A Few Pearls in the Theory of Quasi-Metric Spaces

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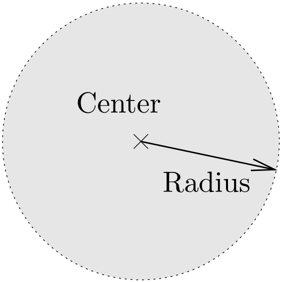
Outline

Outline

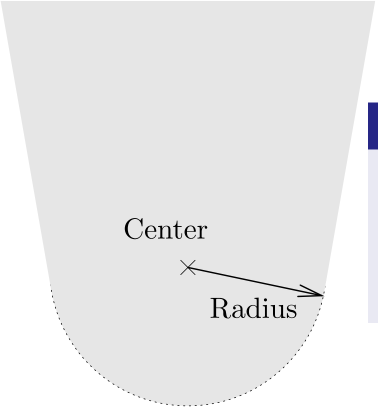
Metric Spaces



Definition (Metric)

- $x = y \Leftrightarrow d(x, y) = 0$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

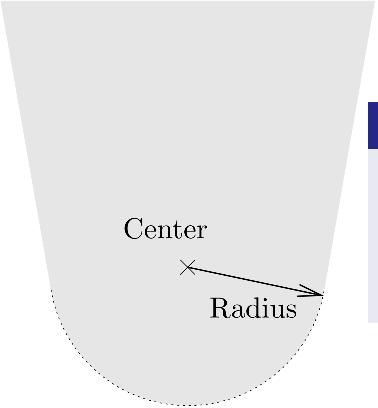
Quasi-Metric Spaces



Definition (Quasi-Metric)

- $x = y \Leftrightarrow d(x, y) = 0$
- ~~$d(x, y) = d(y, x)$~~
- $d(x, y) \leq d(x, z) + d(z, y)$

Hemi-Metric Spaces



Definition (Hemi-Metric)

- $x = y \Rightarrow d(x, y) = 0$
- ~~$d(x, y) = d(y, x)$~~
- $d(x, y) \leq d(x, z) + d(z, y)$

Goals of this Talk

1. Quasi-, Hemi-Metrics a Natural Extension of Metrics
2. Most Classical Theorems Adapt

   ... proved very recently.
3. Non-Determinism and Probabilistic Choice
4. Simulation Hemi-Metrics

Quasi-Metrics are Natural [Lawvere73]

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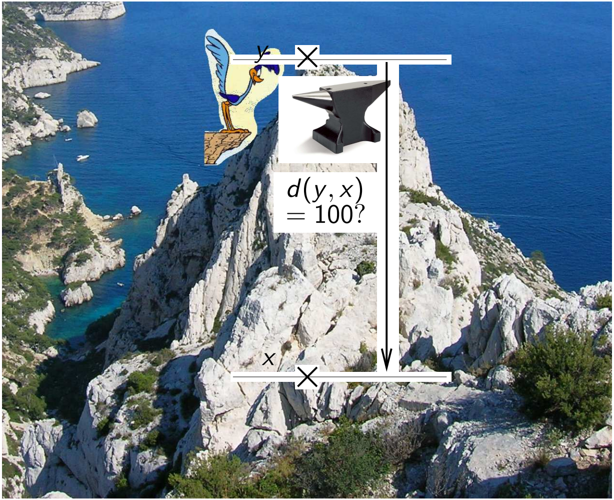
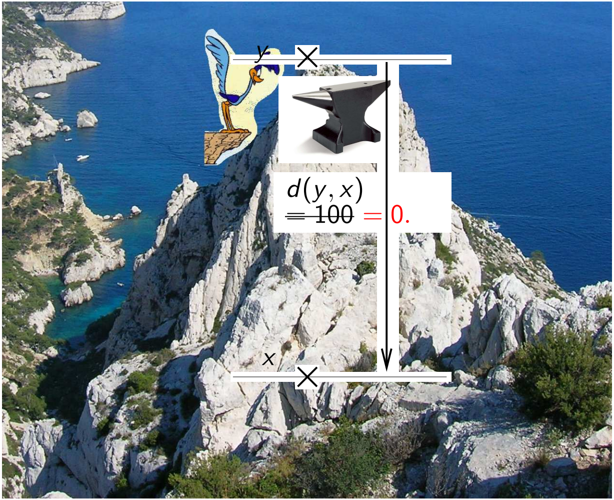


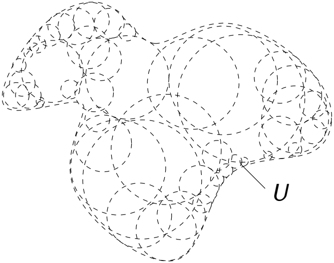
Quasi-Metrics are Natural [Lawvere73]



Outline

The Open Ball Topology

As in the symmetric case, define:



Definition (Open Ball Topology)

An open U is a union of open balls.

