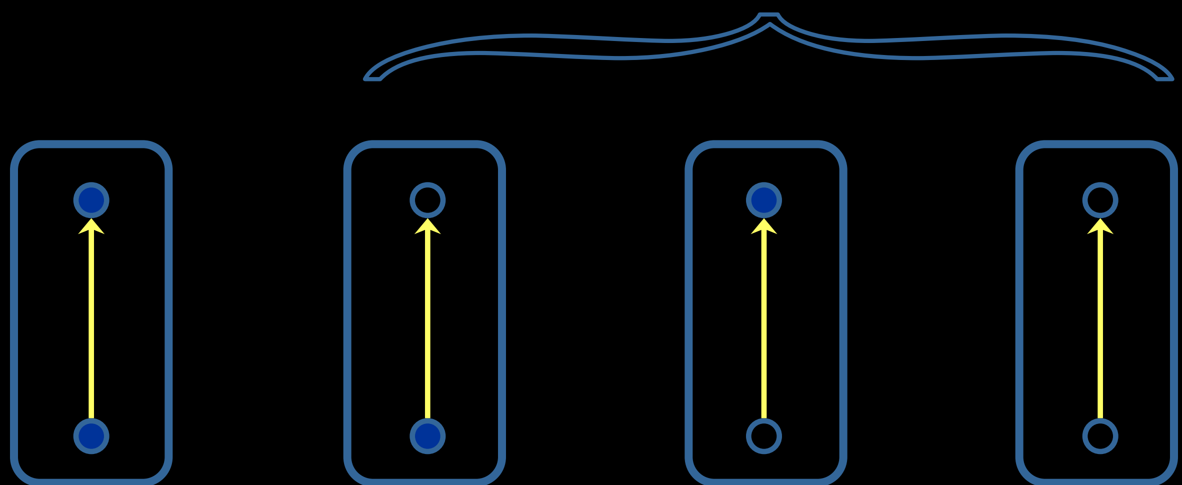
- **S4** is an extension of **wK4** (add reflexivity axiom)
- **S4** has no d-models whatsoever!!
- **S4** is incomplete in d-semantics.

Reason: The relation induced by **d** is always irreflexive:

$$x \notin \mathbf{d}[x]$$

Caprice exemplified

Topological ◊ non-topological ◊



How capricious is Delia?

Definition: Weak partial orders are obtained from partial orders by deleting (some) reflexive arrows.

- For any class of weak partial orders of depth $\leq n$, if there is a root-reflexive frame in this class with the depth exactly n, then the logic of this class is d-incomplete.

Gracious Delia

- Kripke completeness implies d-completeness for extensions of **GL**.

- **GL** is the logic of finite irreflexive trees.

- In d-semantics, **GL** defines the class of scattered topologies

  [Esakia 1981]

- **GL** is d-complete wrt to the class of ordinals.

- **GL** is the d-logic of ω^ω .

  [Abashidze 1988, Blass 1990]

Finite irreflexive trees recursively

- Irreflexive point is an i-tree.
- Irreflexive n-fork is an i-tree.
- Tree sum of i-trees is an i-tree.

Finite irreflexive trees recursively

- Irreflexive point is an i-tree.
- Irreflexive n-fork is an i-tree.
- Tree sum of i-trees is an i-tree.

What is a tree sum?

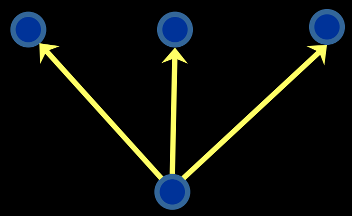
Finite irreflexive trees recursively

- Irreflexive point is an i-tree.
- Irreflexive n-fork is an i-tree.
- Tree sum of i-trees is an i-tree.

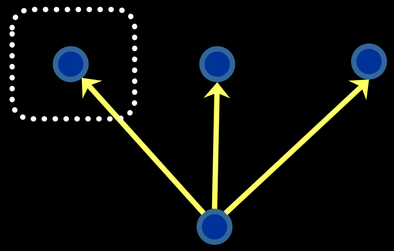
What is a tree sum?

Similar to the ordered sum, but only leaves of a tree can be “blown up” (e.g. substituted by other trees).

