Through the Looking-Glass: Unification, Projectivity, and Duality.

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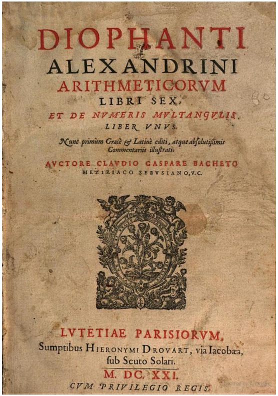
Topology, Algebra and Categories in Logic
Dedicated to the memory of Leo Esakia (1934–2010)
Marseille, 28 July 2011

Prologue

Jacques Herbrand, 1908–1931.

Prologue



Diophantus' Arithmetica, III century AD.
G. Bachet's edition, 1621.

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is an algorithm that yields the **most general solution** to the above system of homogenous equations.
As a **substitution**: $x \mapsto -\frac{3}{2}y$, $y \mapsto y$, $z \mapsto y$.

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A *substitution* is a mapping $\sigma\colon \mathscr{V} \to \mathsf{Term}_{\mathscr{V}}(\mathscr{F})$ that acts identically to within a finite number of exceptions, *i.e.* is such that $\{X \in \mathscr{V} \mid \sigma(X) \neq X\}$ is a finite set.

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The set of equations E axiomatises the *variety of algebras* consisting of the models of the theory E, written $\mathbb{V}_E$.

A *unification problem modulo E* is a **finite** set of pairs

$$\mathscr{E} = \{(s_j, t_j) \mid s_j, t_j \in \mathsf{Term}_{\mathscr{V}}(\mathscr{F})\ ,\ j \in J\},$$

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Definition

The **unification type** of a unifiable problem $\mathscr{E}$ is:

- **unitary**, if $\leqslant$ admits a maximum;
- **finitary**, if $\leqslant$ admits no maximum, but admits finitely many maximal elements such that every unifier for $\mathscr{E}$ lies below one of them;
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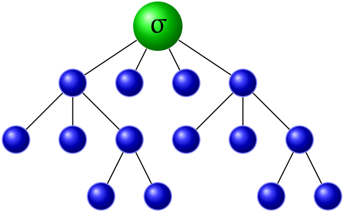

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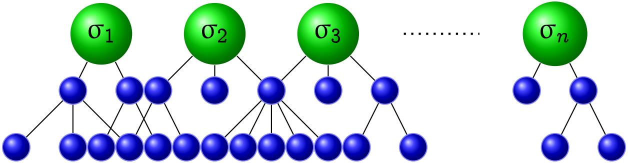

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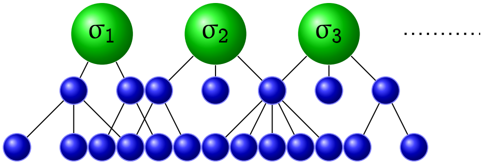

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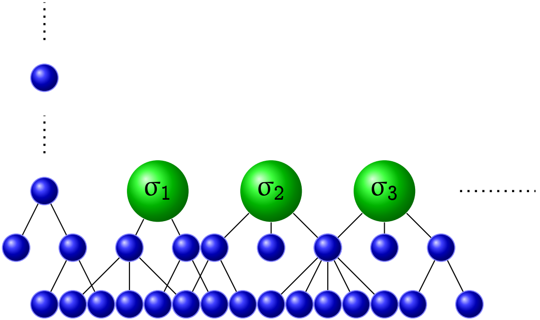

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The list above is arranged in decreasing order of desirability.

- **Unitary**: There is a **most general unifier** (mgu): the set of its lower bouds is the whole poset.
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MV-algebras: next talk by L. Spada.

Distributive lattices: talk by L. Cabrer after Coffee Break.