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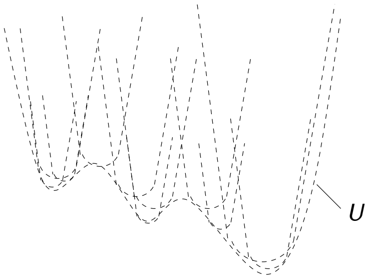


Definition (Open Ball Topology)

An open U is a union of open balls.

. . . but open balls are stranger.

Note: there are more relevant topologies, generalizing the Scott topology [Rutten96,BvBR98], but I'll try to remain simple as long as I can. . .

The Specialization Quasi-Ordering

Definition ($\leq$)

Let $x \leq y$ iff (equivalently):

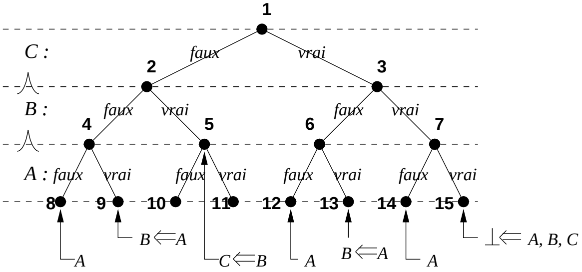
- every open containing x also contains y
- $d(x, y) = 0$.

This would be trivial in the symmetric case.

Example: $\mathrm{d}_{\mathbb{R}}(x, y) = \max(x - y, 0)$ on $\mathbb{R}$.
Then $\leq$ is the usual ordering.

Excuse Me for Turning Everything Upside-Down. . .

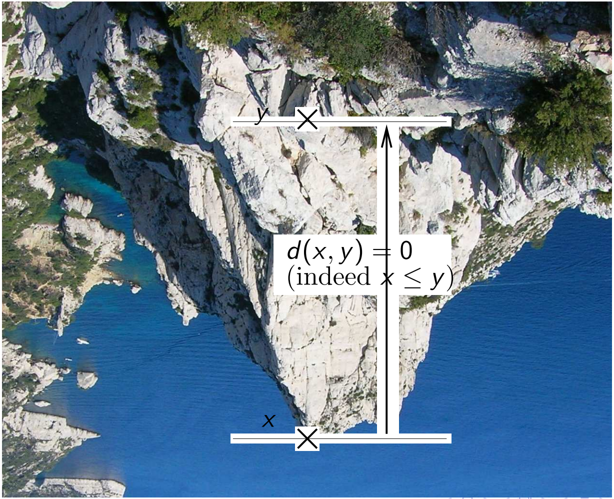
. . . but I'm a computer scientist. To me, trees look like this:



with the root on top, and the leaves at the bottom.

Excuse Me for Turning Everything Upside-Down. . .

. . . but you should really look at hills this way:



Symmetrization

Definition (d^{sym})

If d is a quasi-metric, then

$$d^{sym}(x,y) = \max(d(x,y), \underbrace{d(y,x)}_{d^{op}(x,y)})$$

is a metric.

Example: $\mathrm{d}_{\mathbb{R}}^{sym}(x,y) = |x-y|$ on $\mathbb{R}$.

Motto: A quasi-metric d describes

- a metric d^{sym}
- a partial ordering $\leq$ $\qquad (x \leq y \Leftrightarrow d(x,y) = 0)$
- and possibly more.

My Initial Impetus

Consider two transition systems T_1, T_2.

- Does T_1 simulate T_2? $(T_1 \leq T_2)$
- Is T_1 close in behaviour to T_2? $(d^{sym}(T_1, T_2) \leq \epsilon)$

…notions of *bisimulation metrics* [DGJP04,vBW04]

These questions are subsumed by computing simulation hemi-metrics between T_1 and T_2 [JGL08].

Outline

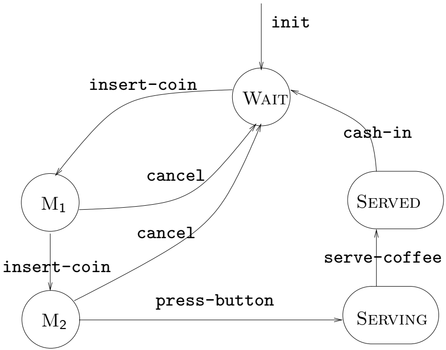
Non-Deterministic Transition Systems

Definition

State space X
Transition map $\delta : X \to \mathbb{P}(X)$

I'll assume:

- $\delta(x) \neq \emptyset$ (no deadlock)
- δ continuous (mathematically practical)
- $\delta(x)$ closed (does not restrict generality)



Lower Vietoris topology on $\mathbb{P}(X)$, generated by

$$\Diamond U = \{A \mid A \cap U \neq \emptyset\}, \quad U \text{ open}$$

The Hausdorff-Hoare Hemi-Metric

Under these conditions, δ is a continuous map from X to the Hoare powerdomain

$$\mathcal{H}(X) = \{F \text{ closed, non-empty}\}$$

with lower Vietoris topology.

