Relativizing the substructural hierarchy.

[Partly based on joint work with a) A. Ciabattoni, K. Terui, b) P. Jipsen, c) R. Horčík.]

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Substructiral logics and residuated lattices

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Their algebraic semantics are *residuated lattices*, and include Boolean algebras, Heyting algebras, MV-algebras, but also lattice-ordered groups.

Outline

Starting from the algebraic properties of residuated lattices, we will:

- Rediscover the substructural hierarchy (Ciabattoni-NG-Terui)
- Rediscover the sequent calculus for FL, and the hypersequent calculus (Avron, Ciabattoni-NG-Terui)
- Rediscover residuated frames (NG-Jipsen)
- Relativize the hierarchy/calculus/frames for the involutive (classical), and distributive cases (NG-Jipsen)
- Survey some recent results
  - cut elimination (admissibility) for FL, InFL, DFL, HFL, HDFL and extensions
- Also, prove two new results
  - FEP for IDFL and extensions (NG)
  - cut elimination for HDFL and extensions (Ciabattoni-NG-Terui)

Residuated lattices

A *residuated lattice*, or *residuated lattice-ordered monoid*, [Blount and Tsinakis] is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ such that

- $\langle A, \wedge, \vee\rangle$ is a lattice,
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Fact. The last condition is equivalent to either one of:

- Multiplication distributes over existing $\bigvee$'s and, for all $a, c \in A$, both $\bigvee\{b : ab \leq c\}$ $(=: a\backslash c)$ and $\bigvee\{b : ba \leq c\}$ $(=: c/a)$ exist.
- (For complete lattices) $\cdot$ distributes over $\bigvee$. [Quantales]
- For all $a, b, c \in A$, $\begin{array}{ll} b \leq a\backslash(ab \vee c) & a \leq (c \vee ab)/b \\ a(a\backslash c \wedge b) \leq c & (a \wedge c/b)b \leq c \end{array}$

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So, residuated lattices form an equational class/variety.

Pointed residuated lattices are expansions with a constant 0. This allows us to define two negations: $\sim x = x\backslash 0$ and $-x = 0/x$.

Examples

- Boolean algebras. $x/y = y\backslash x = y \to x = \neg y \vee x$ and $x \cdot y = x \wedge y$.
- MV-algebras. For $x \cdot y = x \odot y$ and $x\backslash y = y/x = \neg(\neg x \odot y)$.
- Lattice-ordered groups. For $x\backslash y = x^{-1}y$, $y/x = yx^{-1}$; $\neg x = x^{-1}$.
- (Reducts of) relation algebras. For $x \cdot y = x;y$, $x\backslash y = (x^{\cup};y^c)^c$, $y/x = (y^c;x^{\cup})^c$, $1 = id$.
- Ideals of a ring (with 1), where $IJ = \{\sum_{fin} ij \mid i \in I, j \in J\}$ $I/J = \{k \mid kJ \subseteq I\}$, $J\backslash I = \{k \mid Jk \subseteq I\}$, $1 = R$.
- Quantales $(Q, \bigvee, \cdot, 1)$ are (definitionally equivalent) complete residuated lattices.
- The powerset $\langle \mathcal{P}(M), \cap, \cup, \cdot, \backslash, /, \{e\}\rangle$ of a monoid $\mathbf{M} = \langle M, \cdot, e\rangle$, where $X \cdot Y = \{x \cdot y \mid x \in X, y \in Y\}$, $X/Y = \{z \in M \mid \{z\} \cdot Y \subseteq X\}$, $Y\backslash X = \{z \in M \mid Y \cdot \{z\} \subseteq X\}$.

Bi-modules

- $x1 = x = 1x$, $(xy)z = x(yz)$
- $x(y \vee z) = xy \vee xz$ and $(y \vee z)x = yx \vee zx$

So, if $\mathbf{P}$ is a residuated lattice, then $(P, \vee, \cdot, 1)$ is a semiring. [In the complete case, a quantale.]

