The possible values of critical points between varieties of algebras

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$$\Psi(A) = \Psi(\varinjlim \vec{A}) \cong \varinjlim(\Psi \circ \vec{A}) \cong \varinjlim(\vec{S}) \cong S$$

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- This theorem allows to find a lattice, of cardinality $\aleph_1$, with no congruence-preserving extension.
- We can also compare the ranges of two functors $\Phi\colon \mathcal{A} \to \mathcal{S}$ and $\Psi\colon \mathcal{B} \to \mathcal{S}$.

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- In order to prove (1), we only need to prove (2) for diagrams of small objects.
- Or we prove (2) fails, and deduce the guess is wrong (i.e. (1) fails).

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- A (1) follows from B (1) and A (2) follows from B (2).

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Boolean algebras

- We prove B (1) : Diagrams of finite Boolean algebras, indexed by ω, are in the range of Ψ.
- Let $\vec{S} = (S_n, \sigma_n^m \mid m \leq n \text{ in } \omega)$ be a diagram of finite Boolean algebras.
- Let C_0 be a maximal chain of S_0.
- Let C_1 be a maximal chain of S_1 that contains $\sigma_0^1(C_0)$.
- Let C_2 be a maximal chain of S_2 that contains $\sigma_1^2(C_1)$.
- and so on...
- We obtain a diagram $\vec{C}$ with $\Psi \circ \vec{C} \cong \vec{S}$, as wanted.
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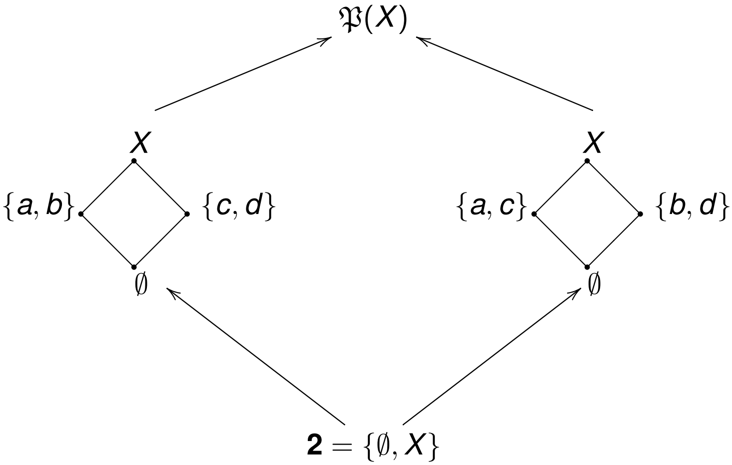
We prove B (2). Let $X = \{a, b, c, d\}$ be a four elements set. We consider the following diagram of Boolean algebras



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The congruence functor

- For $f\colon A \to B$ a morphism of algebras. We put :

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- $\operatorname{Con_c} f$ is a morphism of semilattices.
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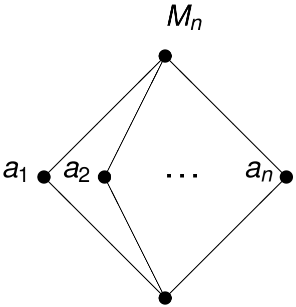

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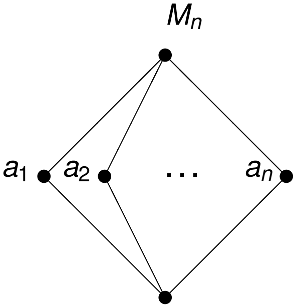

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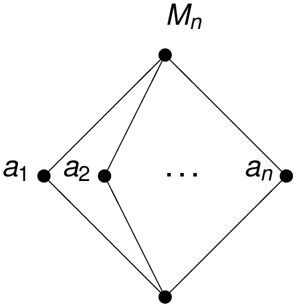