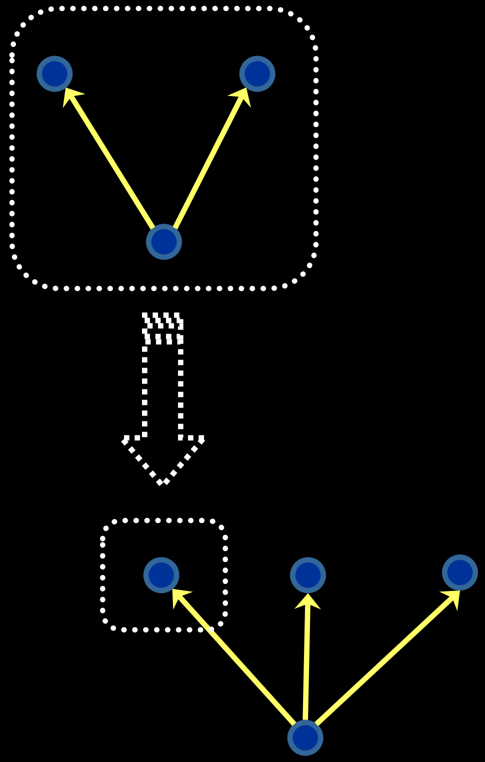
Tree sum exemplified



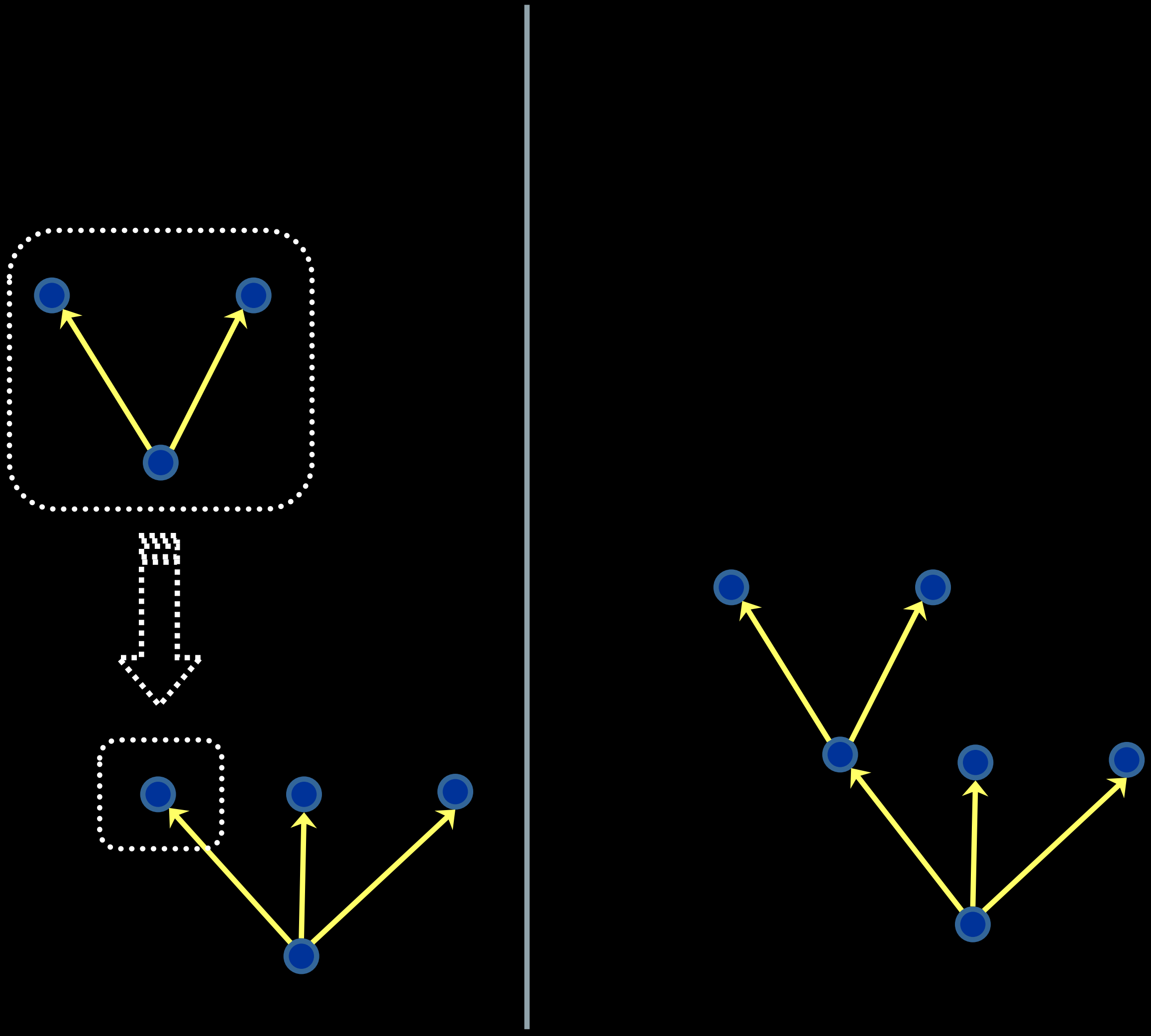
Tree sum exemplified



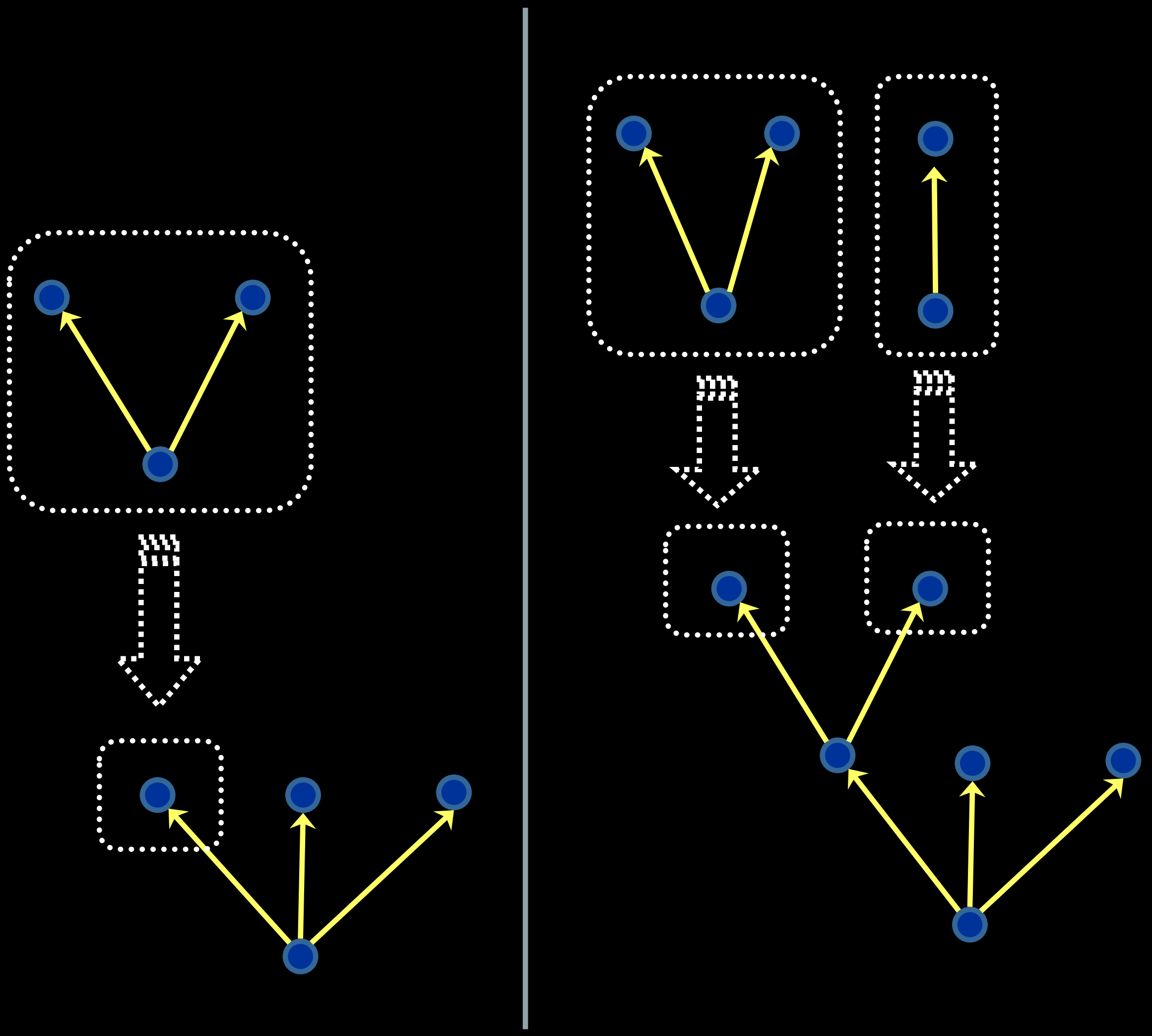
Tree sum exemplified



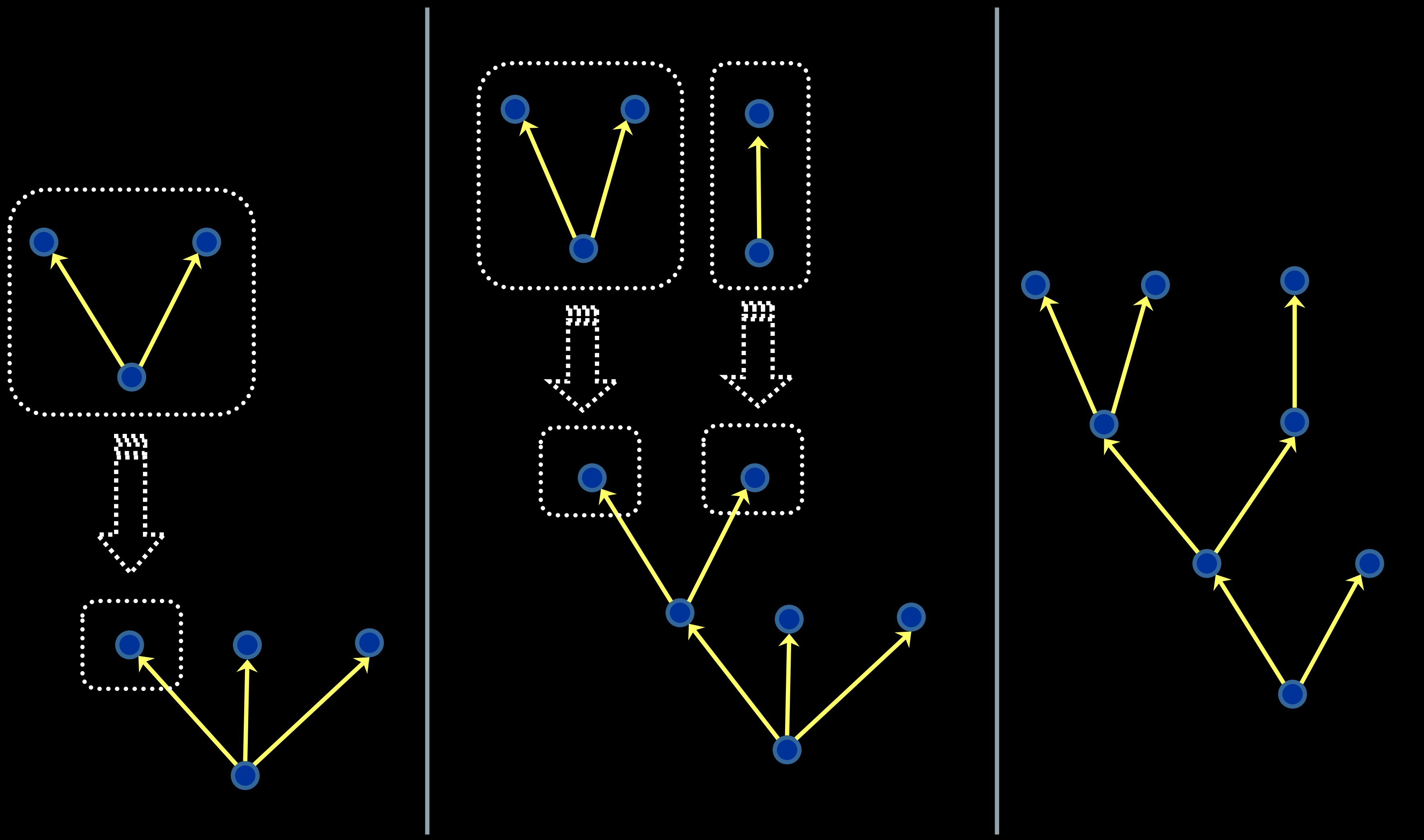
Tree sum exemplified



Tree sum exemplified



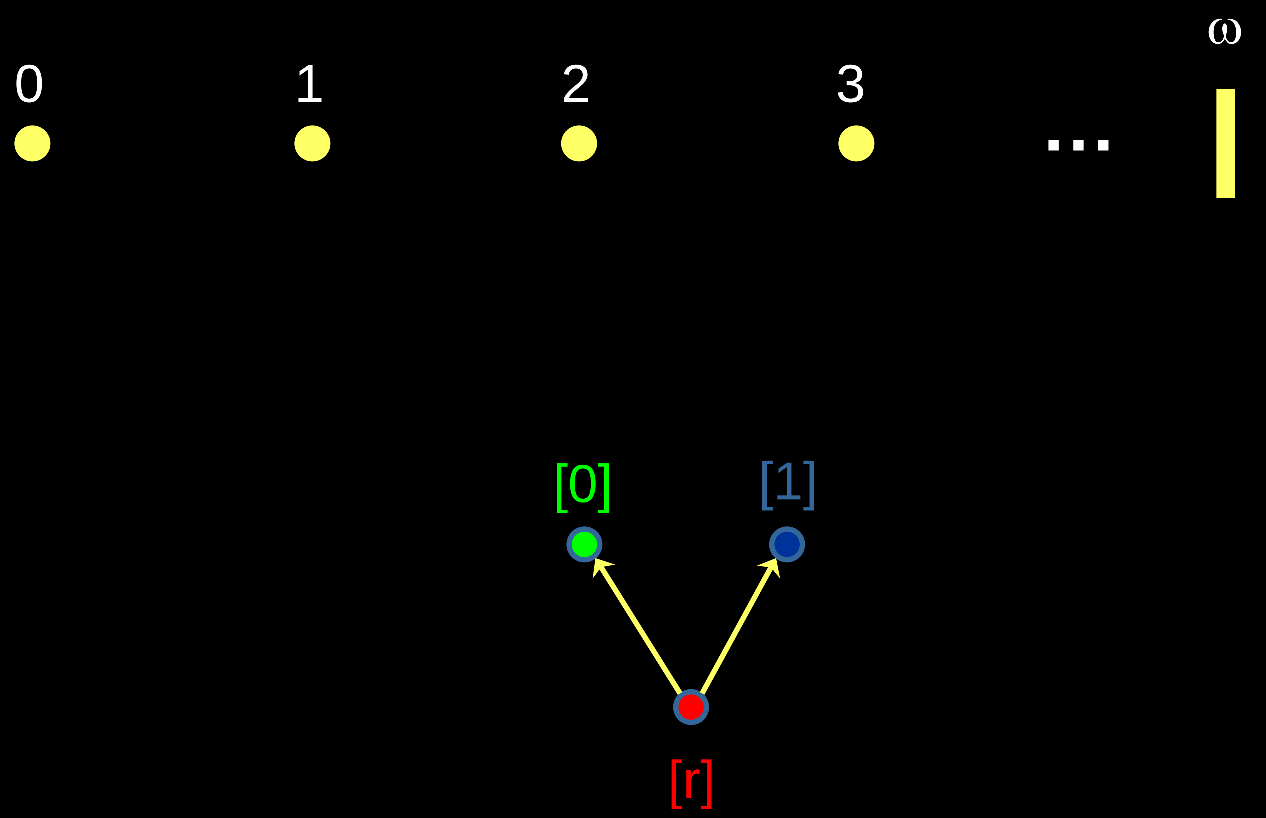
Tree sum exemplified



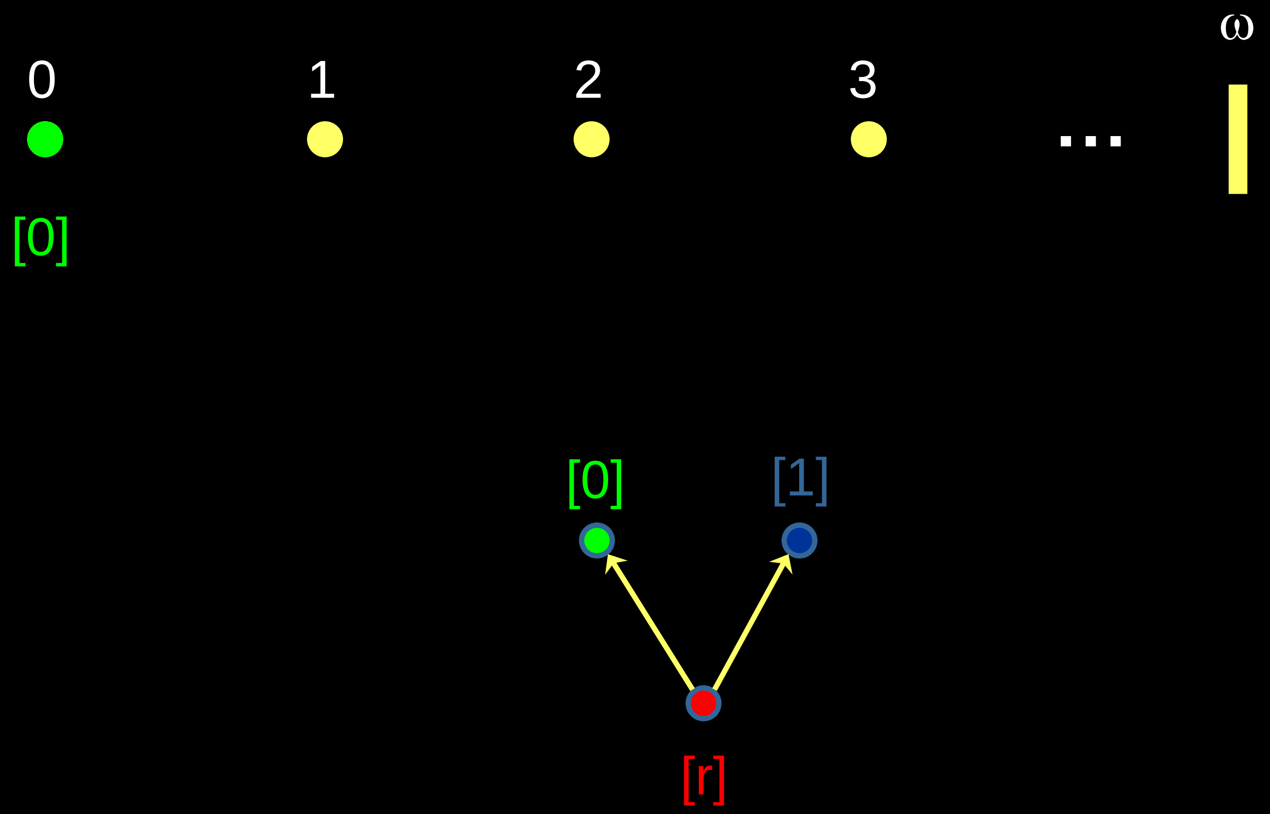
d-maps

- **f: X $\rightarrow$ Y** is a d-map iff:
  - **f** is open
  - **f** is continuous
  - **f** is pointwise discrete
- d-maps preserve d-validity of modal formulas
  - so they anti-preserve (reflect) satisfiability.
- One can show that each finite i-tree is an image of an ordinal via a d-map.
- This gives ordinal completeness of **GL**.