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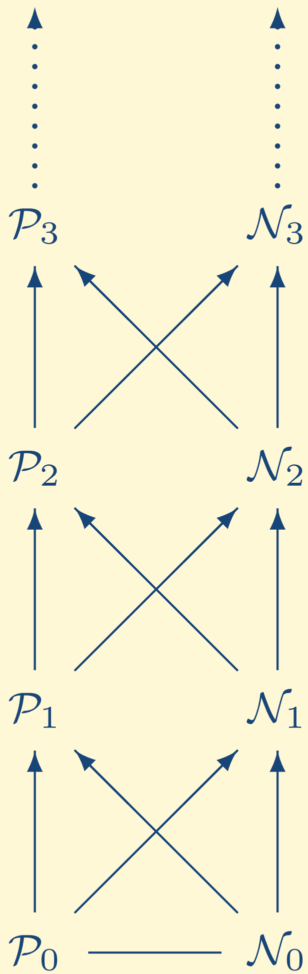
Formula hierarchy



- The sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas are defined by:
  - (0) $\mathcal{P}_0 = \mathcal{N}_0 =$ the set of variables
  - (P1) $\mathcal{N}_n \subseteq \mathcal{P}_{n+1}$
  - (P2) $\alpha, \beta \in \mathcal{P}_{n+1} \;\Rightarrow\; \alpha \vee \beta, \alpha \cdot \beta, 1 \in \mathcal{P}_{n+1}$
  - (N1) $\mathcal{P}_n \subseteq \mathcal{N}_{n+1}$
  - (N2) $\alpha, \beta \in \mathcal{N}_{n+1} \;\Rightarrow\; \alpha \wedge \beta \in \mathcal{N}_{n+1}$
  - (N3) $\alpha \in \mathcal{P}_{n+1}, \beta \in \mathcal{N}_{n+1} \;\Rightarrow\; \alpha \backslash \beta, \beta / \alpha, 0 \in \mathcal{N}_{n+1}$
- $\mathcal{P}_{n+1} = \langle \mathcal{N}_n \rangle_{\vee, \prod}$; $\mathcal{N}_{n+1} = \langle \mathcal{P}_n \rangle_{\wedge, \mathcal{P}_{n+1} \backslash, / \mathcal{P}_{n+1}}$
- $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}, \mathcal{N}_n \subseteq \mathcal{N}_{n+1}, \bigcup \mathcal{P}_n = \bigcup \mathcal{N}_n = Fm$
- $\mathcal{P}_1$-reduced: $\bigvee \prod p_i$
- $\mathcal{N}_1$-reduced: $\bigwedge (p_1 p_2 \cdots p_n \backslash r / q_1 q_2 \cdots q_m)$
  $p_1 p_2 \cdots p_n q_1 q_2 \cdots q_m \leq r$
- Sequent: $a_1, a_2, \ldots, a_n \;\Rightarrow\; a_0 \; (a_i \in Fm)$

A. Ciabattoni, NG, K. Terui. From axioms to analytic rules in nonclassical logics, Proceedings of LICS'08, 229-240, 2008.

Submodules and nuclei

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If $P = N$ is the underlying set of a residuated lattice $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$, a nucleus is just a closure operator that satisfies $\gamma(x) \cdot \gamma(y) \leq \gamma(x \cdot y)$. (Cf. phase spaces.)

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All complete RLs arise as submodules of $\mathcal{P}(\mathbf{M})$, where $\mathbf{M}$ is a monoid, namely via nuclei on powersets (of monoids). (Each RL can be embedded into a complete one.) *Residuated frames* arise from studying submodules of $\mathcal{P}(\mathbf{M})$.They form relational semantics for substructural logics and are the most important tool in Algebraic Proof Theory.

Lattice frames

A *lattice frame* is a structure $\mathbf{F} = (L, R, N)$ where L and R are sets and N is a binary relation from L to R.

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For $X \subseteq L$ and $Y \subseteq R$ we define

$$X^{\rhd} = \{b \in R : x \; N \; b, \text{ for all } x \in X\}$$
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The maps ${}^{\rhd} : \mathcal{P}(L) \to \mathcal{P}(R)$ and ${}^{\lhd} : \mathcal{P}(R) \to \mathcal{P}(L)$ form a Galois connection. The map $\gamma_N : \mathcal{P}(L) \to \mathcal{P}(L)$, where $\gamma_N(X) = X^{\rhd\lhd}$, is a closure operator.