The Hausdorff-Hoare Hemi-Metric

Under these conditions, δ is a continuous map from X to the Hoare powerdomain

$$\mathcal{H}(X) = \{F \text{ closed, non-empty}\}$$

with lower Vietoris topology. When X, d is quasi-metric:

Definition ("One Half of the Hausdorff Metric")

$$d_{\mathcal{H}}(F, F') = \sup_{x \in F} \inf_{x' \in F'} d(x, x')$$

Theorem (JGL08)

If X^{op} is compact (more generally, precompact), then

$$\textit{lower Vietoris} = \textit{open ball topology of } d_{\mathcal{H}} \textit{ on } \mathcal{H}(X)$$

Probabilistic Transition Systems

Definition

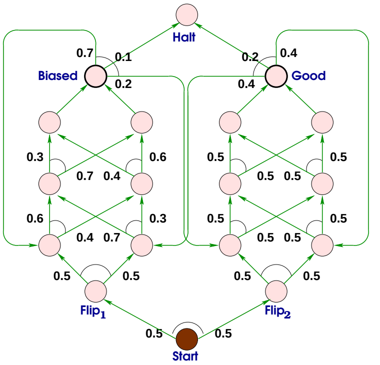
State space X

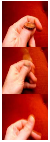
Transition map

$\delta : X \to \mathbf{V}_1(X)$

I'll assume:

- $\mathbf{V}_1(X)$ space of probabilities
- δ continuous (mathematically practical)





Probabilistic choice

Weak topology on $\mathbf{V}_1(X)$, generated by

$$[f > r] = \{\nu \in \mathbf{V}_1(X) \mid \int_X f(x)d\nu > r\}, \quad f \text{ lsc}, \ r \in \mathbb{R}^+$$

The Hutchinson Hemi-Metric

Call $f : X \to \mathbb{R}$ c-Lipschitz iff

$$\mathrm{d}_{\mathbb{R}}(f(x), f(y)) \leq c \times d(x, y) \quad (\text{i.e., } f(x) - f(y) \leq d(x, y))$$

When X, d is quasi-metric:

Definition (à la Kantorovich-Hutchinson)

$$d_{\mathrm{H}}(\nu, \nu') = \sup_{f \text{ 1-Lipschitz}} \mathrm{d}_{\mathbb{R}}(\textstyle\int_x f(x) d\nu, \int_x f(x) d\nu')$$

Theorem (JGL08)

If X is totally bounded (e.g. X^{sym} compact)

$$\textit{Weak} = \textit{open ball topology of } \mathrm{d}_{\mathrm{H}} \textit{ on } \mathbf{V}_1(X)$$

in the symmetric case, replace $\mathrm{d}_{\mathbb{R}}$ by $\mathrm{d}_{\mathbb{R}}^{sym}$

...replace total boundedness by separability+completeness?

A Unifiying View

Represent spaces of non-det./prob. choice as previsions [JGL07], i.e., certain functionals $\underbrace{\langle X \to \mathbb{R}^+ \rangle}_{lsc} \to \mathbb{R}^+$

- $\nu \in \mathbf{V}_1(X)$ by $\lambda h \in \langle X \to \mathbb{R}^+ \rangle \cdot \int_x h(x) d\nu$ (Markov)
- $F \in \mathcal{H}(X)$ by $\lambda h \in \langle X \to \mathbb{R}^+ \rangle \cdot \sup_{x \in F} h(x)$

Theorem

$\mathbf{V}_1(X) \cong$ *linear previsions* $(F(h + h') = F(h) + F(h'))$
$\mathcal{H}(X) \cong$ *sup-preserving previsions*

Leads to natural generalization. . .

A Unifying View

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$\mathbf{V}_1(X) \cong$ *linear previsions* $(F(h + h') = F(h) + F(h'))$
$\mathcal{H}(X) \cong$ *sup-preserving previsions*

Leads to natural generalization...

Definition and Theorem (Hoare Prevision)

$\Delta\,\mathbf{P}(X)$ = sublinear previsions $(F(h + h') \leq F(h) + F(h'))$
encode both $\mathcal{H}$ and $\mathbf{V}_1$, their sequential compositions, and no more.