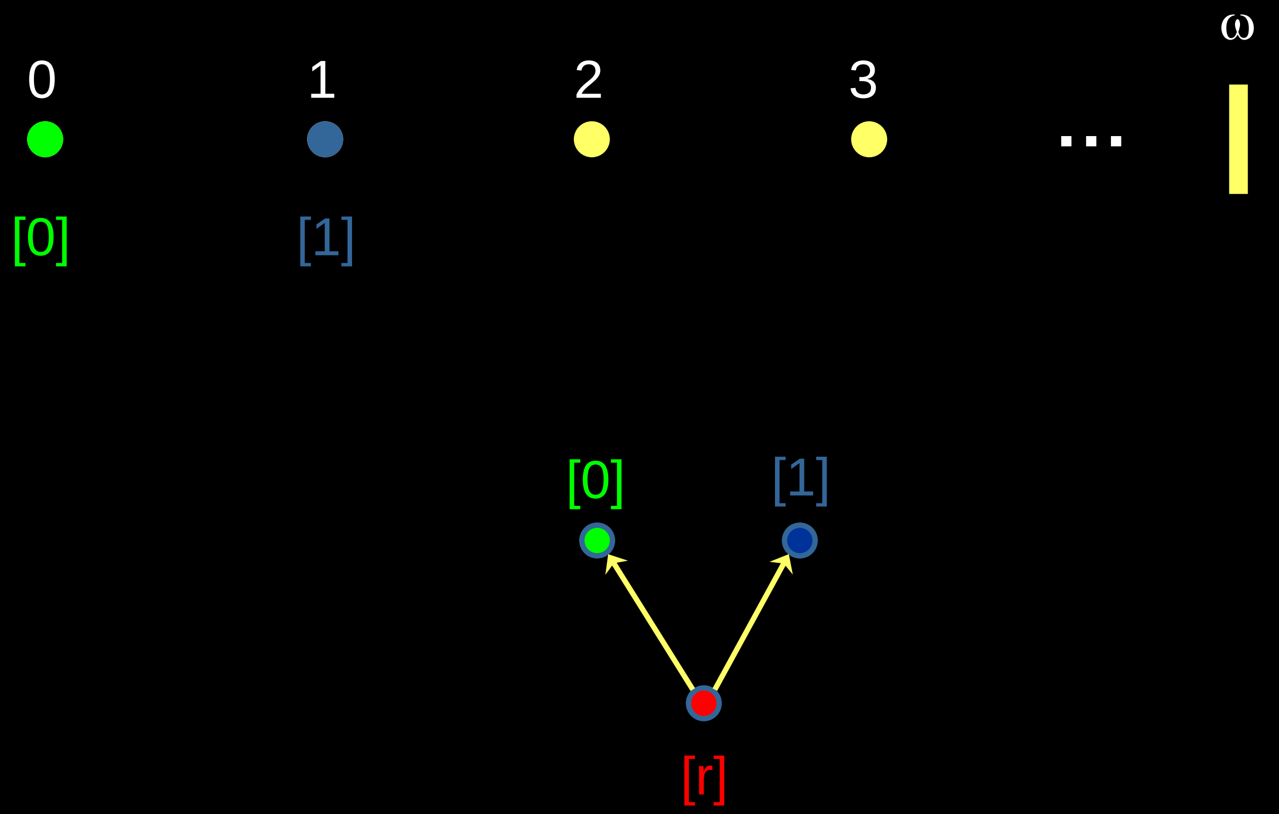
Mapping ordinals to i-trees



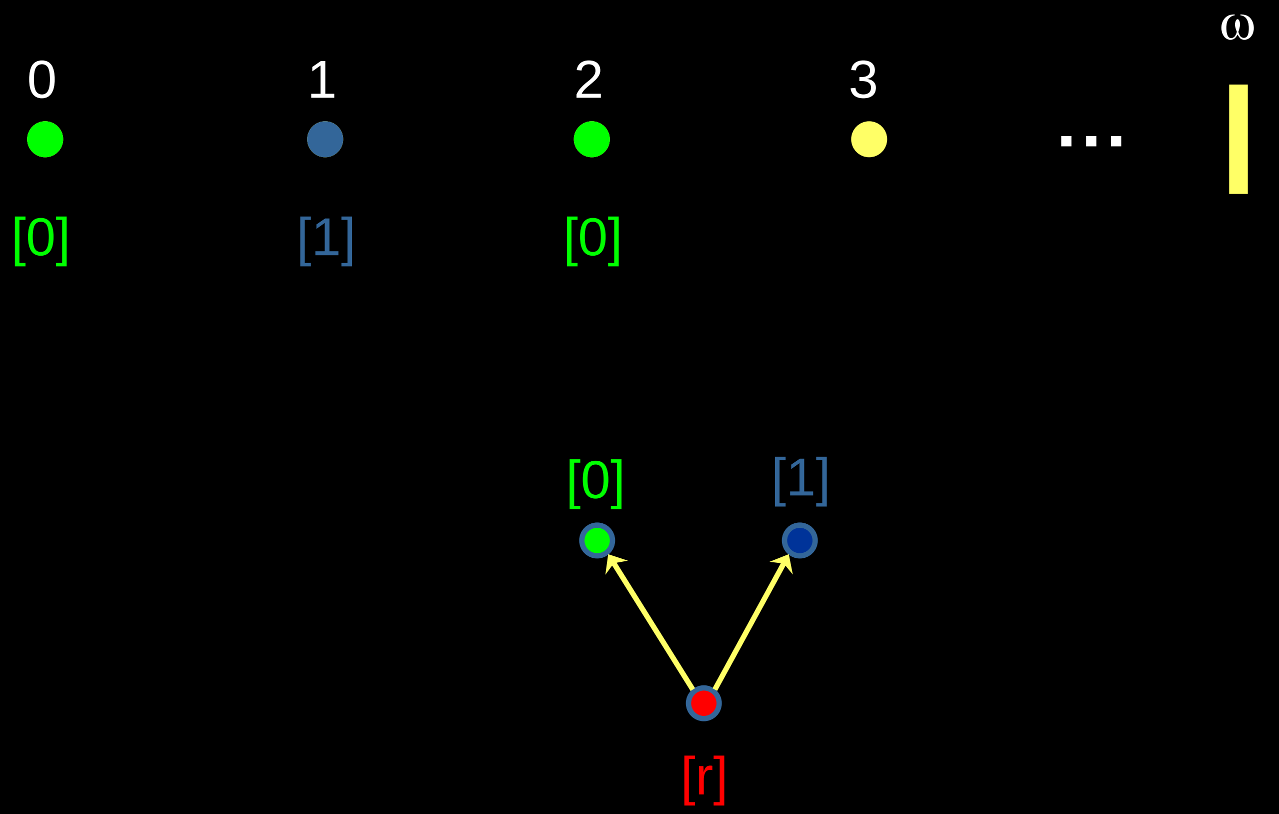
Mapping ordinals to i-trees



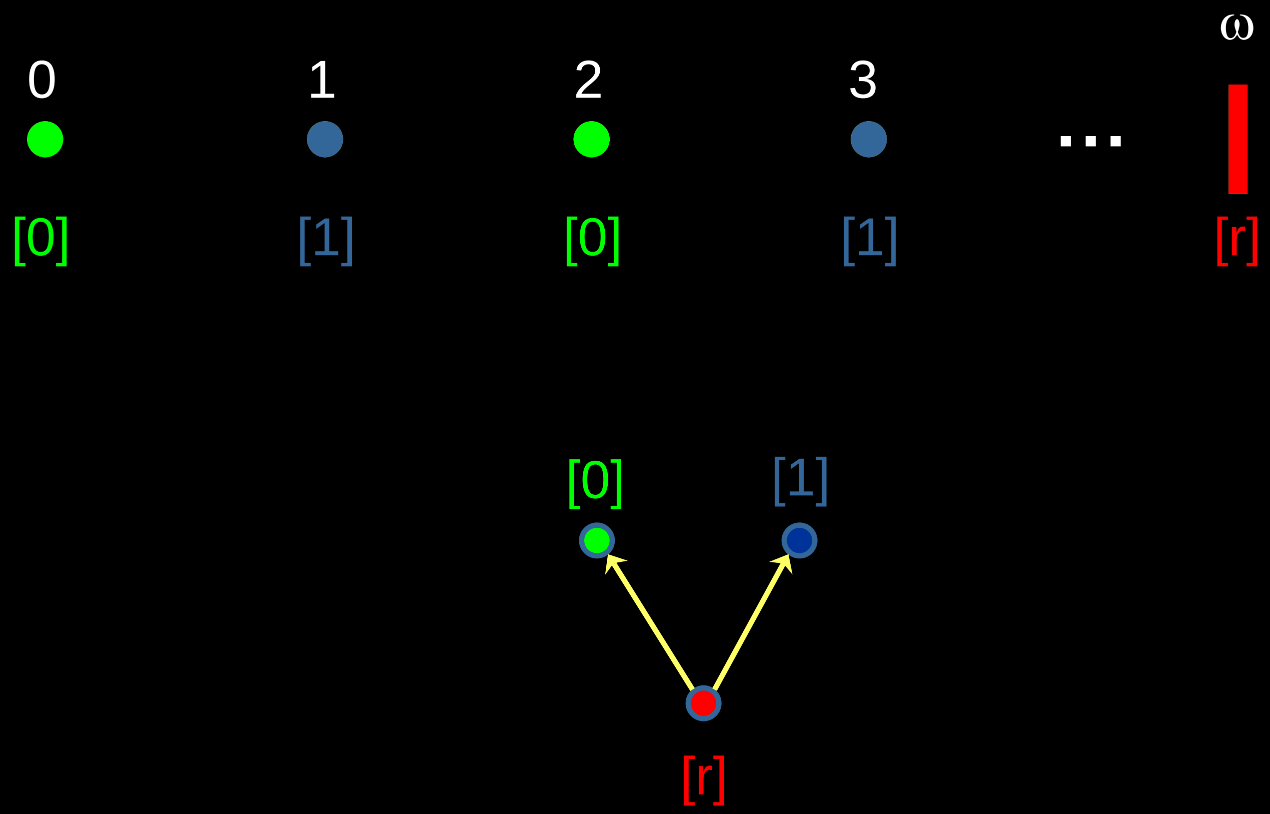
Mapping ordinals to i-trees



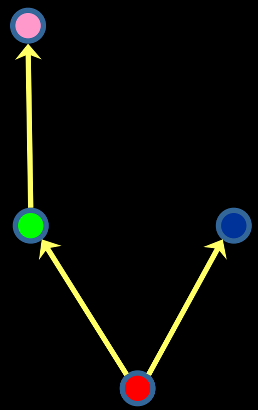
Mapping ordinals to i-trees



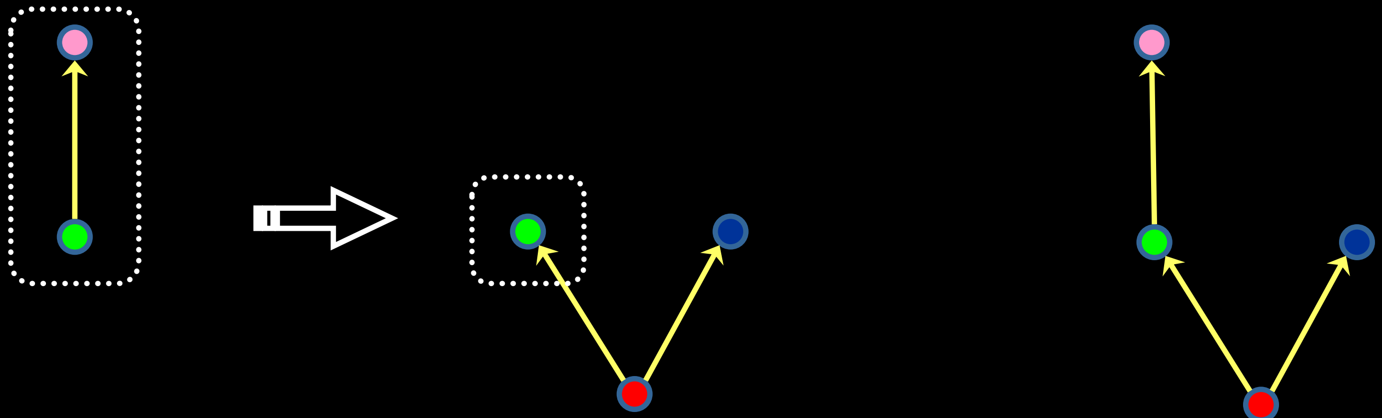
Mapping ordinals to i-trees



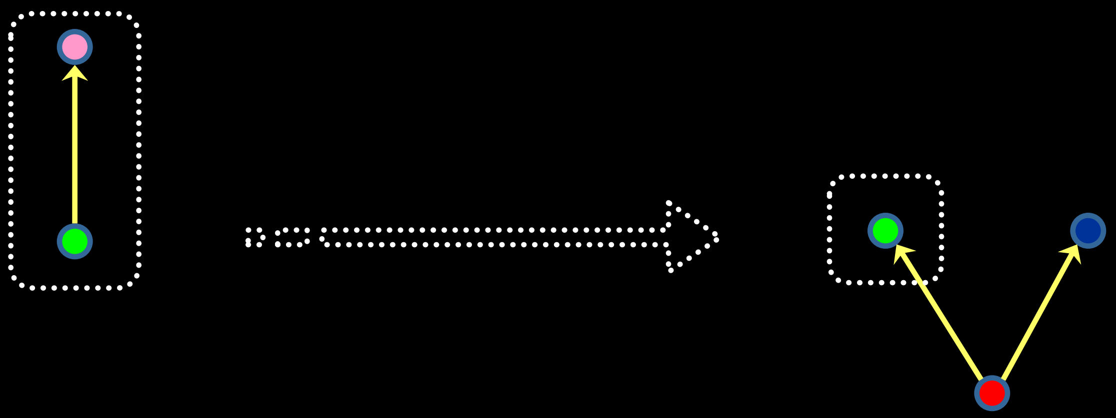
Mapping ordinals to i-trees



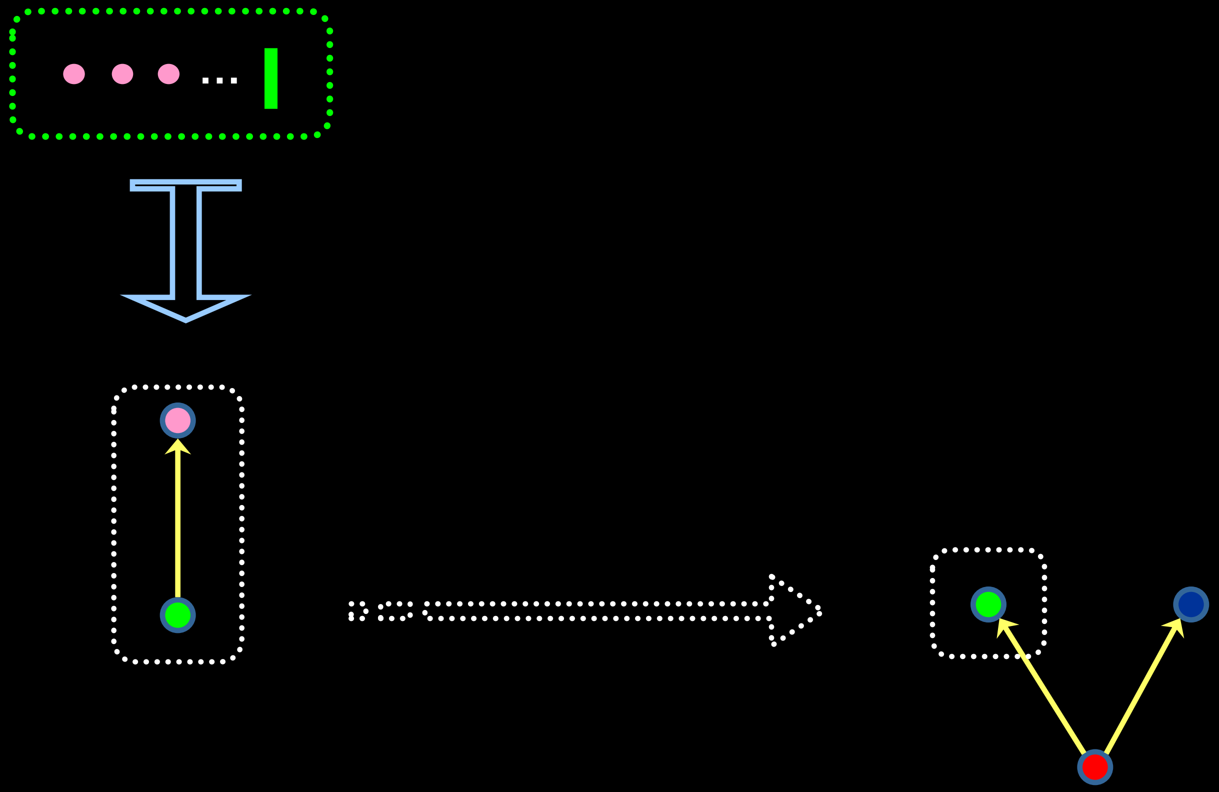
Mapping ordinals to i-trees



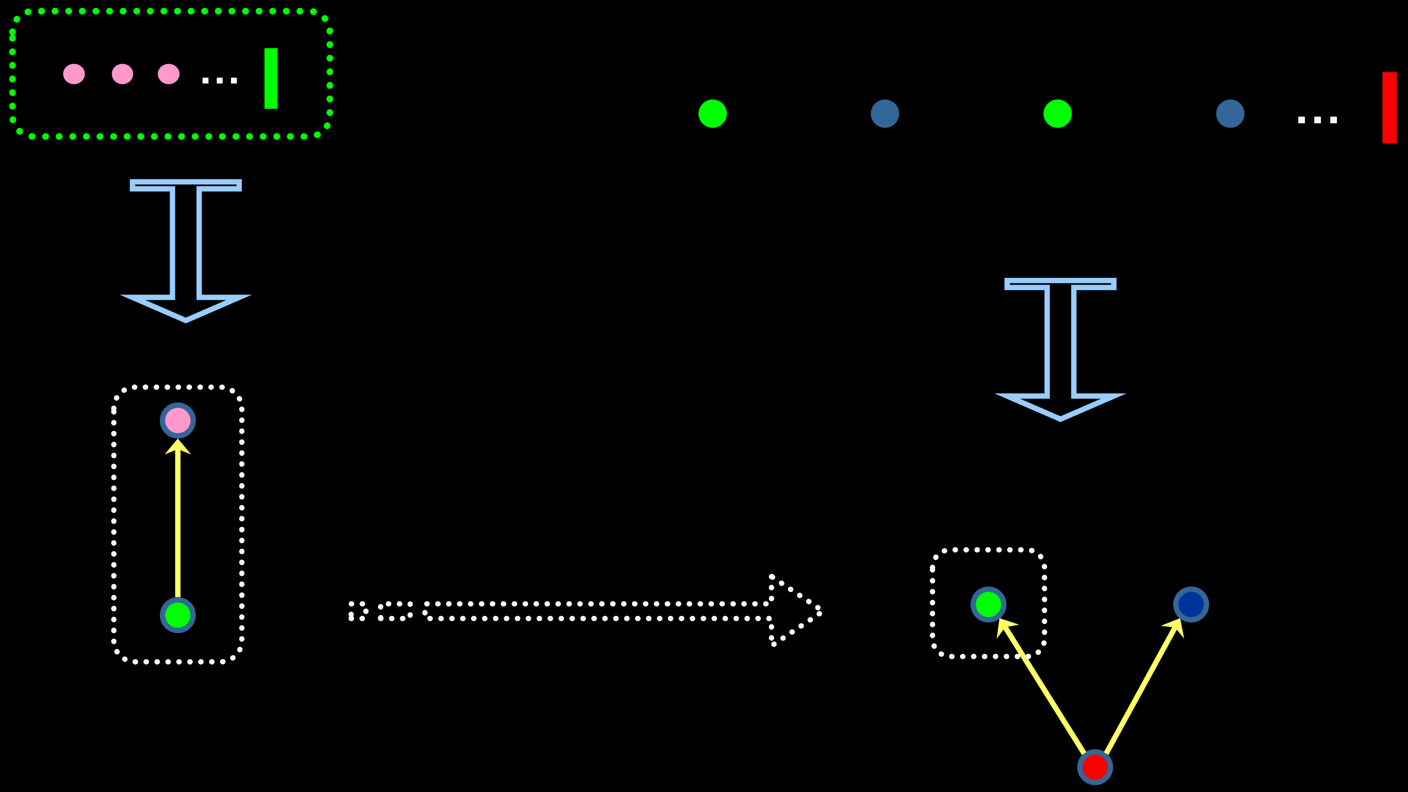
Mapping ordinals to i-trees



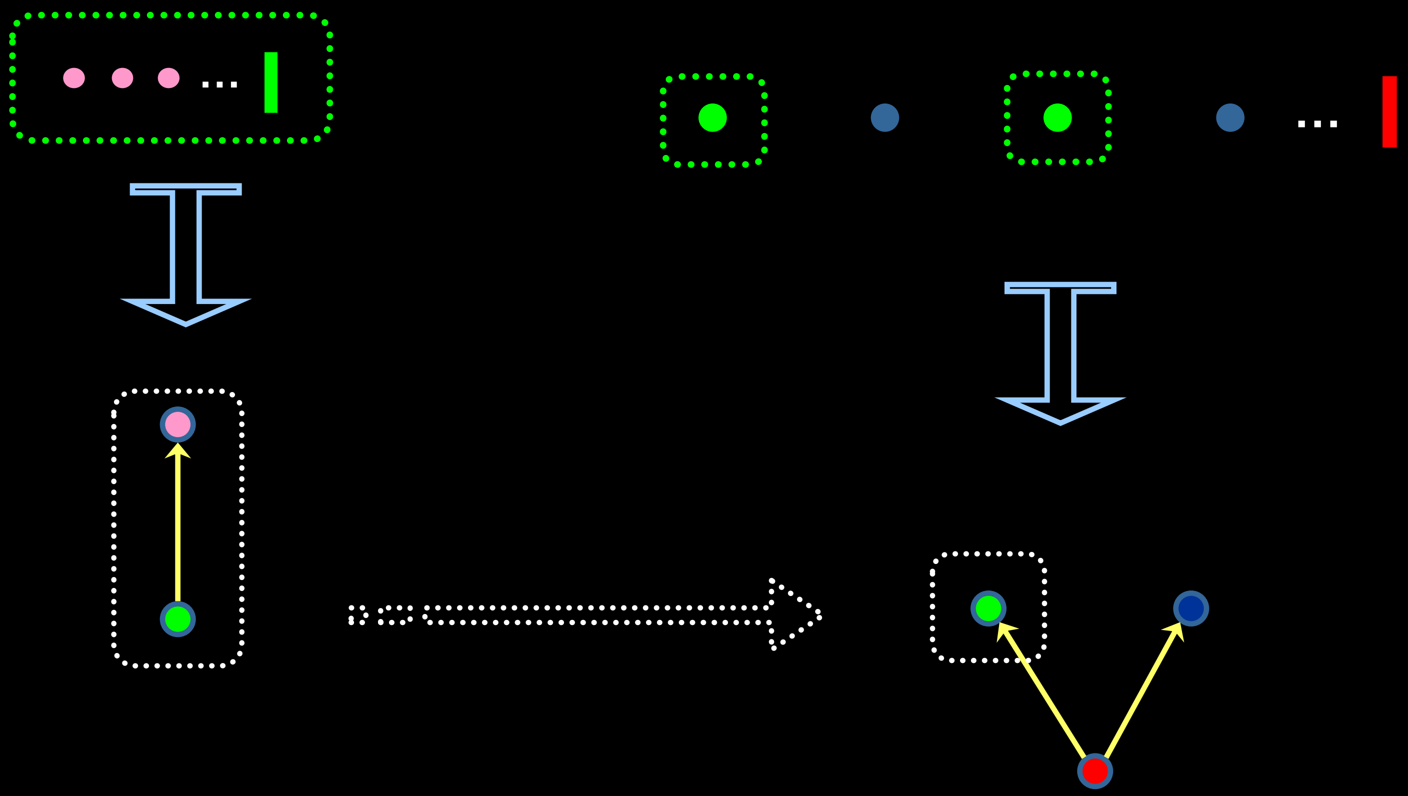
Mapping ordinals to i-trees



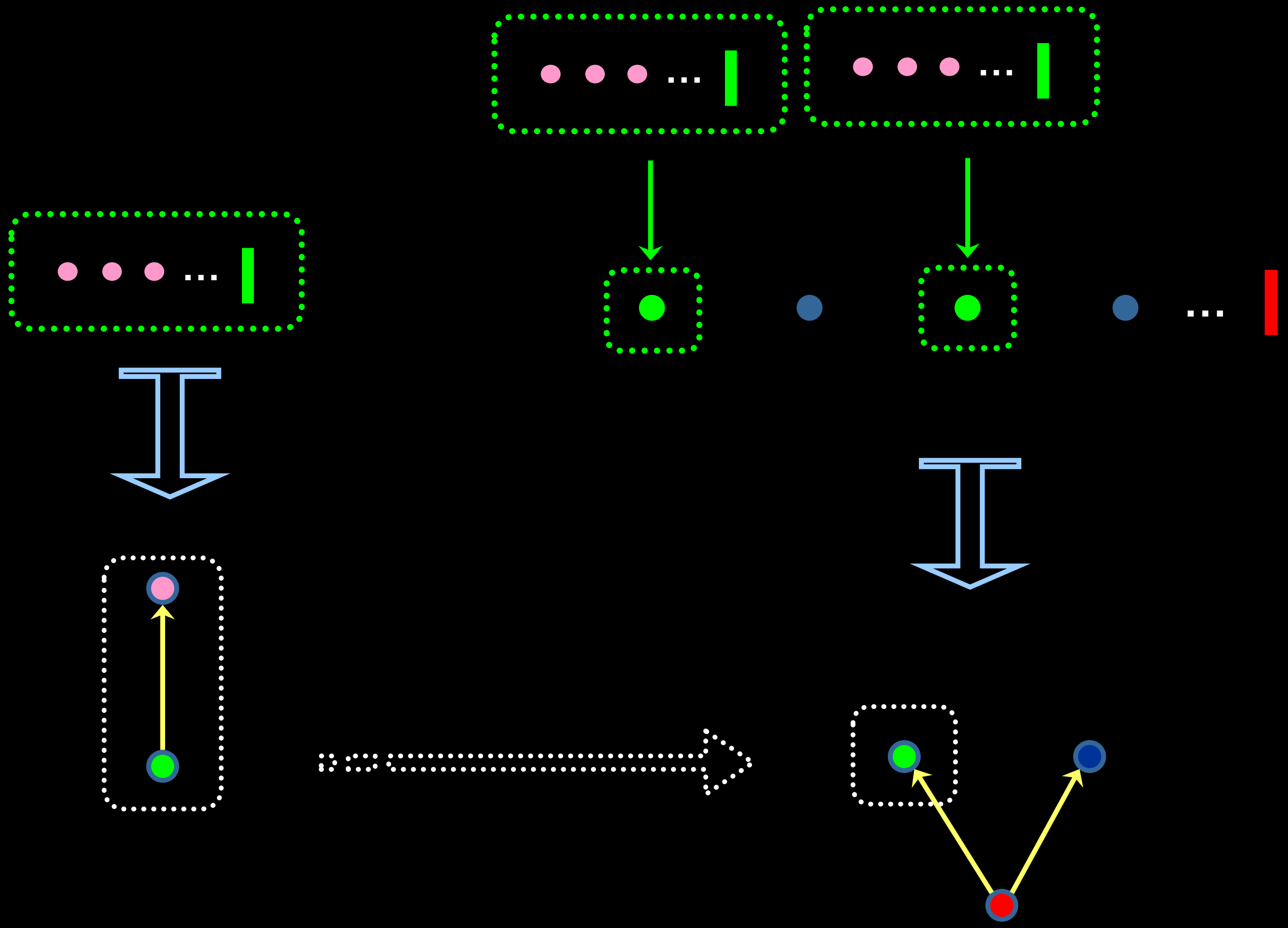
Mapping ordinals to i-trees



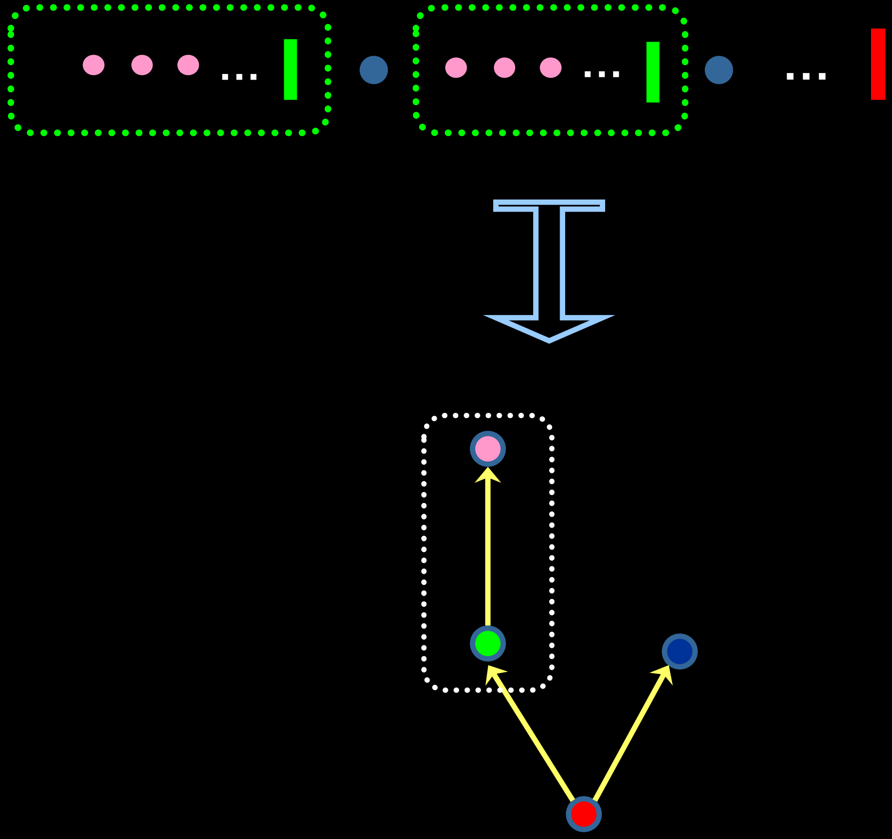
Mapping ordinals to i-trees



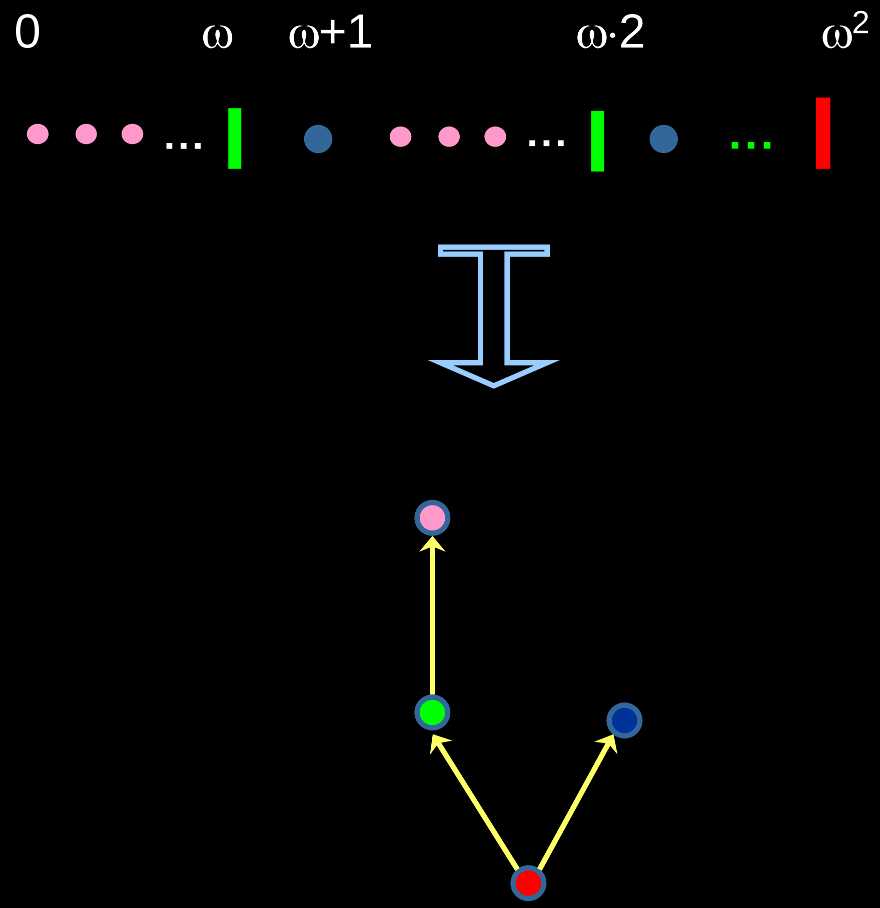
Mapping ordinals to i-trees



Mapping ordinals to i-trees



Mapping ordinals to i-trees



Ordinals recursively

- 0 is an ordinal
- $\omega + 1$ is an ordinal
- ordinal sums of ordinals are ordinals

Ordinals recursively

- 0 is an ordinal
- $\omega + 1$ is an ordinal
- ordinal sums of ordinals are ordinals

What is an ordinal sum?

Ordinals recursively

- 0 is an ordinal
- $\omega + 1$ is an ordinal
- ordinal sums of ordinals are ordinals

What is an ordinal sum?

Roughly: take an ordinal, take it's isolated points and plug in other spaces in place of them.

In the sum, a set is open if:

(a) It's trace on the original ordinal is open (externally).

(b) it's intersection with each plugged space is open (internally)

d-morphisms

$f: X \to F$ is a d-morphism if:

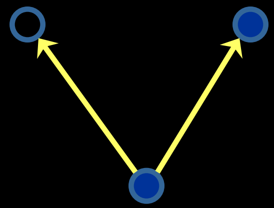
(a) $f: X \to F^+$ is an interior map.

(b) f is i-discrete (preimages of irreflexive points are discrete)