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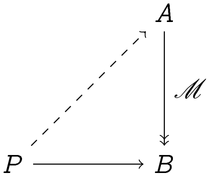

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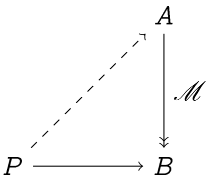

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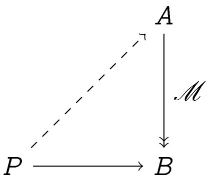

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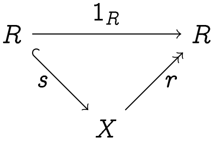

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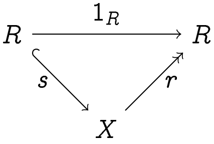

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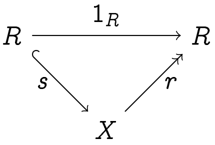

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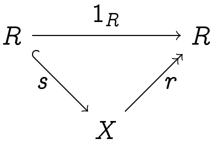

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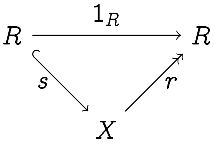

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In **varieties**: retractions are surjections; sections are injections.

Easy facts.

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- In particular, free algebras are projective.

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Thus σ induces a homomorphism from A to a f.p. projective algebra.
Ghilardi's insight was that there is also a converse: *From any such homomorphism one can extract a unifier for $\mathscr{E}$.*

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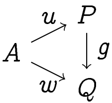

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The relation $\preceq$ is a pre-order on the set $U(A)$ of algebraic unifiers for A. Let $\leqslant$ be the associated partial order. Then we obtain the **poset of algebraic unifiers for A.**

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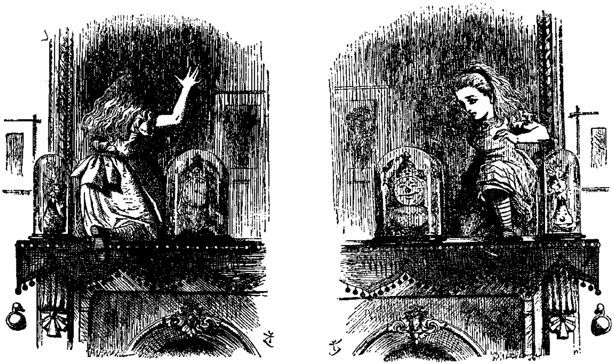
Through the Looking-Glass: Duality



Alice Through the Looking-Glass, Sir J. Tenniel, 1871.

Duality's Grandpa

Marshall Stone, 1903–1989.

$$\mathsf{StoneSp}^{\mathrm{op}} \longleftrightarrow \mathsf{BoolAlg}$$

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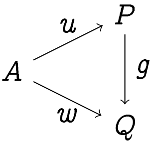

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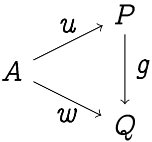

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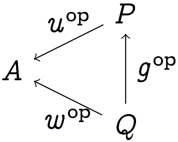

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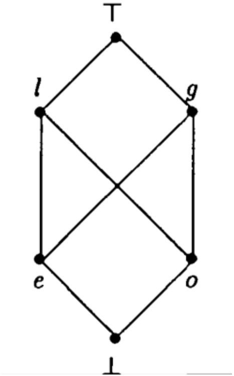
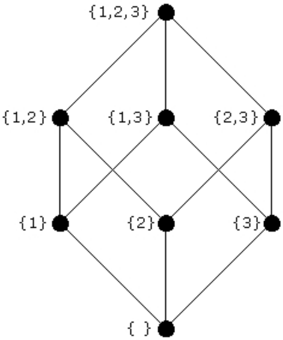
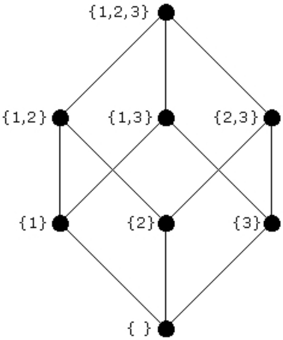


The non-injective poset G.