Mixed Non-Det./Prob. Transition Systems

Definition

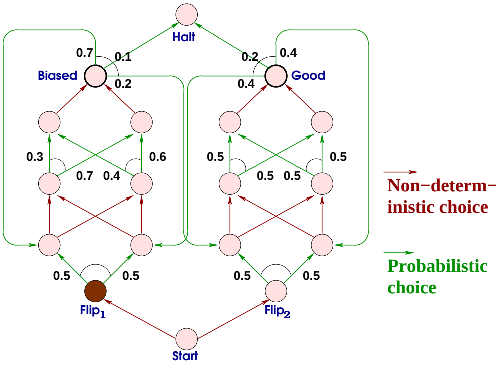
State space X
Transition map
$\delta : X \to \triangle \mathbf{P}(X)$

I'll assume:

- $\triangle \mathbf{P}(X)$ space of Hoare previsions
- δ continuous (mathematically practical)

Weak topology on $\triangle \mathbf{P}(X)$, generated by

$$[f > r] = \{F \in \triangle \mathbf{P}(X) \mid F(f) > r\}, \quad f \text{ lsc}, \ r \in \mathbb{R}^+$$



Pearl 1: The Hutchinson Hemi-Metric … on Previsions

Motto: replace $\int_X f(x) d\nu$ by $F(f)$ ("generalized average")
When X, d is quasi-metric:

Definition (à la Kantorovich-Hutchinson)

$$d_{\mathrm{H}}(F, F') = \sup_{f \text{ 1-Lipschitz}} \mathrm{d}_{\mathbb{R}}(F(f), F'(f))$$

Theorem (JGL08)

If X is totally bounded (e.g. X^{sym} compact)

$$\textit{Weak} = \textit{open ball topology of } d_{\mathrm{H}} \textit{ on } \triangle\, \mathbf{P}(X)$$

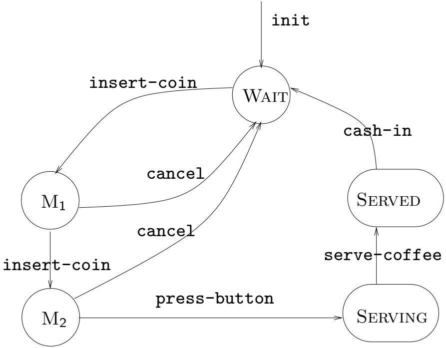
Also, we retrieve the usual hemi-metrics/topologies on $\mathcal{H}(X)$, $\mathbf{V}_1(X)$ through the encoding as previsions

Prevision Transition Systems

Add action labels $\ell \in L$, to control system:

Definition (PrTS)

A prevision transition system π is a family of continuous maps $\pi_\ell : X \to \Delta\, \mathbf{P}(X)$, $\ell \in L$.



- L is a set of actions that P has control over;
- $\pi_\ell(x)(h)$ is the generalized average gain when, from state x, we play $\ell \in L$—receiving $h(y)$ if landed on y.

Remark. Notice the similarity with Markov chains. We just replace probabilities by previsions.

Evaluating Generalized Average Payoffs

As in Markov Decision Processes ($1\frac{1}{2}$-player games), let:

- P pick action ℓ at each step
- gets reward $r_\ell(x) \in \mathbb{R}$ (from state x)
- Discount $\gamma \in (0, 1)$

The generalized average payoff at state x in internal state q:

$$V_q(x) \quad = \quad \sup_\ell \left[r_\ell(x) + \gamma \widehat{\pi}_\ell(x)(V_{q'}) \right]$$

Generalizes classical fixpoint formula for payoff in MDPs.

Simulation Hemi-Metric

Recall $V_q(x) = \sup_\ell \left[r_\ell(x) + \gamma \widehat{\pi}_\ell(x)(V_{q'}) \right]$

Definition (Simulation Hemi-Metric d_π)

$d_\pi(x, y) = \sup_\ell [\mathrm{d}_{\mathbb{R}}(r_\ell(x), r_\ell(y)) + \gamma \times (d_\pi)_{\mathrm{H}}(\pi_\ell(x), \pi_\ell(y))]$
—a least fixpoint over the complete lattice of all hemi-metrics on X.

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Proposition

$$V_q(x) - V_q(y) \leq d_\pi(x, y)$$

Simulation Hemi-Metric

In particular, close points have near-equal payoffs

Bounding Deviation

$$|V_q(x) - V_q(y)| \leq d_\pi^{sym}(x, y)$$

And simulated states have higher payoffs

Simulation

Let x simulate y iff $x \leq^{d_\pi} y$ (i.e., $d_\pi(x, y) = 0$)
If $x \leq^{d_\pi} y$, then $V_q(x) \leq V_q(y)$.