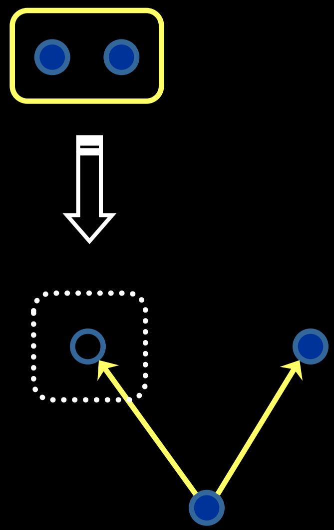
(c) f is r-dense (preimages of reflexive points are dense-in-itself)

- d-morphisms preserve validity.
- We use d-morphisms to obtain d-completeness from Kripke completeness.

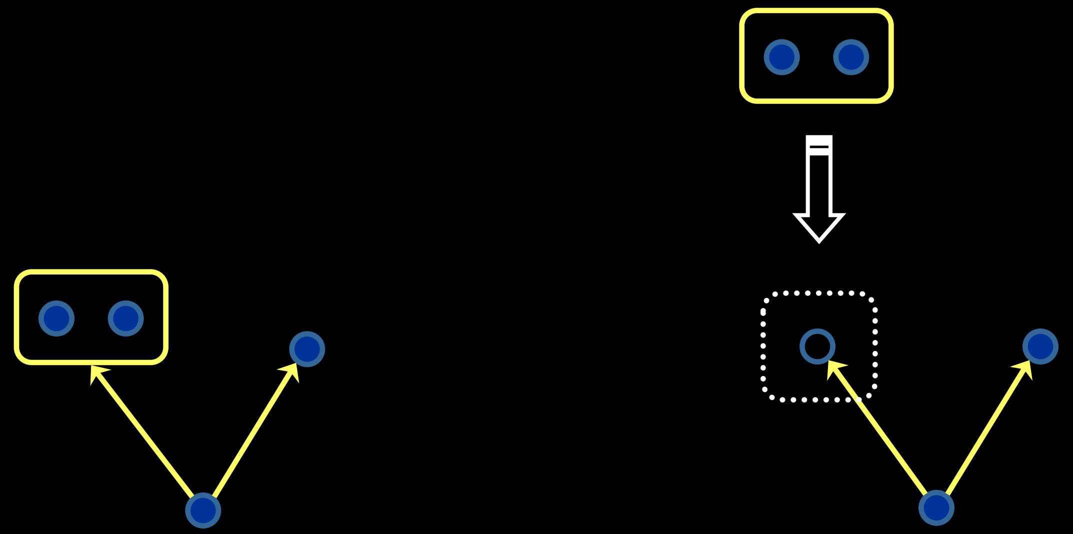
d-completeness for wK4



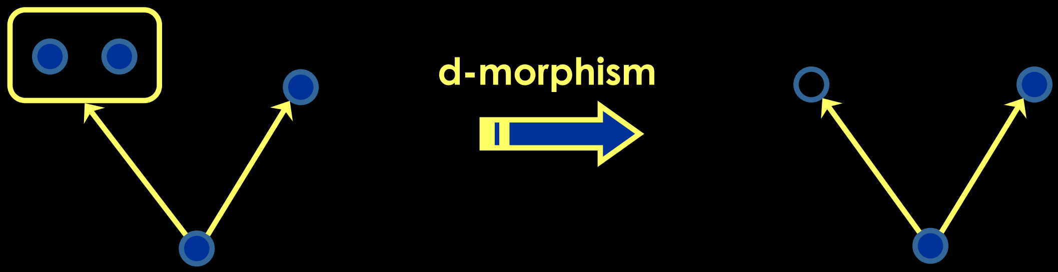
d-completeness for wK4



d-completeness for wK4



d-completeness for wK4



Recipe: Substitute each reflexive point with a two-point irreflexive cluster.

d-completeness for K4.Grz

- **K4.Grz** doesn't admit two-point clusters at all.
- Kripke models for **K4.Grz** are weak partial orders.
- Finite weak trees suffice.
- How to build a K4.Grz-space that maps d-morphically onto a given finite weak tree?
- Toy (but key) example: single reflexive point 

El'kin space

- A set **E**, together with a free ultrafilter U.
- nonempty O⊆A is open iff O∈U
- **E** is dense-in-itself
- **E** is a **K4.Grz**-space (no subset can be decomposed into two disjoint dense in it sets)

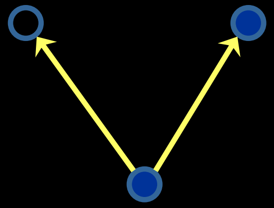
Pictorial representation:

El'kin space

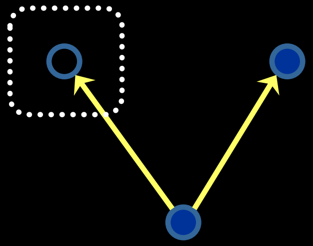
- A set **E**, together with a free ultrafilter U.
- nonempty $O \subseteq A$ is open iff $O \in U$
- **E** is dense-in-itself
- **E** is a **K4.Grz**-space (no subset can be decomposed into two disjoint dense in it sets)

Pictorial representation: 

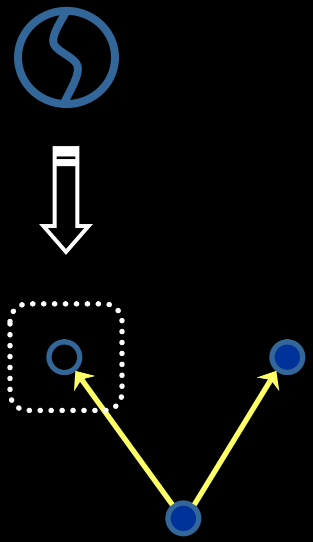
Building K4.Grz-space preimages



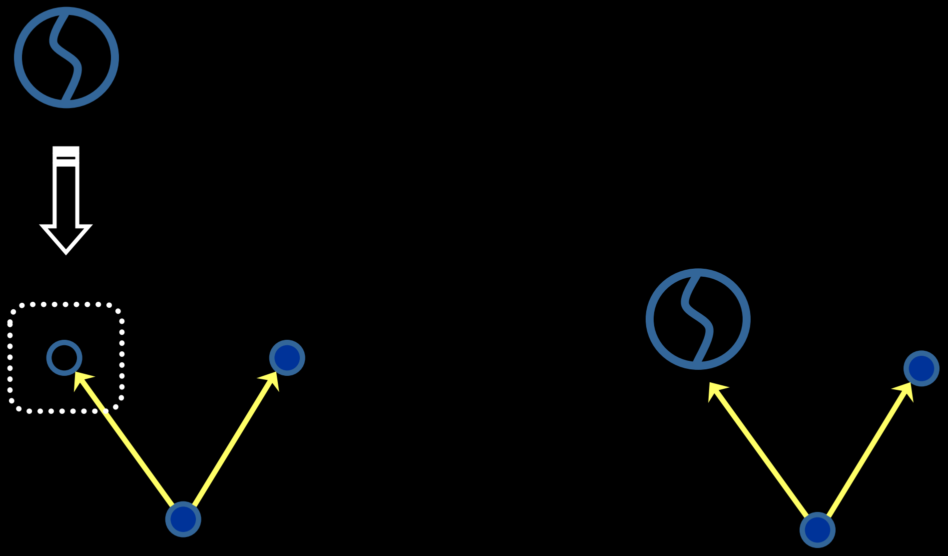
Building K4.Grz-space preimages



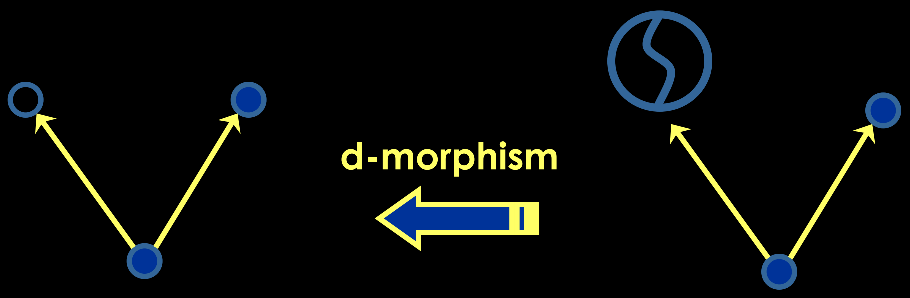
Building K4.Grz-space preimages



Building K4.Grz-space preimages



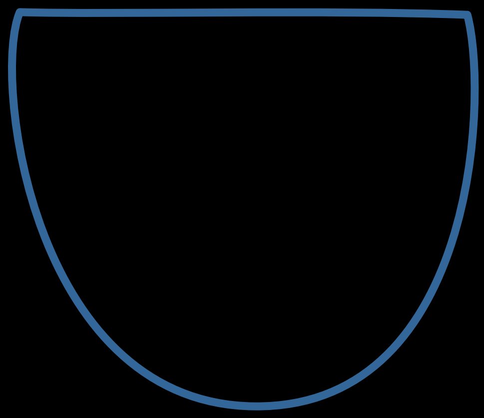
Building K4.Grz-space preimages



Recipe: Substitute each reflexive point with a copy of Elkin's Space.