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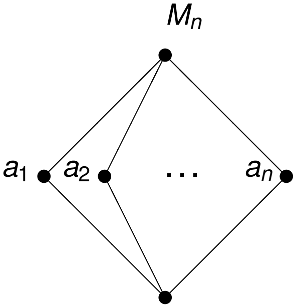

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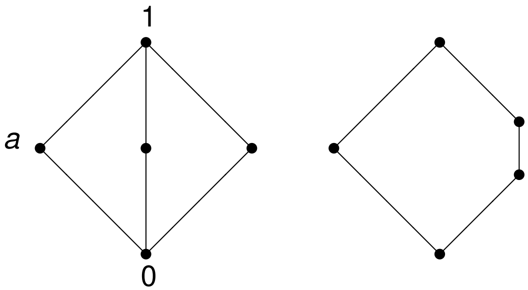

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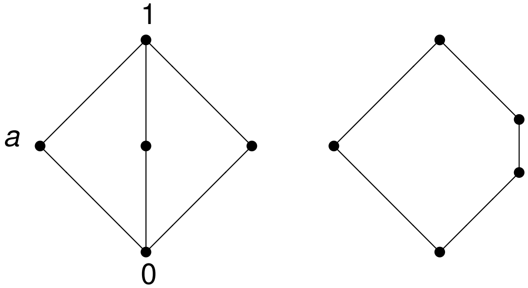

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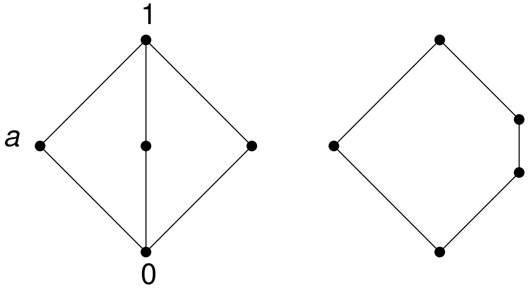

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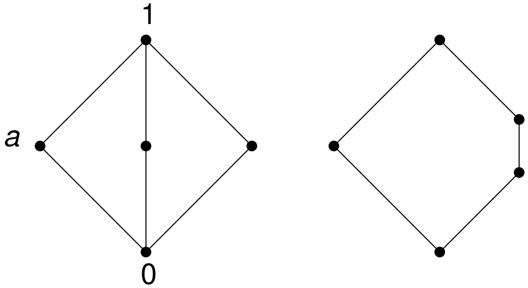

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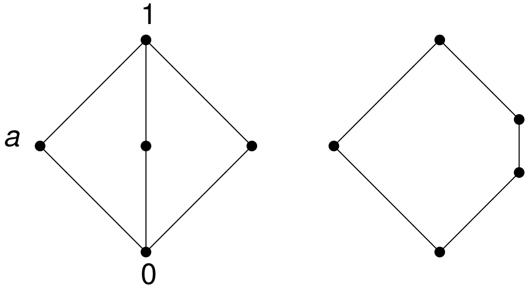

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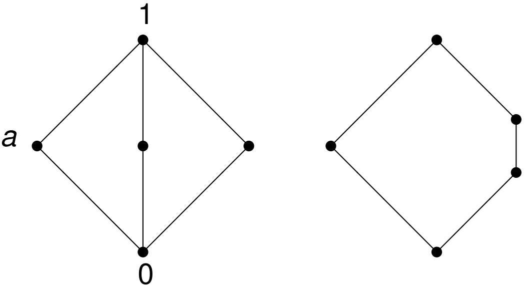

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Then $\mathrm{Con_c}\, \mathcal{V} \subseteq \mathrm{Con_c}\, \mathcal{W}$.

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- But for lattices we can do more.

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- There is a condensate A of $\vec{A}$ of cardinality $\aleph_2$ such that $\operatorname{Con_c} A$ has no lifting in $\mathcal{W}$.

That is all

Thank you for your attention.
Any questions ?