A Note on Bisimulation Metrics

We retrieve the bisimulation (pseudo-)metric of [DGJP04,vBW04,FPP05]) as:
$d^*_\pi(x, y) = \sup_\ell [\mathrm{d}_{\mathbb{R}}^{sym}(r_\ell(x), r_\ell(y)) + \gamma \times (d^*_\pi)_{\mathrm{H}}(\pi_\ell(x), \pi_\ell(y))]$

- Of course, our simulation quasi-metrics are inspired by their work
- But simulation required more:

  Even in metric spaces, simulation quasi-metric spaces require the theory of quasi-metric spaces, with properly generalized Hutchinson quasi-metric

Outline

The Theory of Quasi-Metric Spaces

- I hope I have convinced you there was a need to study quasi-metric spaces, not just metric spaces
- Fortunately, a lot has happened recently
- I'll mostly concentrate on notions of completeness
- But let's start with an easy pearl.

Pearl 2: Wilson's Theorem

Remember the following classic?

Theorem (Urysohn-Tychonoff, Early 20th Century)

For countably-based spaces, metrizability ⇔ regular Hausdorff.

Proof: hard. □

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Theorem (Urysohn-Tychonoff, Early 20th Century)

For countably-based spaces, metrizability ⇔ regular Hausdorff.

Proof: hard. □

We have the much simpler:

Theorem (Wilson31)

For countably-based spaces, hemi-metrizability ⇔ TRUE.

Proof: let $(U_n)_{n\in\mathbb{N}}$ be countable base.
Define $d_n(x, y) = 1$ iff $x \in U_n$ and $y \notin U_n$; 0 otherwise.
Together $(d_n)_{n\in\mathbb{N}}$ define the original topology.
Then let $d(x, y) = \sup_{n\in\mathbb{N}} \frac{1}{2^n} d_n(x, y)$. □

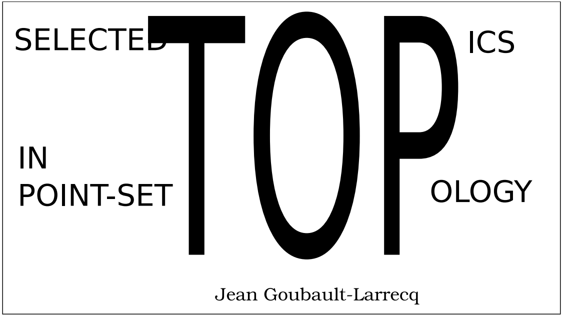
Outline

Completeness

- Completeness is an important property of metric spaces.
- Many generalizations available:
  - Čech-completeness
  - Choquet-completeness
  - Dieudonné-completeness
  - Rudin-completeness
  - Smyth-completeness
  - Yoneda-completeness
  - ...
- I was looking for a unifying notion.
- I failed, but Smyth [Smyth88] and Yoneda [BvBR98] are the two most important for quasi-metric spaces.

A Shameless Ad

Most of this in Chapter 5 of:

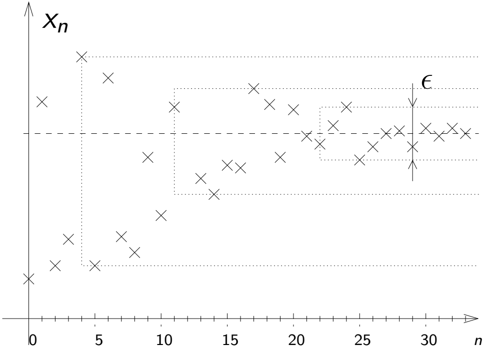


... a book on topology (mostly non-Hausdorff) with a view to domain theory (but not only).

Completeness in the Symmetric Case

Definition

A metric space is complete ⇔ every Cauchy net has a limit.



Definition (Cauchy)

$\forall \epsilon > 0$,
for $i \leq j$ large enough,
$d(x_i, x_j) < \epsilon$

i.e.,
$\limsup_{i \leq j} d(x_i, x_j) = 0$

Basic Results in the Symmetric Case

The following are complete/preserve completeness:

- $\mathbb{R}^{sym}$ (i.e., with $d_{\mathbb{R}}^{sym}(x, y) = |x - y|$)
- every compact metric space
- closed subspaces
- arbitrary coproducts
- countable topological products
- categorical products (sup metric)
- function spaces (all maps/u.cont./c-Lipschitz maps)

Complete Quasi-Metric Spaces

For quasi-metric spaces, two proposals:

Definition (Smyth-c. [Smyth88])	Definition (Yoneda-c. [BvBR98])
Every Cauchy net has a d^{op}-limit	Every Cauchy net has a d-limit