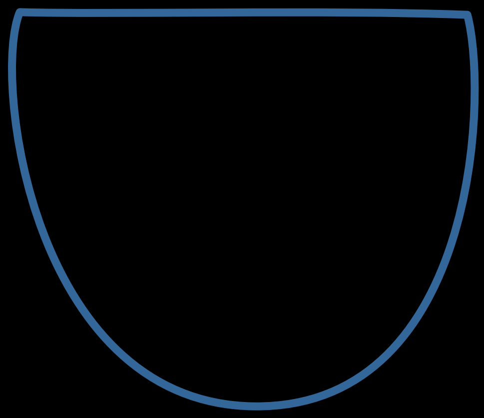
Topo-sums of spaces

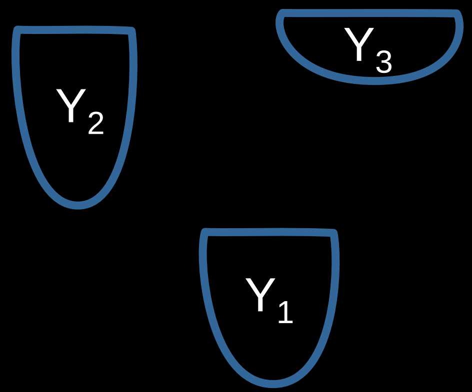
A space X
(Skeleton)



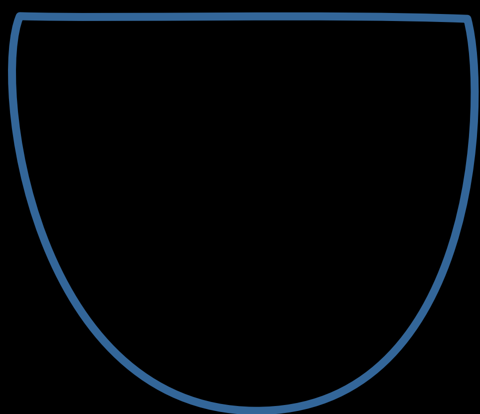
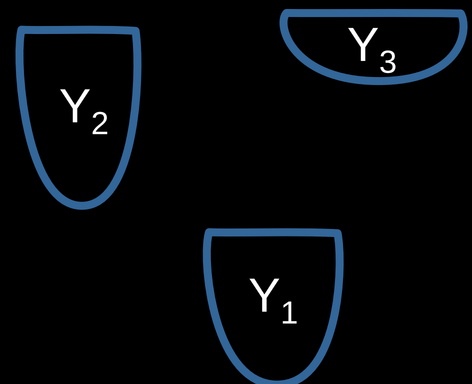
Topo-sums of spaces

A space X (Skeleton)	Family of spaces $(Y_i)_{i \in X}$ indexed by X (Components)	





Topo-sums of spaces

A space X (Skeleton)	Family of spaces $(Y_i)_{i \in X}$ indexed by X (Components)	X-ordered sum of (Y_i) $\mathbf{Y = \oplus_X Y_i}$
		

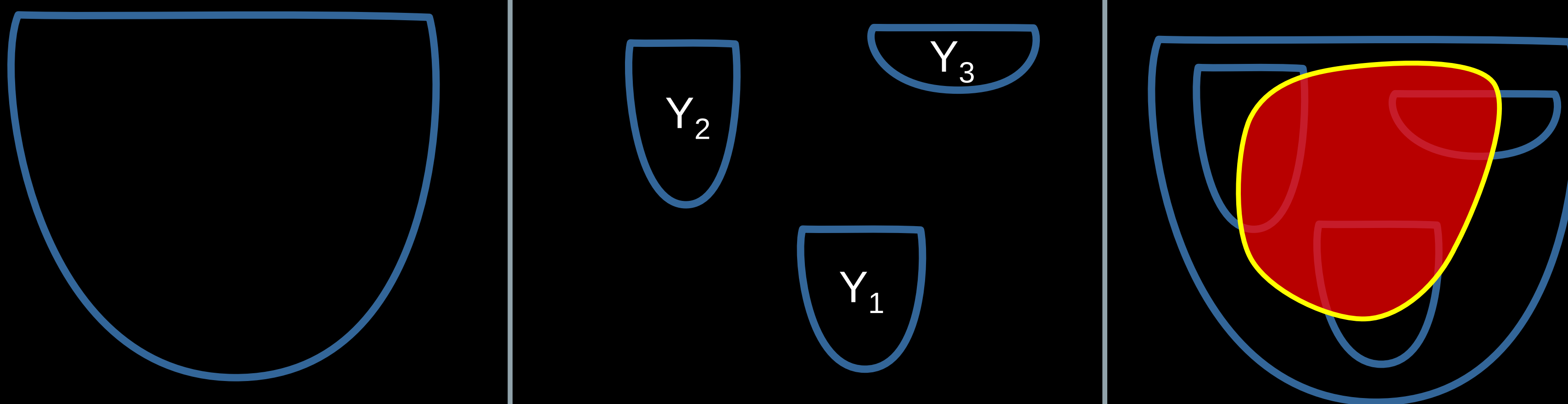
Topo-sums of spaces

A space X
(Skeleton)

Family of spaces $(Y_i)_{i \in X}$ indexed by X
(Components)

X-ordered sum of (Y_i)

$$Y = \oplus_X Y_i$$



A set $U \subseteq Y$ is open iff it's trace on the skeleton is open and its traces on all the components are open.

Some results

- d-completeness of some extensions of **K4.Grz** "with a provability smack"

  [Bezhanishvili, Esakia, Gabelaia 2010]
- d-logics of maximal, submaximal, nodec spaces.

  [Bezhanishvili, Esakia, Gabelaia, Studia 2005]
- d-logic of Stone spaces is **K4**.

  [Bezhanishvili, Esakia, Gabelaia, RSL 2010]
- d-logic of Spectral spaces.

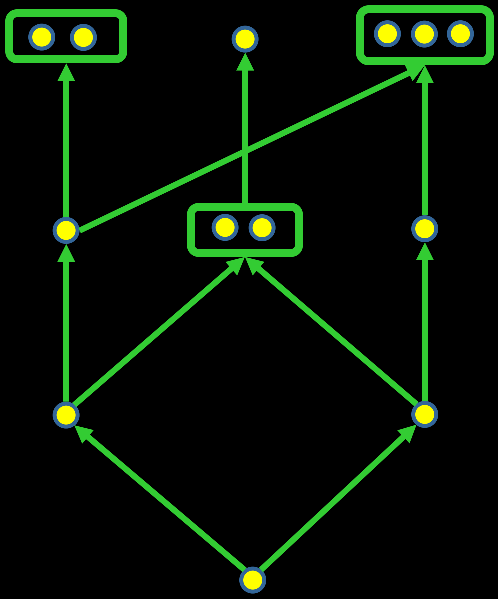
  [Bezhanishvili, Esakia, Gabelaia 2011]
- d-definability of T_0 separation axiom.

  [Bezhanishvili, Esakia, Gabelaia 2011]
- d-completeness of the **GLP**.

  [Beklemishev, Gabelaia 201?]

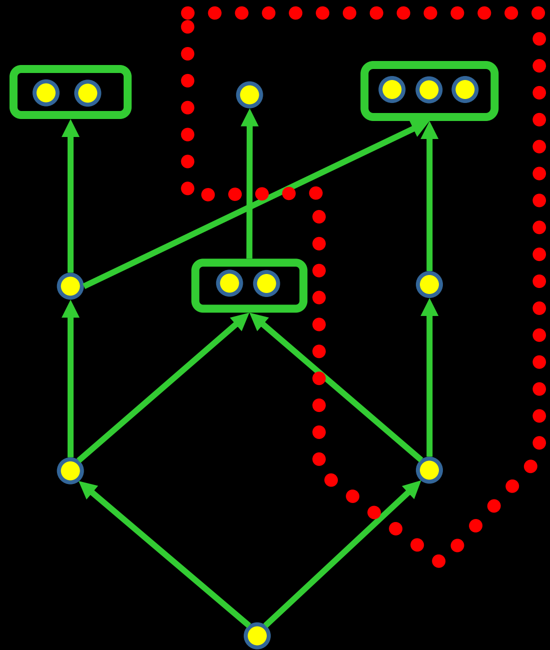
Quasiorders as topologies

- **Interior** is the largest open contained in a set.



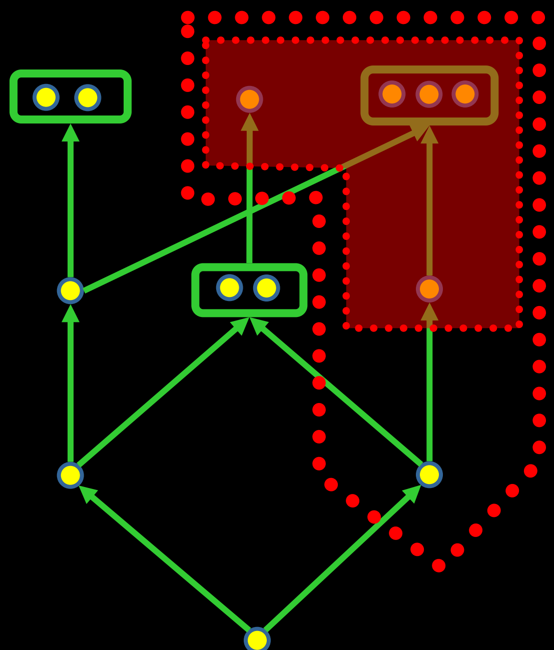
Quasiorders as topologies

- **Interior** is the largest open contained in a set.



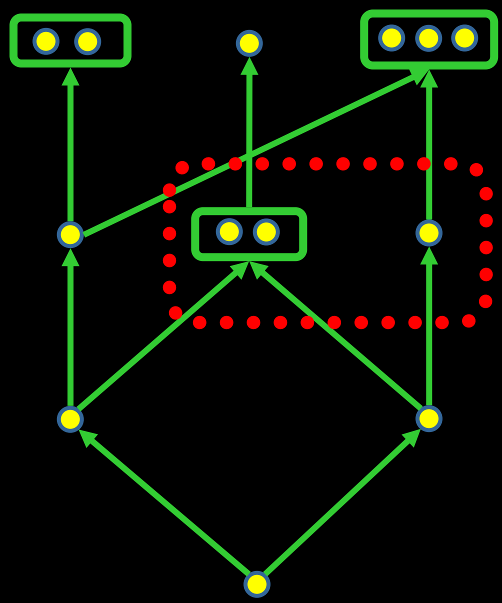
Quasiorders as topologies

- **Interior** is the largest open contained in a set.



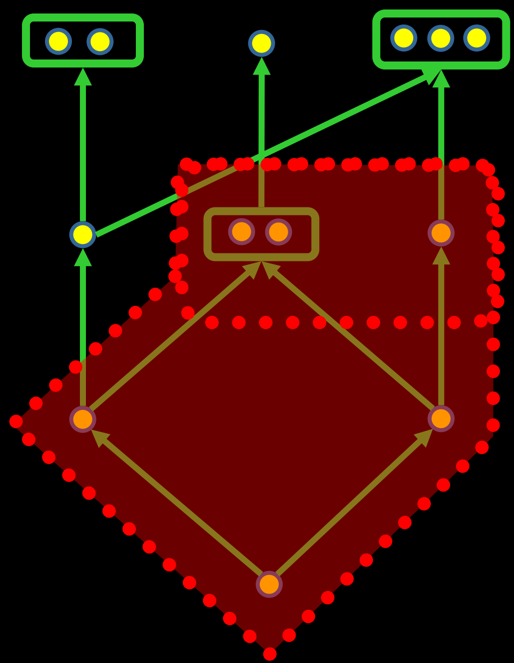
Quasiorders as topologies

- **Closure** takes all the points below.



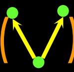
Quasiorders as topologies

- **Closure** takes all the points below.



Interior fields of sets

Some examples of Interior Fields of Sets in **R** and their logics:

- B(Op(**R**)) Boolean combinations of opens — S4
- C (**R**) Finite unions of convex sets — S4.Grz
- C (OD(**R**)) Boolean comb. of open dense subsets — S4.Grz.2
- B(C$^{\infty}$(**R**)) Countable unions of convex sets — Log()
- All subsets of **R** with small boundary — S4.1
- Nowhere dense and interior dense subsets of **R** — S4.1.2

Question: Which logics arise in this way from **R**?

Theorem

Suppose L is an extension of S4 with fmp. Then the following conditions are equivalent:

(1) L arises from a subalgebra of $\mathbf{R}^+$.

(2) L is the logic of a path-connected quasiorder.

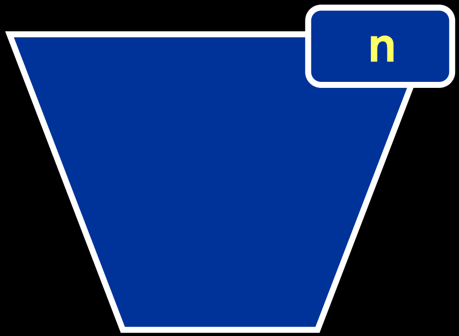
(3) L is the logic of a connected space.

(4) L is a logic of a connected Closure Algebra.

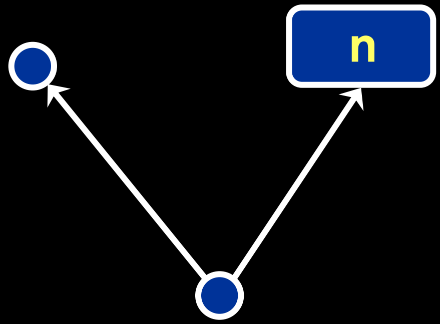
Corollary: All logics extending S4.1 with the finite model property arise from a subalgebra of $\mathbf{R}^+$.

Glueing the finite frames

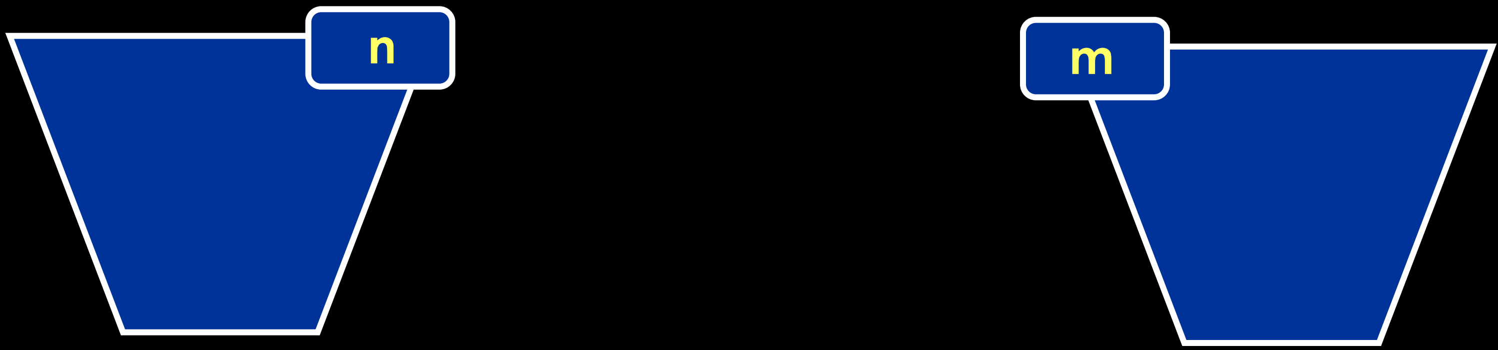
Suppose L admits the frame:



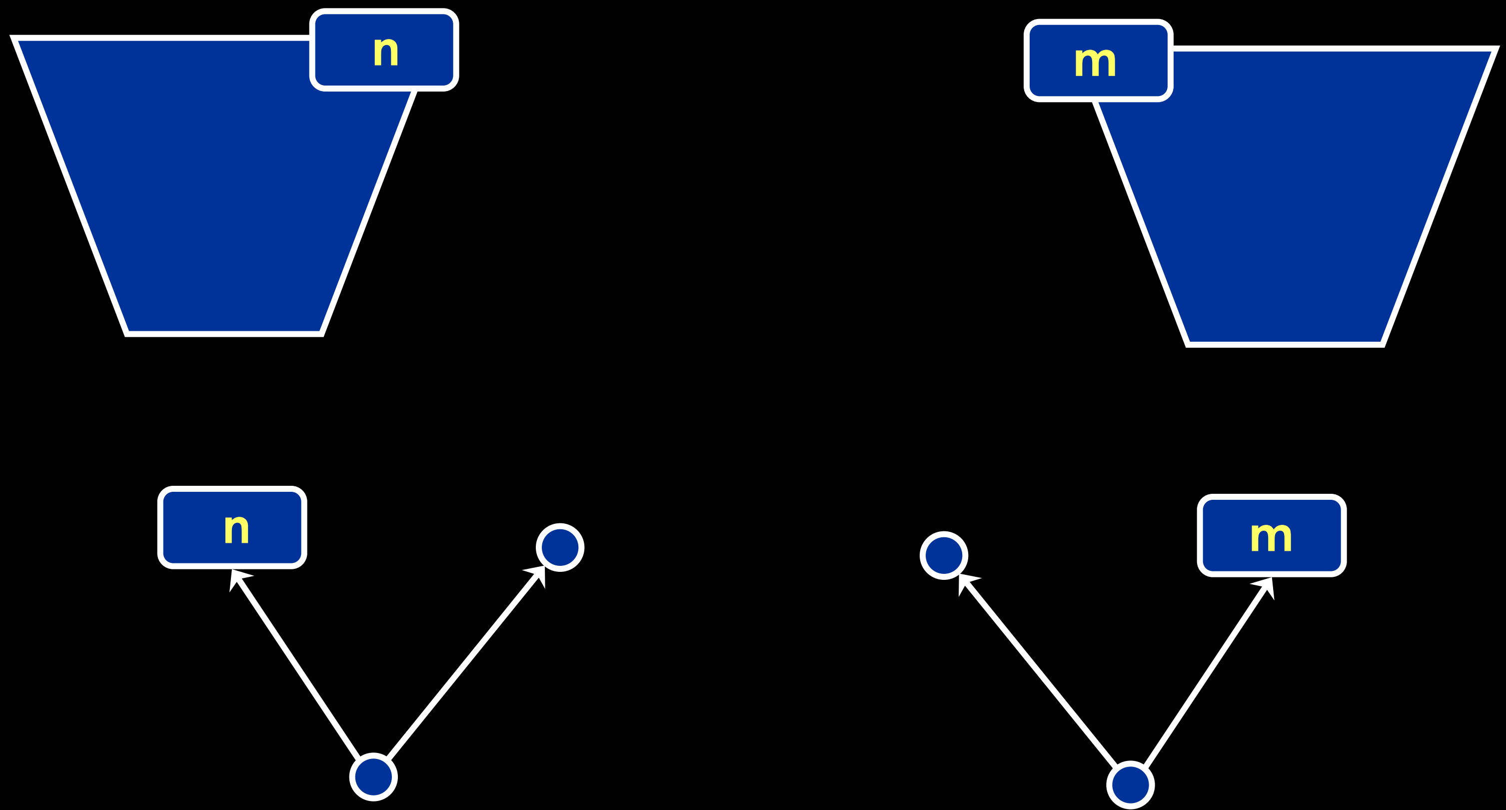
Then L also admits the frame:



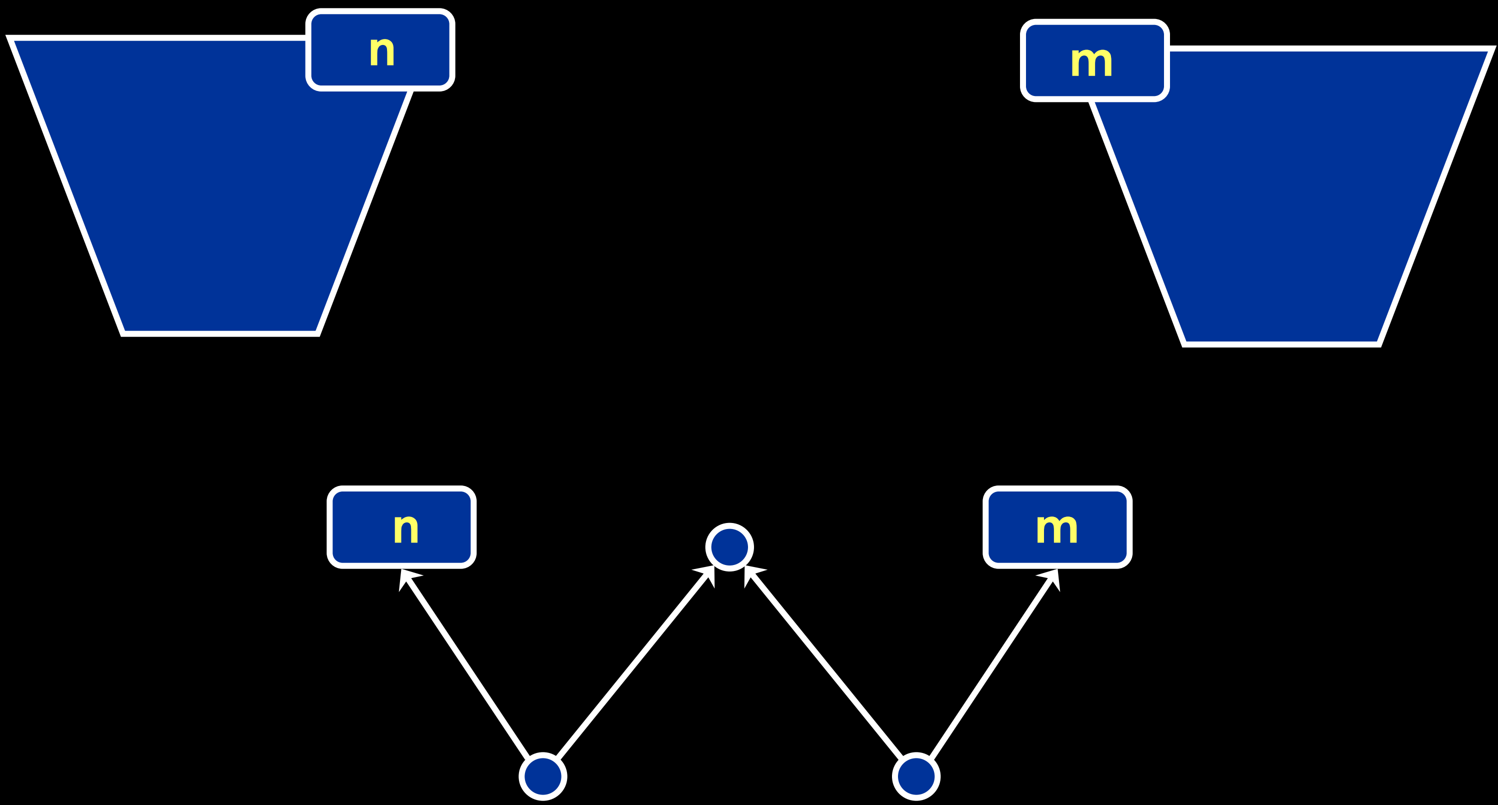
Glueing the finite frames



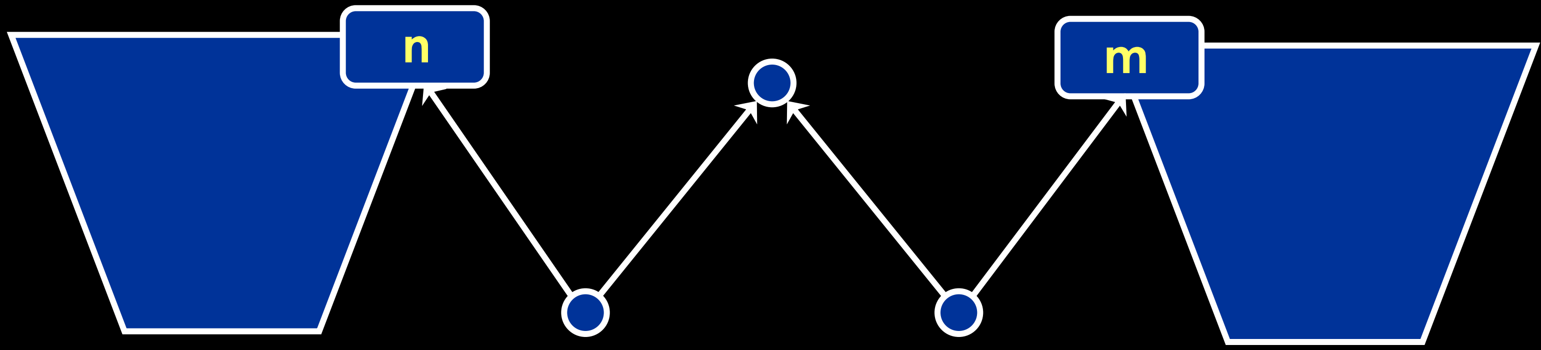
Glueing the finite frames



Glueing the finite frames



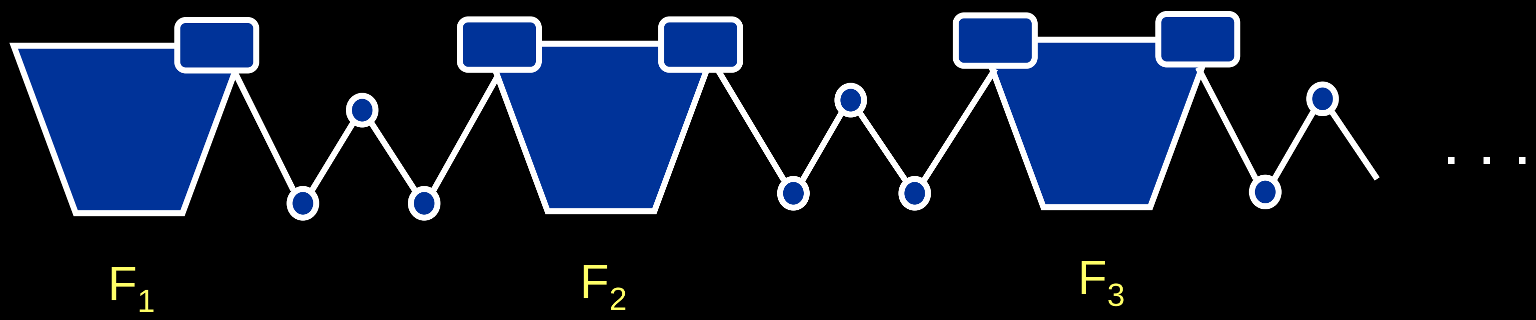
Glueing the finite frames



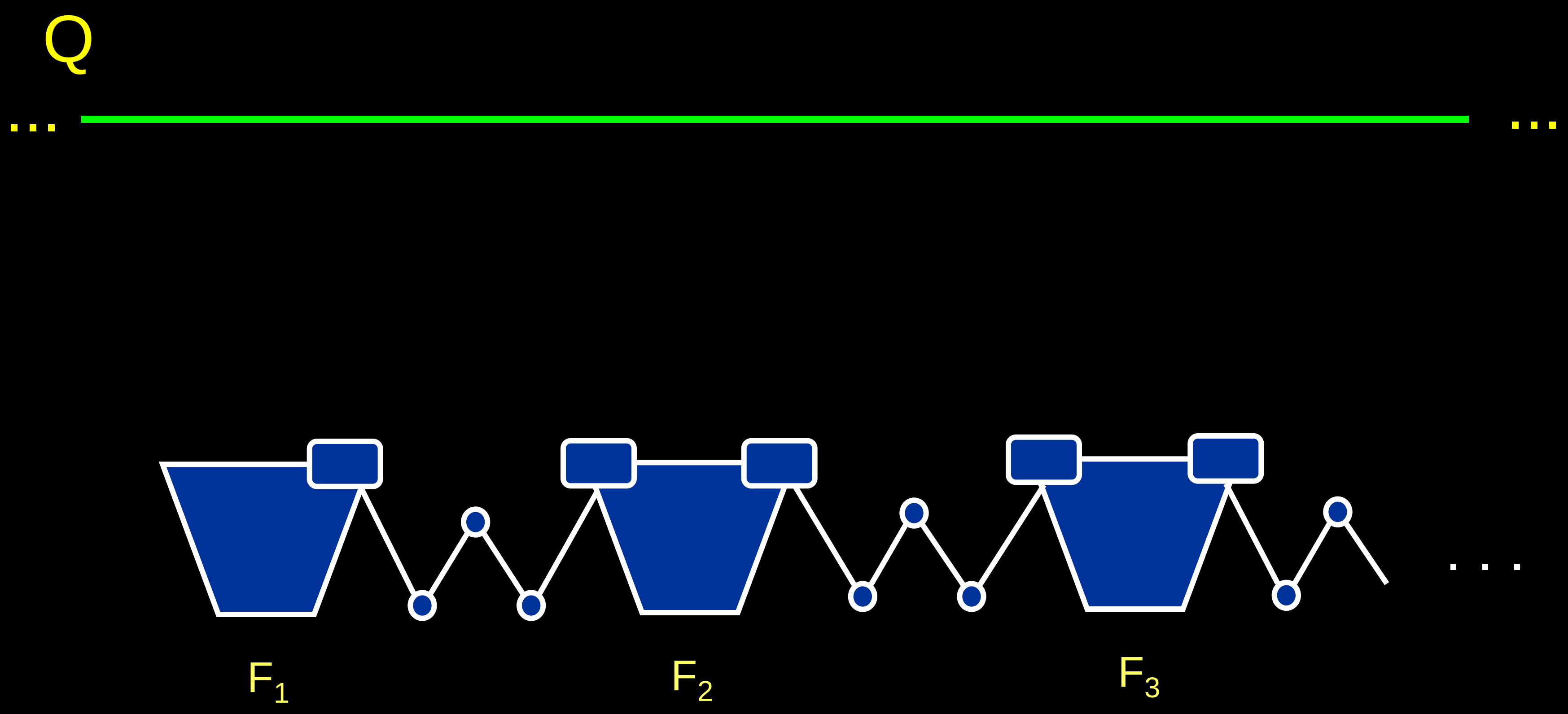
Glueing all finite frames

Glueing all finite frames

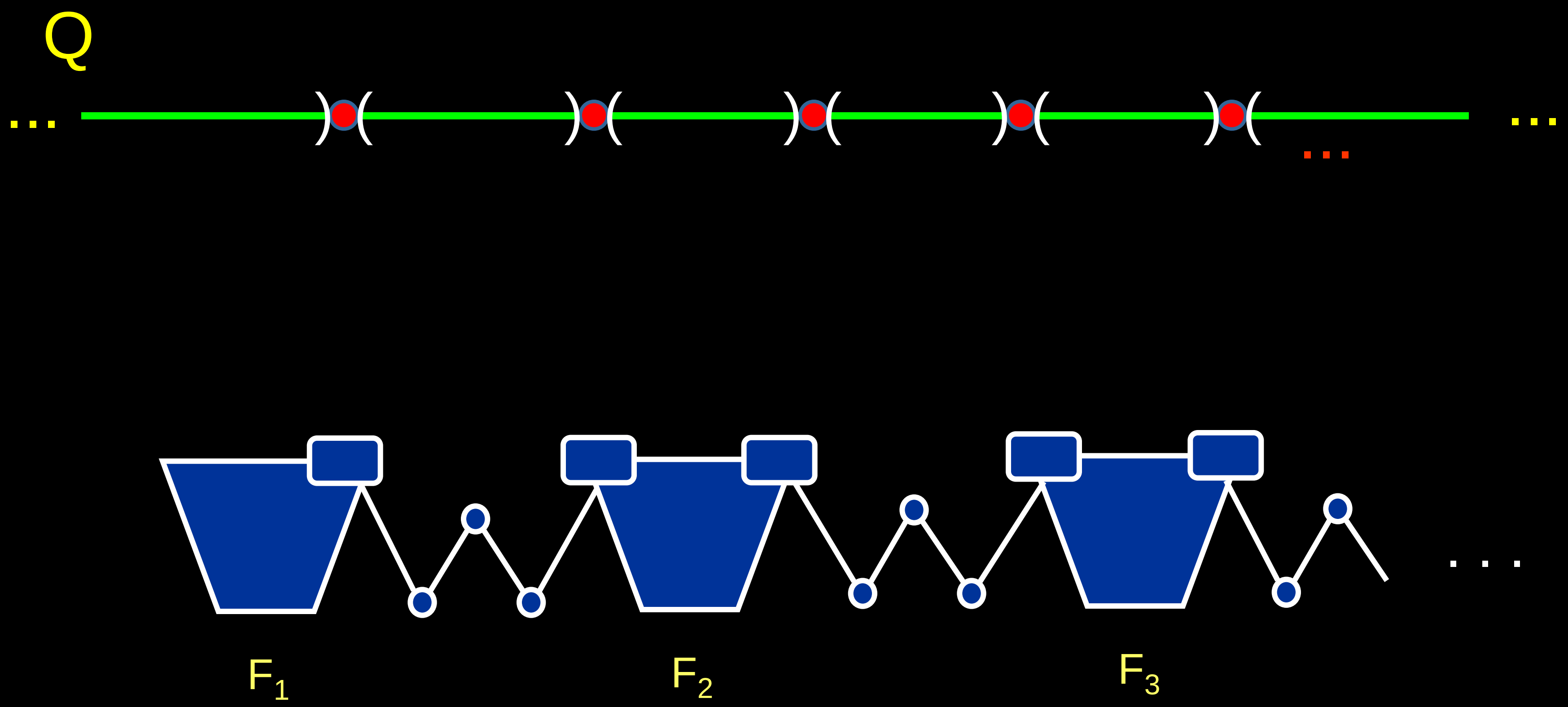
Glueing all finite frames



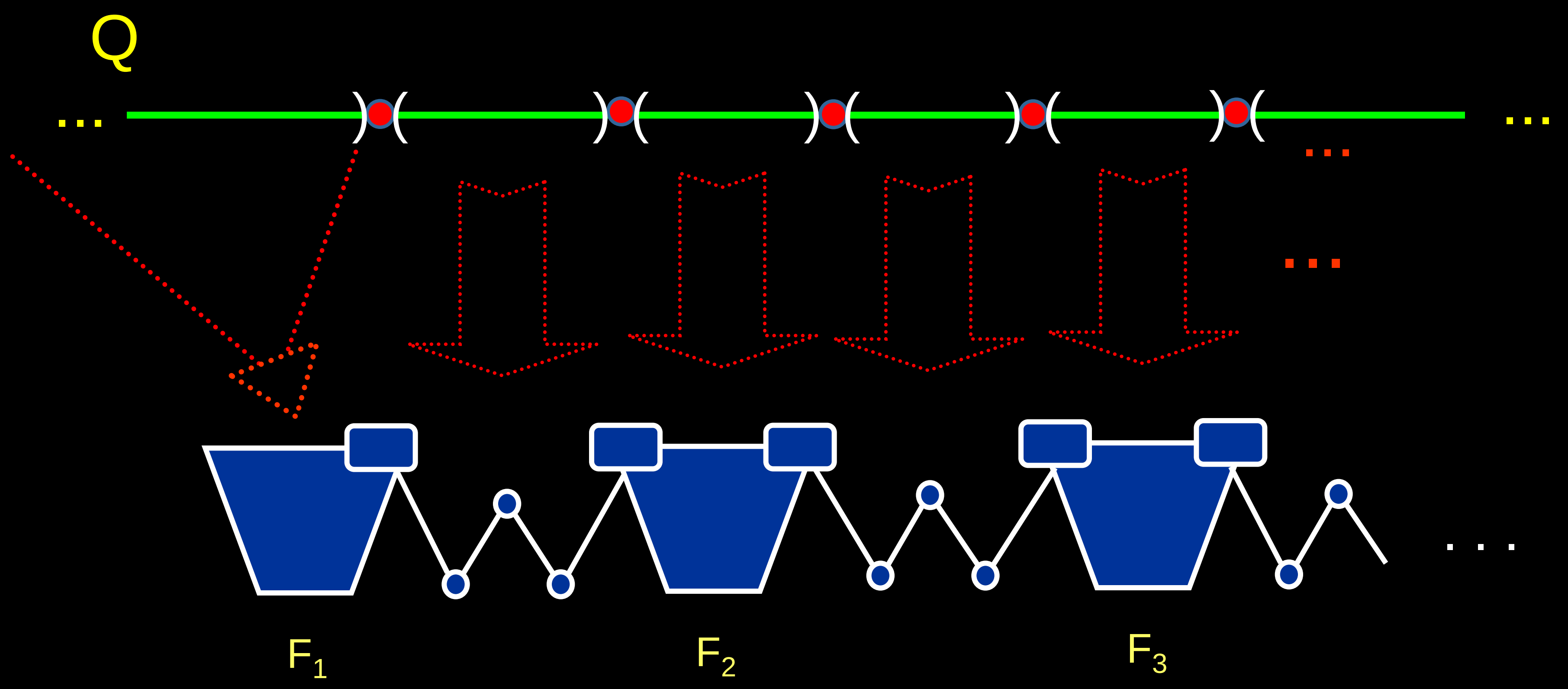
Glueing interior maps



Glueing interior maps



Glueing interior maps



Going from algebras to topologies

Each closure algebra $(A, \Diamond)$ is isomorphic to a subalgebra of X^+ for some topological space (X,τ).