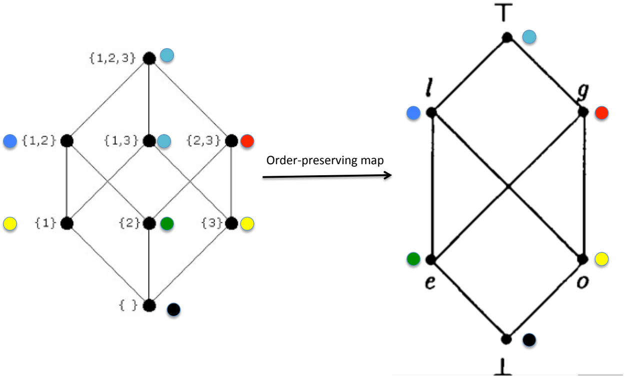


The injective poset $\mathscr{P}(\underline{3})$.



The injective poset $\mathscr{P}(\underline{3})$.

In general: For each integer $n \geqslant 1$, consider the Boolean lattice $\mathscr{P}(\underline{n})$. It is the dual of a projective distributive lattice, by Balbes' Lemma — well, it is the dual of Free_n.



A co-unifier for G.

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- $\perp$, if S is empty;
- $\top$, if S has at least three elements or if it has two elements which are both odd or even;
- g, if S has two elements, say i and j, such that i is odd, j is even and $i > j$;
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In general: For each integer $n \geqslant 1$, define a (clever) o.p. map (=co-unifier) $u_n\colon \mathscr{P}(\underline{n}) \to G$. For each $S \in \mathscr{P}(\underline{n})$, let $u_n(S)$ be:

- $\bot$, if S is empty;
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1. "Projective object" makes sense in any category.
2. "Finitely presented object" does, too, as we now explain.

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I will use directed colimits; the difference with filtered colimits is immaterial. Unraveling the definition yields the following.

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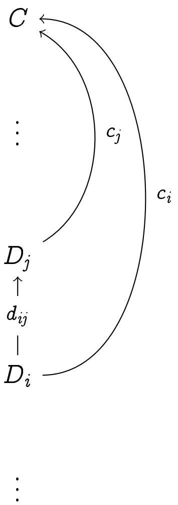

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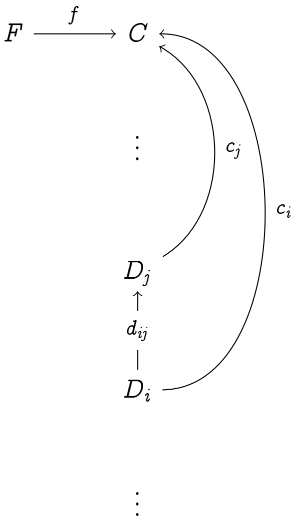

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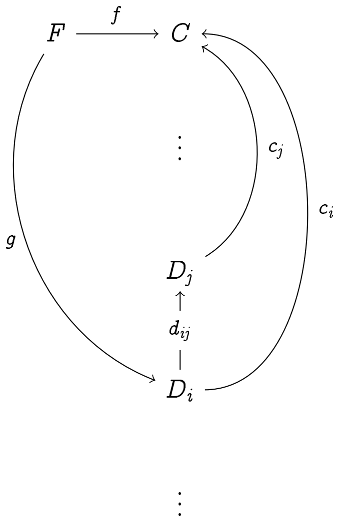

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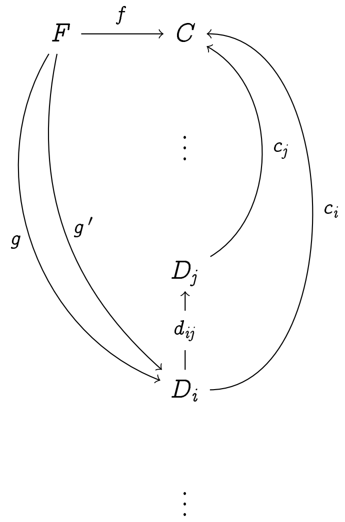