Smyth-complete	Yoneda-complete
complete metric spaces	complete metric spaces
$\mathbb{R}$, ~~$\mathbb{R} \cup \{+\infty\}$~~, $[a, b]$	$\mathbb{R}$, $\mathbb{R} \cup \{+\infty\}$, $[a, b]$
... with	$d_{\mathbb{R}}(x, y) = \max(x - y, 0)$
symcompact spaces i.e., X^{sym} compact	Smyth-complete spaces e.g., symcompact spaces
finite products	categ./countable products
all coproducts	all coproducts
~~function spaces~~	function spaces (all/c-Lip.)

d-Limits

Used in the less demanding Yoneda-completeness:

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy net
x is a d-limit $\Leftrightarrow \forall y,\ d(x, y) = \limsup_{n\to+\infty} d(x_n, y)$.

Example: if d metric, d-limit=ordinary limit.

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Example: given ordering $\leq$, $d_{\leq}(x,y) = \begin{cases} 0 & \text{if } x \leq y \\ 1 & \text{else} \end{cases}$:
Cauchy=eventually monotone, d-limit=sup.
In this case, Yoneda-complete=dcpo.

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Cauchy=eventually monotone, d-limit=sup.
In this case, Yoneda-complete=dcpo.

Warning: in general, d-limits are not limits (wrt. open ball topol.—need generalization of Scott topology [BvBR98]).

d^{op}-Limits

Used for the stronger notion of Smyth-completeness.
Easier to understand topologically:

Fact

Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy net in X.
Its d^{op}-limit (if any) is its ordinary limit in X^{sym} (if any).

Is there an alternate/more elegant characterizations of these notions of completeness? What do they mean?

Outline

Formal Balls

- Introduced by [WeihrauchSchneider81]
- Characterize completeness through domain theory [EdalatHeckmann98]

  . . . for metric spaces
- A natural idea:
  - Start all over again,
  - look for new relevant definitions of completeness

    . . . this time for quasi-metric spaces,
  - based on formal balls.

Formal Balls

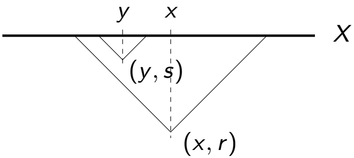
Definition

A formal ball is a pair (x, r), $x \in X$, $r \in \mathbb{R}^+$.

The poset $\mathbf{B}(X)$ of formal balls is ordered by

$$(x, r) \sqsubseteq (y, s) \Leftrightarrow d(x, y) \leq r - s$$

(Not reverse inclusion of corresponding closed balls)



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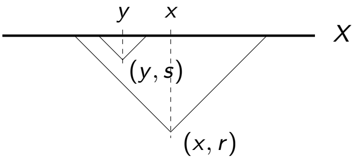

Theorem (EdalatHeckmann98)

Let X be metric. X complete $\Leftrightarrow$ $\mathbf{B}(X)$ dcpo.

Pearl 3: the Kostanek-Waszkiewicz Theorem

Let us generalize to quasi-metric spaces.
How about defining completeness as follows?

Definition (Proposal)

Let X be quasi-metric. X complete $\Leftrightarrow$ $\mathbf{B}(X)$ dcpo.

Why not, but. . .

Pearl 3: the Kostanek-Waszkiewicz Theorem

Let us generalize to quasi-metric spaces.
How about defining completeness as follows?

Definition (Proposal)

~~Let X be quasi-metric. X complete $\Leftrightarrow$ $\mathbf{B}(X)$ dcpo.~~

Why not, but. . . this is a theorem:

Theorem (Kostanek-Waszkiewicz10)

Let X be quasi-metric. X Yoneda-complete $\Leftrightarrow$ $\mathbf{B}(X)$ dcpo.

Moreover, given chain of formal balls $(x_n, r_n)_{n \in \mathbb{N}}$, with sup (x, r):

- $r = \inf_{n \in \mathbb{N}} r_n$,
- $(x_n)_{n \in \mathbb{N}}$ is Cauchy,
- x is the d-limit of $(x_n)_{n \in \mathbb{N}}$.

The Continuous Poset of Formal Balls

Let us return to metric spaces for a moment.

Theorem (EdalatHeckmann98)

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The Continuous Poset of Formal Balls

Let us return to metric spaces for a moment.

Theorem (EdalatHeckmann98)

Let X be metric. X complete $\Leftrightarrow$ $\mathbf{B}(X)$ dcpo. Moreover,

- $\mathbf{B}(X)$ *is then a continuous dcpo*
- *and* $(x, r) \ll (y, s) \Leftrightarrow d(x, y) < r - s$ *(not* $\leq$*)*

A typical notion from domain theory:

- way-below: $B \ll B'$ iff for every chain $(B_i)_{i \in I}$ such that $B' \leq \sup_i B_i$, then $B \leq B_i$ for some i.
- continuous dcpo = every B is directed sup of all $B_i \ll B$.