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If $\mathbf{A}$ is a lattice, $\mathbf{F_A} = (A, A, \leq)$ is a lattice frame.

For $X \subseteq L$ and $Y \subseteq R$ we define

$$X^{\triangleright} = \{b \in R : x \; N \; b, \text{ for all } x \in X\}$$
$$Y^{\triangleleft} = \{a \in L : a \; N \; y, \text{ for all } y \in Y\}$$

The maps ${}^{\triangleright} : \mathcal{P}(L) \to \mathcal{P}(R)$ and ${}^{\triangleleft} : \mathcal{P}(R) \to \mathcal{P}(L)$ form a Galois connection. The map $\gamma_N : \mathcal{P}(L) \to \mathcal{P}(L)$, where $\gamma_N(X) = X^{\triangleright\triangleleft}$, is a closure operator.

Lemma. If $\mathbf{A} = (A, \wedge, \vee)$ is a lattice and γ is a cl.op. on $\mathbf{L}$, then $(\gamma[A], \wedge, \vee_\gamma)$ is a lattice. [$x \vee_\gamma y = \gamma(x \vee y)$.]

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Corollary. If $\mathbf{F}$ is a lattice frame then the *Galois algebra* $\mathbf{F}^+ = (\gamma_N[\mathcal{P}(L)], \cap, \cup_{\gamma_N})$ is a complete lattice.

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If $\mathbf{A}$ is a lattice, $\mathbf{F}^+_{\mathbf{A}}$ is the Dedekind-MacNeille completion of $\mathbf{A}$ and $x \mapsto \{x\}^{\lhd}$ is an embedding.

A *residuated frame* is a structure $\mathbf{F} = (L, R, N, \circ, \varepsilon, \backslash\!\!\backslash, /\!\!/)$ where

- (L, R, N) is a lattice frame,
- $(L, \circ, \varepsilon)$ is a monoid and
- $\backslash\!\!\backslash : L \times R \to R$, $/\!\!/ : R \times L \to R$ are such that

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N. Galatos and P. Jipsen. Residuated frames and applications to decidability, to appear in the Transactions of the AMS.

$$\frac{xNa \quad aNz}{xNz}\ \text{(CUT)} \qquad \frac{}{aNa}\ \text{(Id)}$$

GN

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$$\frac{xNa \quad bNz}{x \circ (a\backslash b)Nz} \qquad \frac{xNa \quad bN(v \mathbin{\backslash\!\backslash} c \mathbin{/\!/} u)}{x \circ (a\backslash b)N(v \mathbin{\backslash\!\backslash} c \mathbin{/\!/} u)}$$

$$\frac{x N a \quad a N z}{x N z}\ (\mathsf{CUT}) \qquad \frac{}{a N a}\ (\mathsf{Id})$$

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$$\frac{x N a \quad b N z}{x \circ (a \backslash b) N z} \qquad \frac{x N a \quad b N (v \,\backslash\!\backslash\, c \,/\!/\, u)}{x \circ (a \backslash b) N (v \,\backslash\!\backslash\, c \,/\!/\, u)} \qquad \frac{x N a \quad v \circ b \circ u N c}{v \circ x \circ (a \backslash b) \circ u N c}$$

So, we get the sequent calculus $\mathbf{FL}$, for $a, b, c \in Fm$, $x, y, u, v \in Fm^*$, $z \in S_L \times Fm$.

FL

$$\frac{x \Rightarrow a \quad y \circ a \circ z \Rightarrow c}{y \circ x \circ z \Rightarrow c} \text{ (cut)} \qquad \frac{}{a \Rightarrow a} \text{ (Id)}$$

$$\frac{y \circ a \circ z \Rightarrow c}{y \circ a \wedge b \circ z \Rightarrow c} (\wedge \mathrm{L}\ell) \quad \frac{y \circ b \circ z \Rightarrow c}{y \circ a \wedge b \circ z \Rightarrow c} (\wedge \mathrm{L}r) \quad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b} (\wedge \mathrm{R})$$