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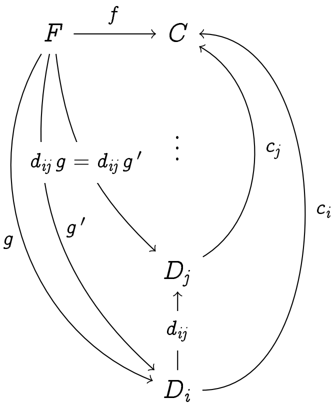

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- The lattice of congruences of any algebra in any variety is algebraic. Compact element=finitely generated congruence. Hence: finitely presentable congruence=finitely generated congruence.
- Finitely presentable topological space=finite and discrete topological space.

Theorem (Gabriel-Ulmer, 1971)

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Caution. This theorem is a **minimal** justification for accepting the Gabriel-Ulmer generalisation: it just says that we are abstracting **one** property of f.p. algebras.

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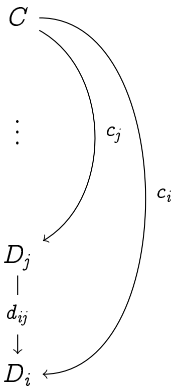

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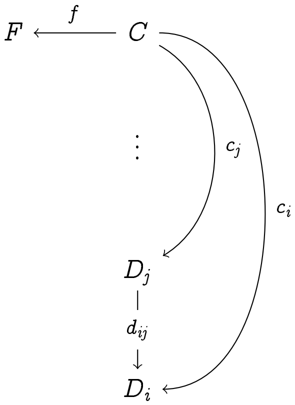

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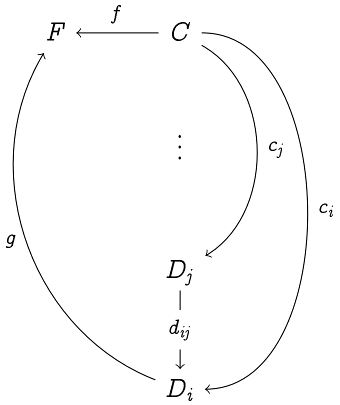

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Compact Hausdorff spaces

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There is no hope to understand the co-unification type of KHaus if we do not address these questions.

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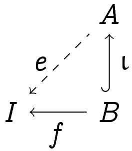

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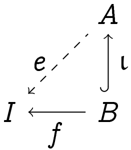

Here, e is just an extension of f from the subspace B to the subspace A.

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The following generalisation of the Tietze Extension Theorem is essentially due to Borsuk:

Injectives in KHaus

The injective objects in KHaus are precisely the retracts of Tychonoff cubes.

Finitely co-presentable compact Hausdorff spaces are not quite as polite. We need to introduce polyhedra, as a preliminary.

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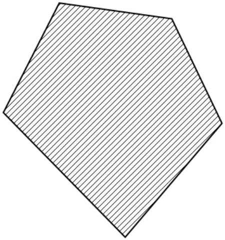
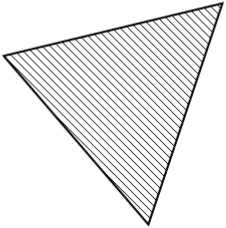
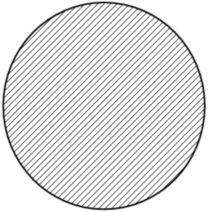


A polytope in $\mathbb{R}^2$*.*



Another such polytope, a good old triangle.



A compact convex set that is not a polytope.