[McKinsey&Tarski, 1944]

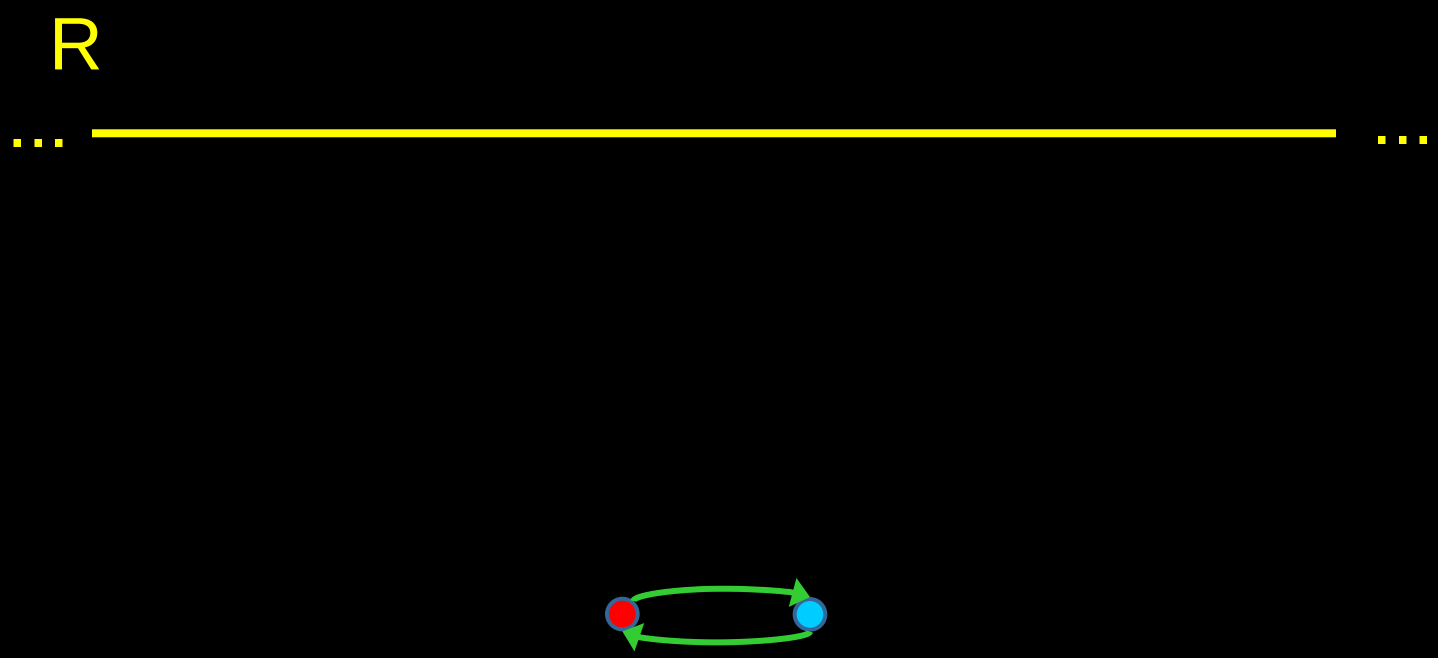
Each closure algebra $(A, \Diamond)$ is isomorphic to a subalgebra of $(\wp(X), R^{-1})$ for some quasiorder (X,R).

[Jonsson&Tarski, 1951]

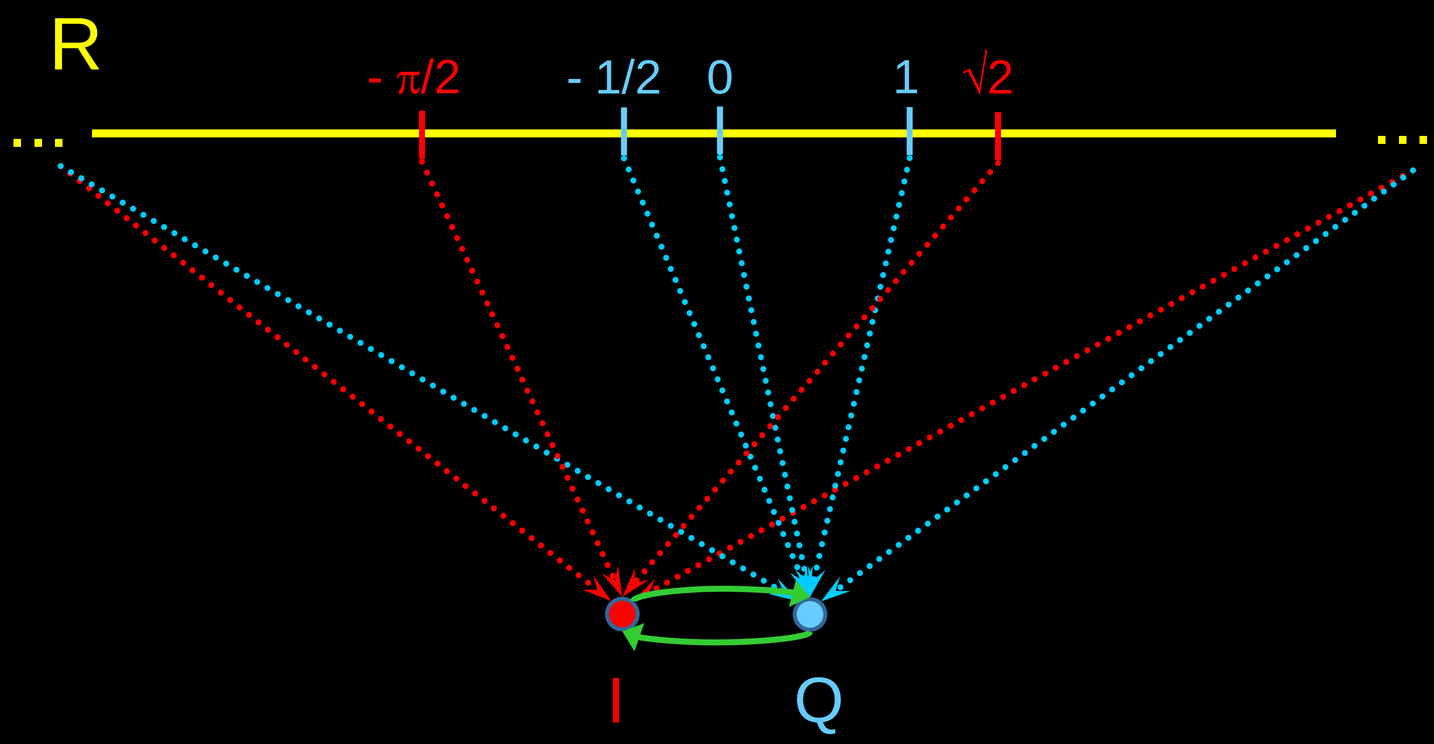
X is a set of Ultrafilters of A and $(X, \tau\cap\tau_R)$ is a Stone space of A.

[Bezhanishvili, Mines, Morandi, 2006]

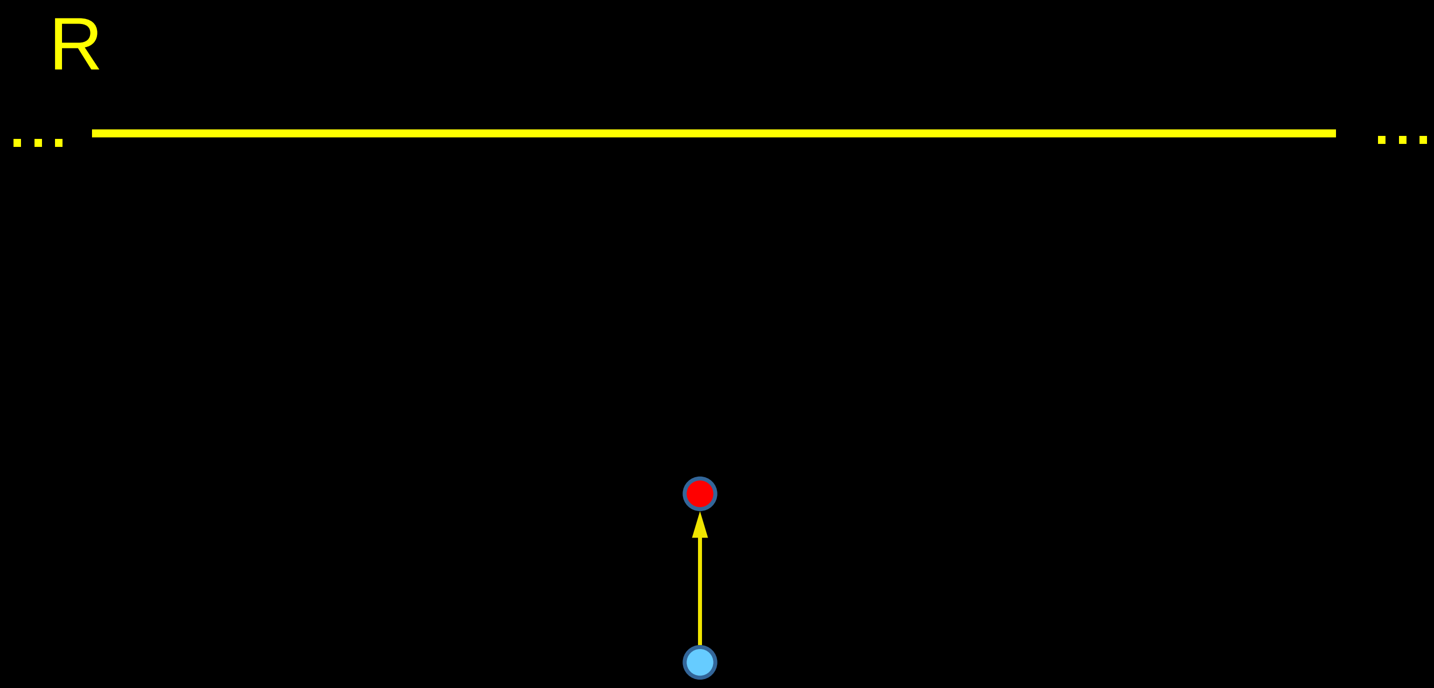
Mapping R onto finite connected quasiorders



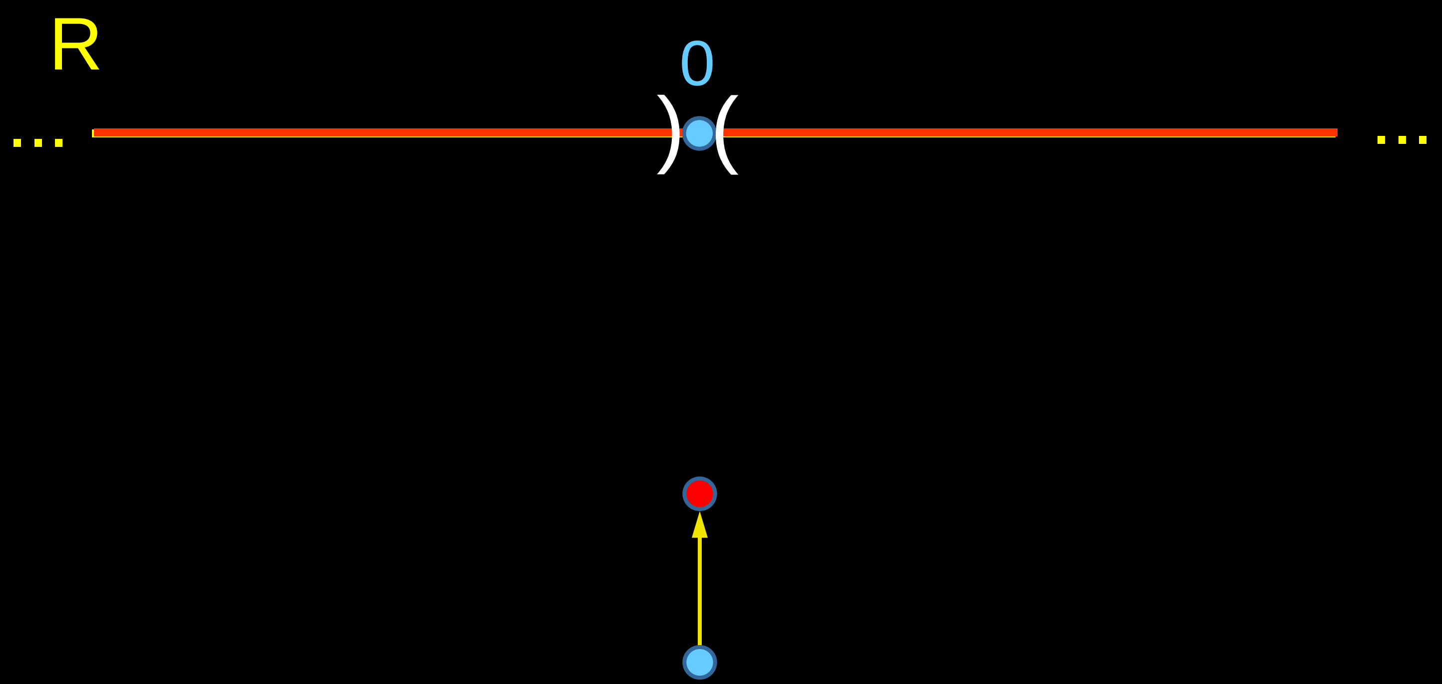
Mapping R onto finite connected quasiorders



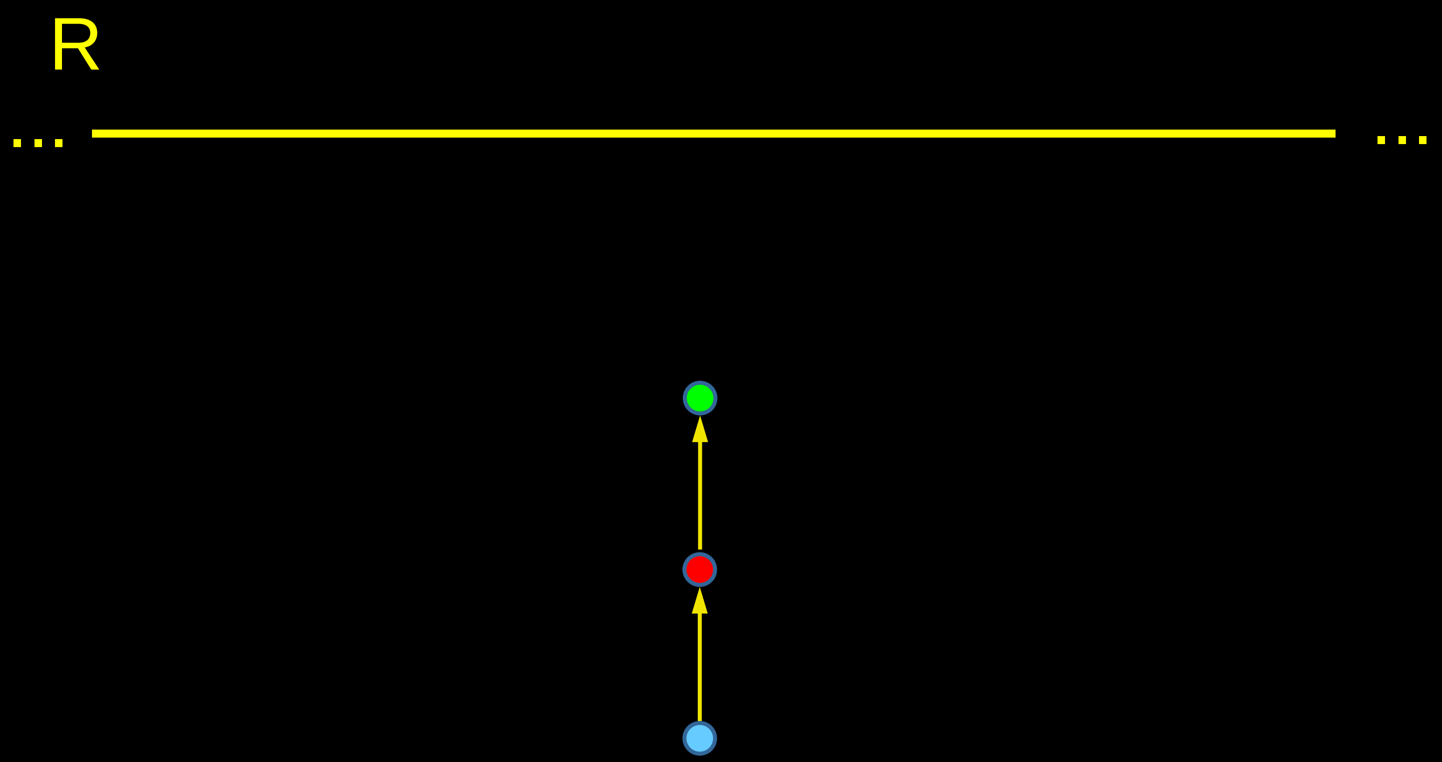
Mapping R onto finite connected quasiorders