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$$\frac{x \Rightarrow a \quad y \circ b \circ z \Rightarrow c}{y \circ x \circ (a \backslash b) \circ z \Rightarrow c} (\backslash \mathrm{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a \backslash b} (\backslash \mathrm{R})$$

$$\frac{x \Rightarrow a \quad y \circ b \circ z \Rightarrow c}{y \circ (b/a) \circ x \circ z \Rightarrow c} (/\mathrm{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a} (/\mathrm{R})$$

$$\frac{y \circ a \circ b \circ z \Rightarrow c}{y \circ a \cdot b \circ z \Rightarrow c} (\cdot \mathrm{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b} (\cdot \mathrm{R})$$

$$\frac{y \circ z \Rightarrow a}{y \circ 1 \circ z \Rightarrow a} (1\mathrm{L}) \qquad \frac{}{\varepsilon \Rightarrow 1} (1\mathrm{R})$$

where $a, b, c \in Fm$, $x, y, z \in Fm^*$.

Gentzen frames

$\mathbf{F_{FL}}$ is the free frame generated by the formulas Fm ($L = Fm^*$, $R = S_L \times Fm$), whith $x \; N \; (u, a)$ iff $\vdash_{\mathbf{FL}} u(x) \; \Rightarrow \; a$.

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The following properties hold for $\mathbf{F_A}$, $\mathbf{F_{FL}}$ (and $\mathbf{F_{A,B}}$, later):

1. $\mathbf{F}$ is a residuated frame (freely) generated by B
2. $\mathbf{B}$ is a (partial) algebra of the same type, ($\mathbf{B} = \mathbf{A}, \mathbf{Fm}$)
3. N satisfies $\mathbf{GN}$, for all $a, b \in B$, $x, y \in L$, $z \in R$.

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We call such pairs $(\mathbf{F}, \mathbf{B})$ *Gentzen frames*. A *cut-free Gentzen frame* is not assumed to satisfy the (CUT)-rule.

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Theorem. (NG-Jipsen) Given a Gentzen frame $(\mathbf{F}, \mathbf{B})$, the map $\{\}^{\triangleleft} : \mathbf{B} \to \mathbf{F}^+, \;\; b \mapsto \{b\}^{\triangleleft}$ is a (partial) homomorphism.
(Namely, if $a, b \in B$ and $a \bullet b \in B$ ($\bullet$ is a connective) then $\{a \bullet_{\mathbf{B}} b\}^{\triangleleft} = \{a\}^{\triangleleft} \bullet_{\mathbf{F}^+} \{b\}^{\triangleleft}$).

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For cut-free Genzten frames, we get only a *quasihomomorphism*.
$a \bullet_{\mathbf{B}} b \in \{a\}^{\lhd} \bullet_{\mathbf{F}^+} \{b\}^{\lhd} \subseteq \{a \bullet_{\mathbf{B}} b\}^{\lhd}$.

Completeness - Cut elimination

For every homomorphism $f : \mathbf{Fm} \to \mathbf{B}$, let $\bar{f} : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{F}^{+}$ be the homomorphism that extends $\bar{f}(p) = \{f(p)\}^{\triangleleft}$ (p: variable.)

Corollary. If $(\mathbf{F}, \mathbf{B})$ is a cf Gentzen frame, for every homomorphism $f : \mathbf{Fm} \to \mathbf{B}$, we have $f(a) \in \bar{f}(a) \subseteq \{f(a)\}^{\triangleleft}$. If we have (CUT), then $\bar{f}(a) = \downarrow f(a)$.

We define $\mathbf{F} \models x \ \Rightarrow \ c$ by $f(x) \ N \ f(c)$, for all f.

Theorem. If $\mathbf{F}^{+}_{\mathbf{FL}} \models x^{\cdot} \leq c$, then $\mathbf{F}_{\mathbf{FL}} \models x \ \Rightarrow \ c$.
Idea: For $f : \mathbf{Fm} \to \mathbf{B}$, $f(x) \in \bar{f}(x) \subseteq \bar{f}(c) \subseteq \{f(c)\}^{\triangleleft}$, so $f(x) \ N \ f(c)$.

Corollary. $\mathbf{FL}$ is complete with respect to $\mathbf{F}^{+}_{\mathbf{FL}}$.

Corollary (CE). $\mathbf{FL}$ and $\mathbf{FL}^{\mathbf{f}}$ prove the same sequents.