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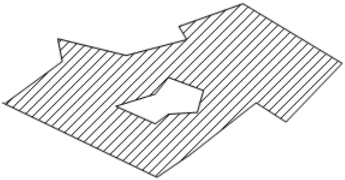

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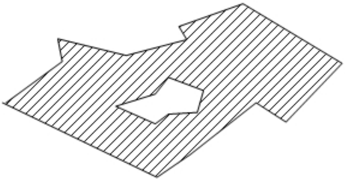

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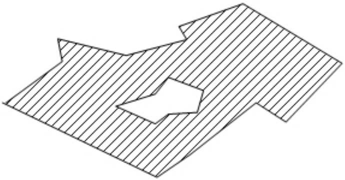

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Any polyhedron is of course a compact Hausdorff space.

Lemma (V.M., unpublished)

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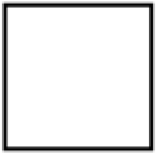
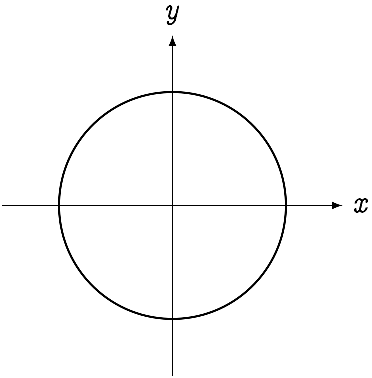
I suspect that the converse of the Lemma holds, too — but do not have a proof of this.

Are finite-dimensional Tychonoff cubes finitely co-presentable?



The Tychonoff square $[-1,1]^2$.

Is the unit circle finitely co-presentable?



The unit circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ *in the plane.*

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If this latter statement were *false*, the Gabriel-Ulmer definition could be seriously questioned — more on this later.

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Hence p_i is a weakly increasing sequence of co-unifiers. So what?

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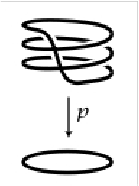
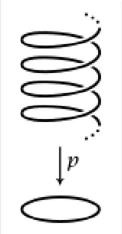
There is an open covering $\{O_i\}$ of X such that, for each i, the inverse image $p^{-1}(O_i)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically by p onto O_i.



A covering space of $\mathbb{S}^1$.

We see that $p \colon \mathbb{R} \to \mathbb{S}^1$ is a covering map upon embedding $\mathbb{R}$ into $\mathbb{R}^3$ as a helix via $t \mapsto (\cos 2\pi t, \sin 2\pi t, t)$.



The universal cover of the circle.

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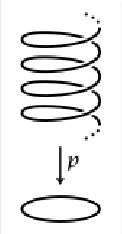
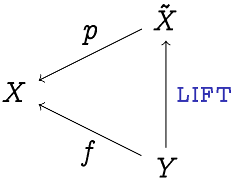


The universal cover of the circle.

Now p acts on the helix simply as the orthogonal projection onto $\mathbb{S}^1$ along the z-axis. So p indeed is a covering map.

The Lifting Lemma

Suppose $p\colon \tilde{X} \to X$ is any covering map, and $f\colon Y \to X$ is a continuous map. If Y is a retract of $[-1,1]^n$, then there exists a **lift** of f to $\tilde{X}$.



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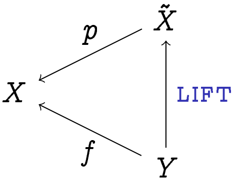

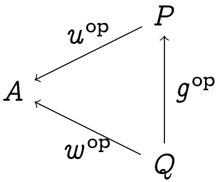

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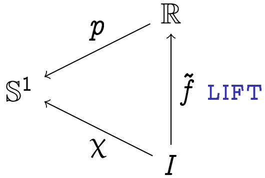

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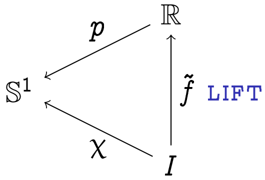

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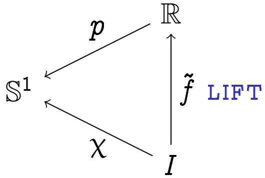

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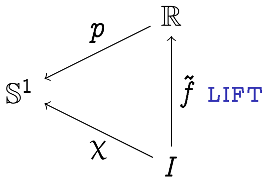