Example: $\overline{\mathbb{R}}^+$ ($r \ll s$ iff $r = 0$ or $r < s$)

Pearl 4: the Romaguera-Valero Theorem

Define $\prec$ by: $(x, r) \prec (y, s) \Leftrightarrow d(x, y) < r - s$
How about defining completeness as follows? (X quasi-metric)

Definition (Proposal)

X complete $\Leftrightarrow$ $\mathbf{B}(X)$ continuous dcpo with way-below $\prec$.

Why not, but. . .

Pearl 4: the Romaguera-Valero Theorem

Define $\prec$ by: $(x, r) \prec (y, s) \Leftrightarrow d(x, y) < r - s$
How about defining completeness as follows? (X quasi-metric)

Definition (Proposal)

~~X complete $\Leftrightarrow \mathbf{B}(X)$ continuous dcpo with way-below $\prec$.~~

Why not, but... this is a theorem:

Theorem (Romaguera-Valero10)

X Smyth-complete $\Leftrightarrow \mathbf{B}(X)$ continuous dcpo with way-below $\prec$.

Moreover, given chain of formal balls $(x_n, r_n)_{n \in \mathbb{N}}$, with sup (x, r):

- $r = \inf_{n \in \mathbb{N}} r_n$,
- $(x_n)_{n \in \mathbb{N}}$ is Cauchy,
- x is the d^{op}-limit of $(x_n)_{n \in \mathbb{N}}$, i.e., its limit in X^{sym}.

The Gamut of Notions of Completeness

		Spaces of formal balls is:
Weaker ↑	Yoneda-complete	a dcpo
	d-continuous Yoneda-complete	a continuous dcpo
	d-algebraic Yoneda-complete	a continuous dcpo with basis (x, r), x d-finite
Stronger ↓	Smyth-complete	a continuous dcpo with $\ll = \prec$

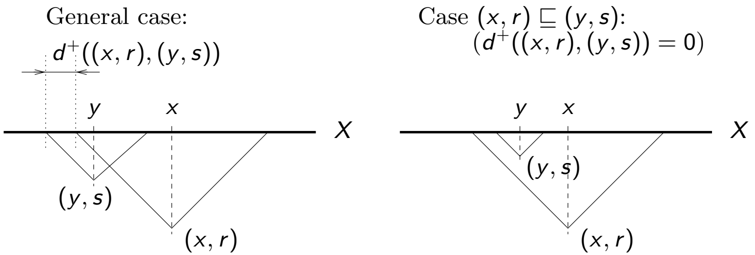
Outline

The Quasi-Metric Space of Formal Balls

Instead of considering $\mathbf{B}(X)$ as a poset, let us make it a quasi-metric space itself.

Definition (Rutten96)

Let $d^+((x,r),(y,s)) = \max(d(x,y) - r + s, 0)$

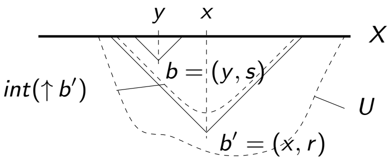


Note: $\sqsubseteq$ is merely the specialization quasi-ordering of d^+.

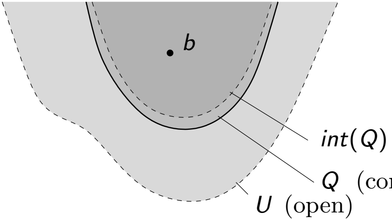
The C-Space of Formal Balls

Theorem

$\mathbf{B}(X)$ *is a c-space, i.e., for all $b \in U$ open in $\mathbf{B}(X)$, $b \in int(\uparrow b')$ for some $b' \in U$*



Key: closed ball around (y, s), radius $\epsilon/2$, is $\uparrow(y, s + \epsilon/2)$

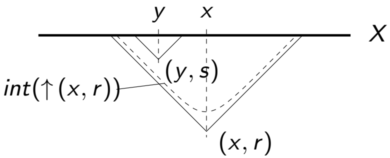


$\sim$ locally compact, where the interpolating compact is $\uparrow b'$ [Ershov73, Erné91]

The Abstract Basis of Formal Balls

Definition (Reminder)

Let $(x, r) \prec (y, s)$ in $\mathbf{B}(X)$
$\Leftrightarrow d(x, y) < r - s$
$\Leftrightarrow (y, s) \in int(\uparrow(x, r))$