Theorem. (Ciabattoni-NG-Terui) For axioms in $\mathcal{N}_2$, the extension of $\mathbf{FL}$ is equivalent to one that admits (modular, infinitary) cut elimination iff the corresponding variety is closed under (MacNeille) completions iff the axiom is *acyclic*.

Frame applications

- DM-completion
- Completeness of the calculus
- Cut elimination
- Finite model property
- Finite embeddability property
- (Generalized super-)amalgamation property (Transferable injections, Congruence extension property)
- (Craig) Interpolation property
- Disjunction property
- Strong separation
- Stability under linear structural rules/equations over $\{\vee, \cdot, 1\}$.

NG and H. Ono, APAL.

NG and P. Jipsen, TAMS.

NG and P. Jipsen, manuscript.

A. Ciabattoni, NG and K. Terui, APAL.

NG and K. Terui, manuscript.

Idea: Express equations over $\{\vee, \cdot, 1\}$ at the frame level.

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$$\frac{x_1 y \leq z \quad x_2 y \leq z \quad y x_1 \leq z \quad y x_2 \leq z}{x_1 x_2 y \leq z}$$

$$\frac{x_1 \circ y \; N \; z \quad x_2 \circ y \; N \; z \quad y \circ x_1 \; N \; z \quad y \circ x_2 \; N \; z}{x_1 \circ x_2 \circ y \; N \; z} \; R(\varepsilon)$$

Simple rules

In the context of $(\mathbf{F_{FL}}, \mathbf{Fm})$, $R(\varepsilon)$ takes the form

$$\frac{x \circ t_1 \circ y \ \Rightarrow \ a \quad \cdots \quad x \circ t_n \circ y \ \Rightarrow \ a}{x \circ t_0 \circ y \ \Rightarrow \ a} \ (R(\varepsilon))$$

We call such equations and rules simple.

Theorem. Let $(\mathbf{F}, \mathbf{B})$ be a cf Gentzen frame and let ε be a $\{\vee, \cdot, 1\}$-equation. Then $(\mathbf{F}, \mathbf{B})$ satisfies R(ε) iff $\mathbf{F}^+$ satisfies ε.

Theorem. All extensions of $\mathbf{FL}$ by simple rules enjoy cut elimination.

K. Terui. Which structural rules admit cut elimination? An algebraic criterion. J. Symbolic Logic 72 (2007), no. 3, 738-754.

N. Galatos and H. Ono. Cut elimination and strong separation for non-associative substructural logics, APAL 161(9) (2010), 1097–1133.

N. Galatos and P. Jipsen. Residuated frames and applications to decidability, to appear in the Transactions of the AMS.

Theorem. Every variety $\mathcal{V}$ of integral RL's $(x \leq 1)$ axiomatized by equartions over $\{\vee, \cdot, 1\}$ has the *finite embeddability property (FEP)*, namely for every $\mathbf{A} \in \mathcal{V}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{V}$.

The frame $\mathbf{F}_{\mathbf{A},\mathbf{B}}$ is generated by B (L is the submonoid of $\mathbf{A}$ generated by B, $R = S_L \times B$) with $x \; N \; (u, b)$ iff $u(x) \leq_{\mathbf{A}} b$.

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- $\mathbf{F}^+_{\mathbf{A},\mathbf{B}} \in \mathcal{V}$
- $\mathbf{B}$ embeds in $\mathbf{F}^+_{\mathbf{A},\mathbf{B}}$ via $\{\_\}^{\lhd} : \mathbf{B} \to \mathbf{F}^+$
- $\mathbf{F}^+_{\mathbf{A},\mathbf{B}}$ is finite

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Hypersequents

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For every rule

$$\frac{s_1 \quad s_2}{s}$$

of $\mathbf{FL}$, the system $\mathbf{HFL}$ is defined to contain the rule

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where H is a (meta)variable for hyprsequents. A hyperstructural rule is of the form

$$\frac{H \mid s_1' \quad H \mid s_2' \quad \dots \quad H \mid s_n'}{H \mid s_1 \mid \cdots \mid s_m}$$