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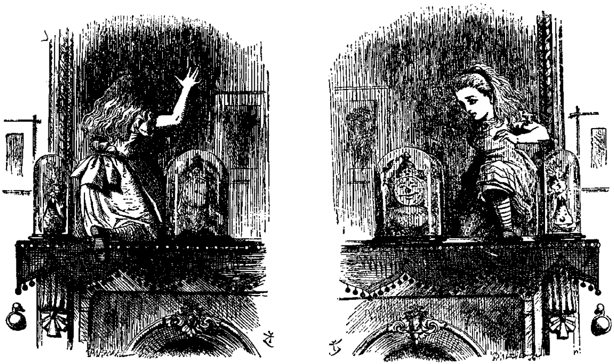
So f factors through one of the p_i's, hence p_i is co-final.
Similar (easier) argument: $p_i \prec p_{i+1}$, hence type 0. **Q.E.D.**

C*-algebras: Through the Looking-Glass, again



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Operations are defined pointwise:

- $f + g$ is given by $(f + g)(x) = f(x) + g(x)$ for all $x \in X$.
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A (real) **C*-algebra** is any commutative ring with unit that is isomorphic to one of the form $\mathrm{C}(X)$, for some compact Hausdorff space X.

C*-algebras can be axiomatized at higher order.

A commutative ring $(C, +, \cdot, 0, 1)$ with unit 1 is C*-algebra if, and only if, the following hold.

1. The Abelian group $(C, +, 0)$ is divisible and torsion free (=a $\mathbb{Q}$-algebra).
2. There exists a partial order on C making it a partially ordered ring in which squares are positive.
3. Some multiple of the unit 1 is larger than any given element.
4. The order is Archimedean (=no infinitesimals): if $1 \geqslant nx$ for all positive integers n, then $x \leqslant 0$.
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We will consider C*-algebras as a full subcategory of commutative rings with unit: their morphisms are just ring homomorphisms.

Grandpa Stone strikes again.

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Sadly, C*-algebras as just defined are **not** just sets with operations: think of the norm-completeness condition.

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References.

E. Manes, *Algebraic Theories*, Springer 1976.

J. Adámek, J. Rosický, E. Vitale, *Algebraic Theories*, CUP 2010.

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What the theorem says is that our chosen **presentation** of KHaus$^{\mathrm{op}}$ is not phrased in terms of operations merely because **we are not using the right, algebraic language**.

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Isbell's result amounts to the following.

Let $f \colon [-1, 1]^{\kappa} \to [-1, 1]$ be any continuous function. Then f can be obtained by a finite number of applications of the Isbell operations, starting form the projection functions $\pi_\alpha(\,(x_\alpha)_{\alpha<\kappa}\,) = x_\alpha$.

Intuition. Choose functions $f_i\colon [-1,1]^\kappa \to [-1,1]$. The series $\iota(f_1, f_2, \ldots)$ converges to a function, and that function is continuous because the series is in fact *uniformly* convergent. (Remember we interpret over $[-1,1]$.)

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Unification problems, substitutions, unifiers, unification type etc. are now defined as before.

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Type zero for C (X) (V.M., unpublished).

The unification problem $s(x, y) = 1$ has type zero: in fact, modulo the algebraic theory of C (X), any unifier can be obtained from another, strictly more general unifier, by instantiation. (That is, no unifier is maximally general.)

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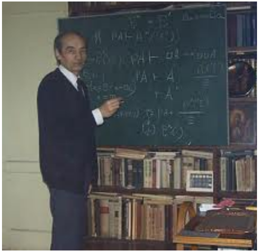
- Unification beyond equational theories is almost entirely unexplored. Why not remedy that.
- More generally: Although first order has its celebrated merits, there is much mathematics at higher order that awaits our attention of logicians and algebraists. (Cf. O. Caramello's talk.)



To Leo Esakia, In Memoriam.

Thank you for your attention.