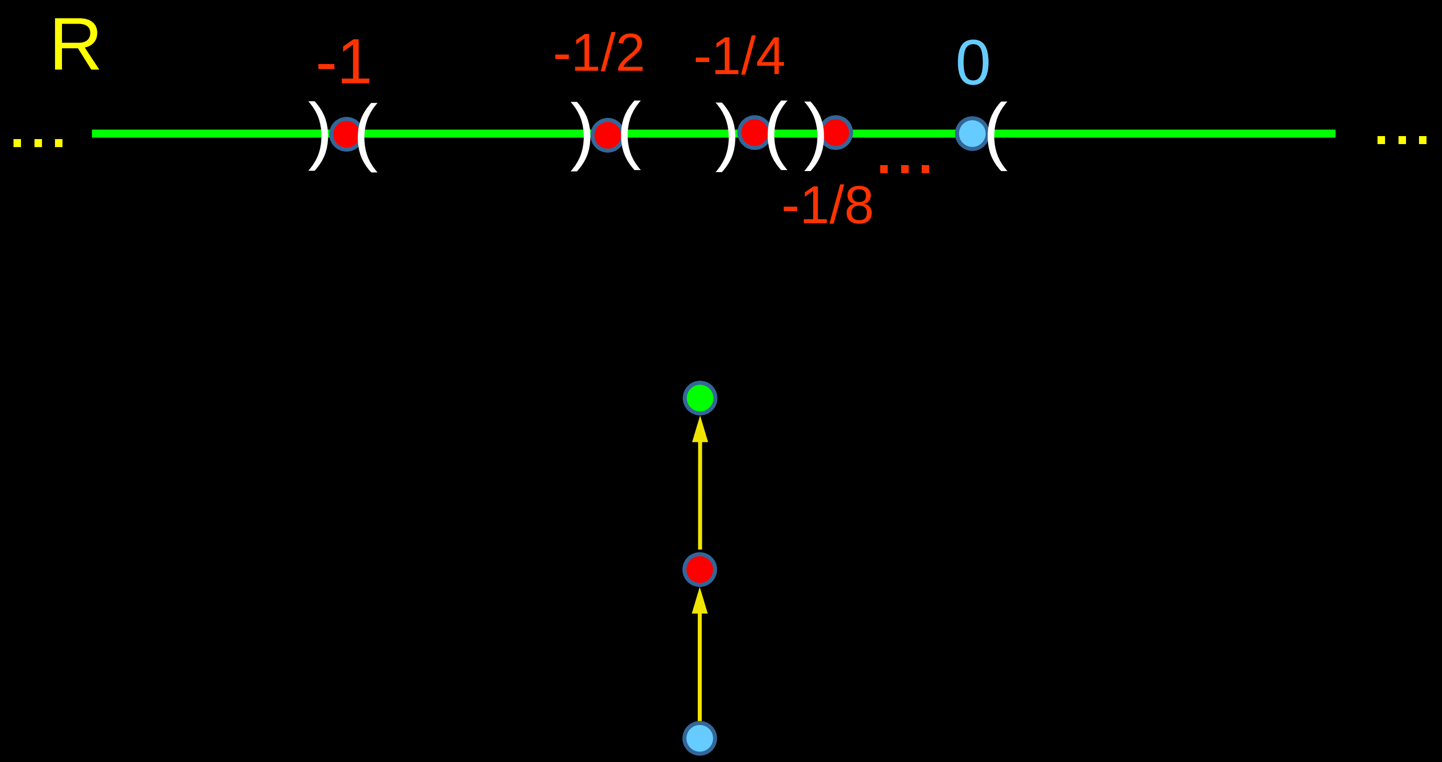
Mapping R onto finite connected quasiorders



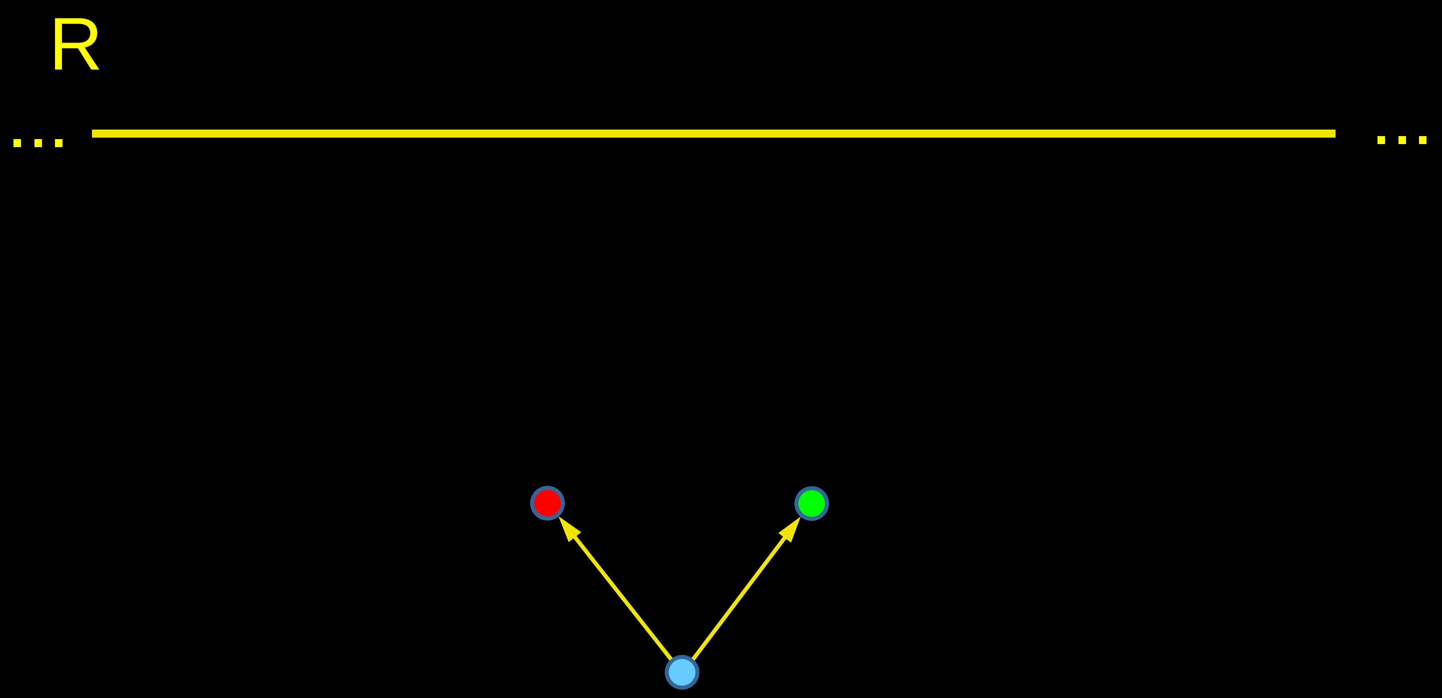
Mapping R onto finite connected quasiorders



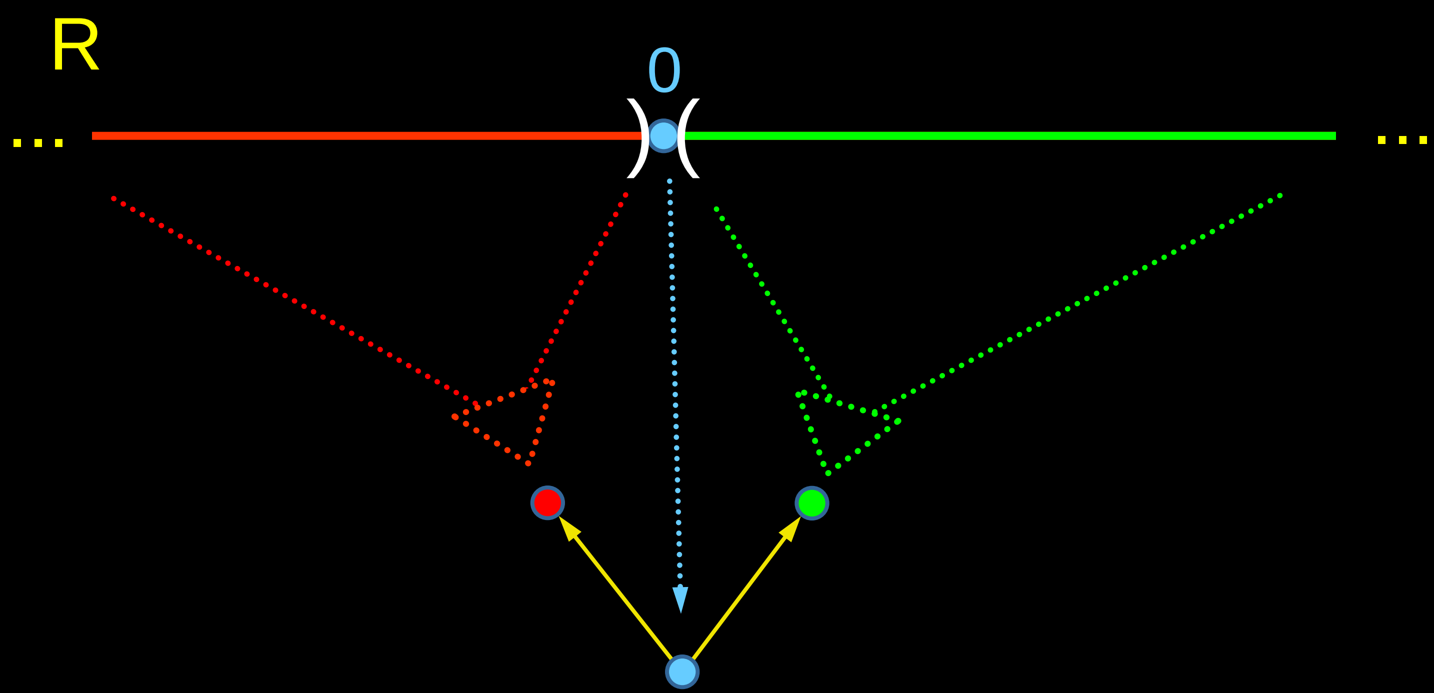
Mapping R onto finite connected quasiorders



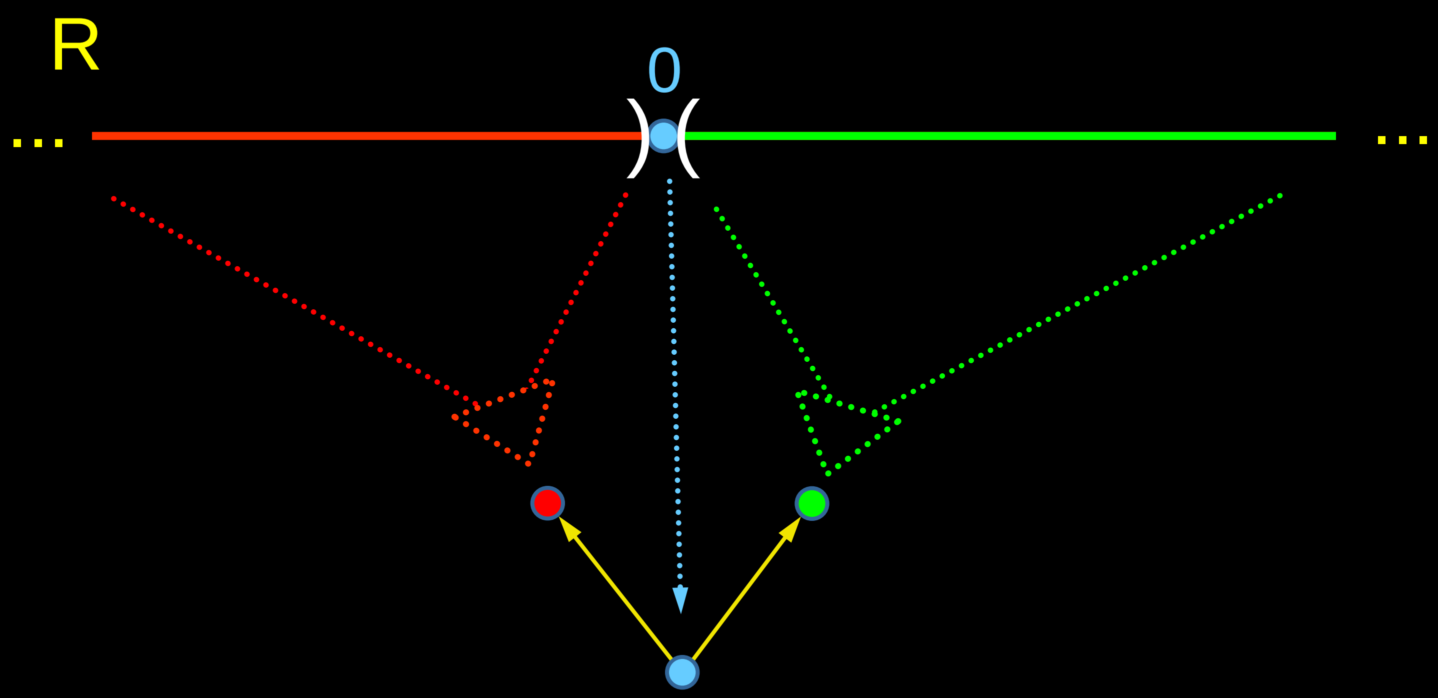
Mapping R onto finite connected quasiorders



Mapping R onto finite connected quasiorders

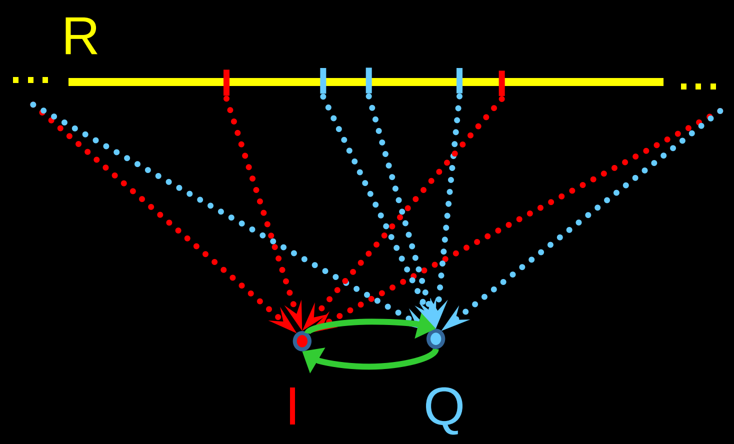


Mapping R onto finite connected quasiorders



We can use this map to falsify formulas on **R**.

Interior fields of sets



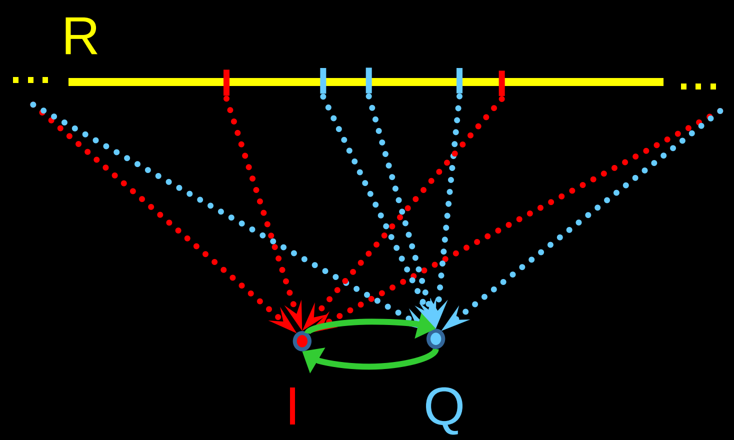
B = {∅, I, Q, R}

□Q = □I = ∅,

□R = R.

(B,□) – Interior Algebra

Interior fields of sets



B = {∅, I, Q, R}

□Q = □I = ∅,

□R = R.

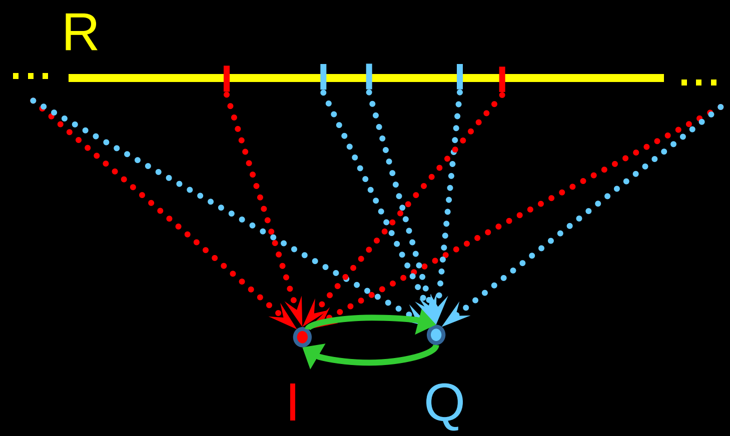
(B,□) – Interior Algebra

◇p→□◇p is valid on B, but not on R.

In R: ∀A ∈ ℘(R).(CA ⊆ ICA) ×

In B: ∀A ∈ B.(CA ⊆ ICA) ✓

Interior fields of sets



$B = \{\emptyset, I, Q, R\}$

$\Box Q = \Box I = \emptyset$,

$\Box R = R$.

$(B,\Box)$ – Interior Algebra

$\Diamond p \rightarrow \Box\Diamond p$ is valid on B, but not on R.

In R: $\forall A \in \wp(R).(CA \subseteq ICA)$ ×

In B: $\forall A \in B.(CA \subseteq ICA)$ ✓

Interior field of sets is a Boolean algebra of subsets which is closed under operators of interior and closure.