Hyper-frames

A *hyperresiduated frame* $\mathbf{H} = (L, R, \vdash, \circ, \varepsilon, \backslash\!\backslash, /\!/, \epsilon)$ is defined by

- $\vdash \subseteq H = (L \times R)^*$. We write $\vdash h$ instead of $h \in \vdash$.
- $(L, \circ, \varepsilon)$ is a monoid and $\epsilon \in R$.
- For all $x, y \in L$, $z \in R$, $h \in H$,

$$\vdash (x \circ y, z) \mid h \iff \vdash (y, x \backslash\!\backslash z) \mid h \iff \vdash (x, z /\!/ y) \mid h.$$

- $\vdash h$ implies $\vdash (x, y) \mid h$ for any $(x, y) \in L \times R$.
- $\vdash (x, y) \mid (x, y) \mid h$ implies $\vdash (x, y) \mid h$ for any $(x, y) \in L \times R$.

We define $r(\mathbf{H}) = (L \times H, R \times H, N, \bullet, (\varepsilon; \emptyset), (\epsilon; \emptyset))$, where $H = (L \times R)^*$. Then $r(\mathbf{H})$ is a residuated frame. We define $\mathbf{H}^+ = r(\mathbf{H})^+$. The *hyper-MacNeille completion* of a residuated lattice $\mathbf{A}$ is $\mathbf{H}^+_{\mathbf{A}}$.

$$\begin{aligned}
(x; h_1) \bullet (y; h_2) &= (x \circ y; h_1 \mid h_2) \\
(x; h_1) \backslash\!\backslash (z; h_2) &= (x \backslash\!\backslash z; h_1 \mid h_2) \\
(z; h_2) /\!/ (x; h_1) &= (z /\!/ x; h_1 \mid h_2) \\
(x; h_1)\ N\ (z; h_2) &\iff \vdash (x, z) \mid h_1 \mid h_2.
\end{aligned}$$

Example. Based on $\mathbf{HFL}$ we define a hyperresiduated frame $\mathbf{H_{HFL}} = (W, W', \vdash, \circ, \varepsilon, \epsilon)$, where

$$\vdash s_1 \mid \ldots \mid s_n \iff \vdash_{\mathbf{HFL}} s_1 \mid \cdots \mid s_n$$

Using the cut-free version of this frame, we can prove cut elimination for $\mathbf{HFL}$.

The Dedekind-MacNeille and the hyper-Dedekind-MacNeille completions for $\mathcal{N}_2$ and $\mathcal{P}_3$ correspond in a strong way to modular cut elimination and to conservativity of the infinitary logic.

A. Ciabattoni, NG, K. Terui. From axioms to analytic rules in nonclassical logics, Proceedings of LICS'08, 229-240, 2008.

A. Ciabattoni, NG, K. Terui. Algebraic proof theory for substructural logics: cut elimination and completions, to appear in APAL.

Relativizing to InFL

Recall that 0 is of type N_n, hence $\sim x, -x : P_n \to N_n$.

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If we add a new type to negations $\sim x, -x : N_n \to P_n$, then we arrive at a new notion of sequent (multiple conclusion). The operations at the frame level corresponding to the negations are denoted by $\{\}^{\sim}$ and $\{\}^{-}$.

$$\frac{x \circ y \Rightarrow z}{y \Rightarrow x^{\sim} \circ z}\ (^{\sim}) \qquad \frac{x \circ y \Rightarrow z}{x \Rightarrow z \circ y^{-}}\ (^{-})$$

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$$\frac{x \circ y \Rightarrow z}{\overline{y \Rightarrow x^\sim \circ z}}\ (^\sim) \qquad \frac{x \circ y \Rightarrow z}{\overline{x \Rightarrow z \circ y^-}}\ (^-)$$

An *involutive (residuated) frame* is a structure of the form $\mathbf{F} = (L = R, N, \circ, \varepsilon, ^\sim, ^-)$, where

- $(L, \circ, \varepsilon, ^\sim, ^-)$ is weakly bi-involutive monoid, namely
  - $(L, \circ, \varepsilon)$ is a monoid
  - $x^{\sim -} = x = x^{-\sim}$
  - $(y^\sim \circ x^\sim)^- = (y^- \circ x^-)^\sim$ $[=: x \oplus y]$
- $x \circ y\ N\ z$ iff $y\ N\ x^\sim \oplus z$ iff $x\ N\ z \oplus y^-$, for all $x, y, z \in L$

Theorem The system $\mathbf{InFL}$ has cut elimination, FMP (and is decidable). Its simple extensions all have cut elimination.

N. Galatos and P. Jipsen. Residuated frames and applications to decidability, to appear in the Transactions of the AMS.

$\mathbf{HInFL_e}$ has cut elimination (via a syntactic argument, for now).

A. Ciabattoni, L. Strassburger and K. Terui. Expanding the realm of systematic proof theory.

Recall that $\wedge : N_n \times N_n \to N_n$.

DFL

Recall that $\wedge : N_n \times N_n \to N_n$. If we add $\wedge : P_n \times P_n \to P_n$ as a new type, then we arrive at a new notion of sequent. The operation at the frame level corresponding to $\wedge$ is denoted by $\textcircled{\wedge}$. We obtain *distributive sequents* (Giambrone, Brady), and the calculus $\mathbf{DFL}$.

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A *distributive residuated frame* (*dr-frame*) is a structure $\mathbf{F} = (L, R, N, \circ, /\!\!/, \backslash\!\!\backslash, \varepsilon, \owedge, \oslash, \obslash)$, where $(L, \circ, \varepsilon)$ is a monoid $(L, \owedge)$ is a semilattice, $N \subseteq L \times R$ and

- $\circ, \owedge : L^2 \to L$, $\backslash\!\!\backslash, \obslash : L \times R \to L$, $/\!\!/, \oslash : R \times L \to R$,
- $x \circ y N z$ iff $x N z /\!\!/ y$ iff $y N x \backslash\!\!\backslash z$.
- $x \owedge y N z$ iff $x N z \oslash y$ iff $y N x \obslash z$.
- $x N w$ implies $x \owedge y N w$; and

Theorem. If $\mathbf{F}$ is a dr-frame then the *Galois algebra* $\mathbf{F}^+ = (\mathcal{P}(L), \cap, \cup, \circ, \backslash, /, 1)_{\gamma_N}$ is a distributive residuated lattice.

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N. Galatos and P. Jipsen. Cut elimination and the finite model property for distributive FL, manuscript.

FEP for IDFL

Let $\mathcal{V}$ be a subvariety of DIRL axiomatized over $\{\vee, \wedge, \cdot, 1\}$. To establish the FEP for $\mathcal{V}$, for every $\mathbf{A}$ in $\mathcal{V}$ and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$, we construct an algebra $\mathbf{D} = \mathbf{F}^{+}_{\mathbf{A},\mathbf{B}}$ such that

- $\mathbf{F}^{+}_{\mathbf{A},\mathbf{B}} \in \mathcal{V}$
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$\mathbf{F}^{+}_{\mathbf{A},\mathbf{B}}$ is defined by taking $(L, \circ, \textcircled{\wedge}, 1)$ to be the $\{\cdot, \wedge, 1\}$-subreduct of $\mathbf{A}$ generated by B, $R = S_L \times B$ and $x \; N \; (u, b)$ iff $u(x) \leq_{\mathbf{A}} b$.

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$\mathbf{F}^+_{\mathbf{A},\mathbf{B}}$ is defined by taking $(L, \circ, \textcircled{\wedge}, 1)$ to be the $\{\cdot, \wedge, 1\}$-subreduct of $\mathbf{A}$ generated by B, $R = S_L \times B$ and $x \; N \; (u, b)$ iff $u(x) \leq_{\mathbf{A}} b$.

Theorem. (NG) Every subvariety of DIRL axiomatized over $\{\vee, \wedge, \cdot, 1\}$ has the FEP.

We consider *distributive hypersequents*, namely multisets $s_1 \mid \cdots \mid s_m$, where s_i's are distributive sequents.

CE for HDFL

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We define distributive hyper-frames by allowing the relation $\vdash$ to 'residuate' with respect to both $\circ$ and $\textcircled{\wedge}$.

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In the process we discover a distributive hyper-MacNeille completion.

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We are working on the multiple conclusion case.

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Are we on our way to a new kind of proof theory?