Fact (Keimel)

c-space = abstract basis

Theorem

$\mathbf{B}(X), \prec$ *is an abstract basis, i.e.:*

- *(transitivity) if* $a \prec b \prec c$ *then* $a \prec c$
- *(interpolation) if* $(a_i)_{i=1}^n \prec c$ *then* $(a_i)_{i=1}^n \prec b \prec c$ *for some* b

C-Spaces and the Romaguera-Valero Thm (Pearl 5)

So $\mathbf{B}(X)$ is a c-space = an abstract basis
Note: sober c-space = continuous dcpo with way-below $\prec$

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Theorem (Romaguera-Valero10)

~~*X Smyth-complete $\Leftrightarrow$ $\mathbf{B}(X)$ continuous dcpo with way-below $\prec$.*~~

Theorem (JGL)

X Smyth-complete $\Leftrightarrow$ $\mathbf{B}(X)$ sober in its open ball topology.

Outline

Notions of Completion

Can we embed any quasi-metric space in a Yoneda/Smyth-complete one?

- Yes: Smyth-completion [Smyth88]
- Yes: Yoneda-completion [BvBR98]

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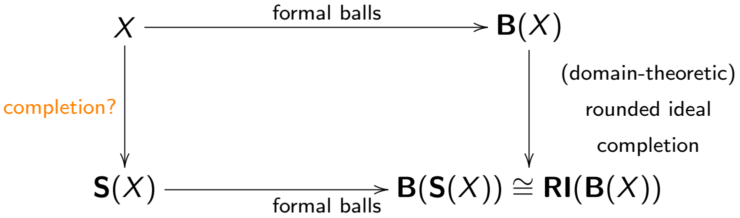
Let us explore another way:



The Theory of Abstract Bases

A rounded ideal D in $B, \prec$ is a non-empty subset of B s.t.:

- (down closed) if $a \prec b \in D$ then $a \in D$
- (directed) if $(a_i)_{i=1}^n \in D$ then $(a_i)_{i=1}^n \prec b$ for some $b \in D$.

Theorem (Rounded Ideal Completion)

The poset $\mathbf{RI}(B, \prec)$ of all rounded ideals, ordered by $\subseteq$ is a continuous dcpo, with basis B.

Note: $\mathbf{RI}(\mathbf{B}(X), \prec)$ is just the sobrification of the c-space $\mathbf{B}(X)$.

The Formal Ball Completion

Definition

The formal ball completion $\mathbf{S}(X)$ is

- space of rounded ideals $D \in \mathbf{RI}(\mathbf{B}(X), \prec)$
  ... with zero aperture $(\inf\{r \mid (x, r) \in D\} = 0)$
- with Hausdorff-Hoare quasi-metric

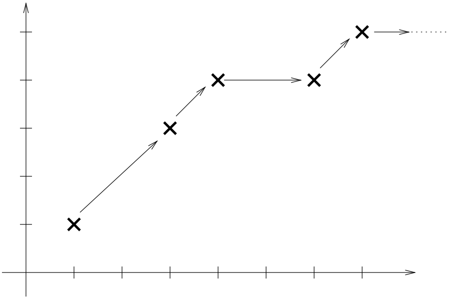
$$d^+_{\mathcal{H}}(D, D') = \sup_{(x,r)\in D} \inf_{(y,s)\in D'} d^+((x, r), (y, s))$$

Theorem

$\mathbf{B}(\mathbf{S}(X)) \cong \mathbf{RI}(\mathbf{B}(X))$

Proof. iso maps (D, r) to $D + r$ ... as expected.

Comparison with Cauchy Completion

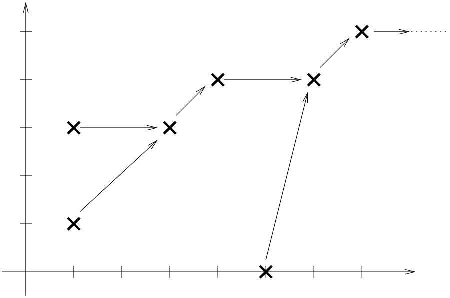


Imagine this is a chain $(x_i, r_i)_{i \in I}$ of formal balls

—

$(x_i)_{i \in I}$ is a Cauchy net

Comparison with Cauchy Completion



Imagine
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family
$(x_i, r_i)_{i \in I}$
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$(x_i)_{i \in I}$ is
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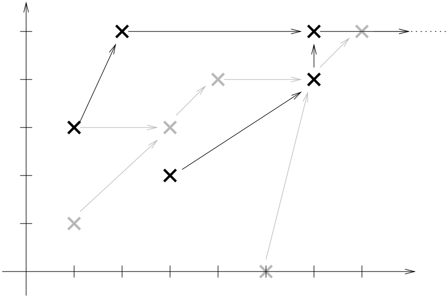
Comparison with Cauchy Completion

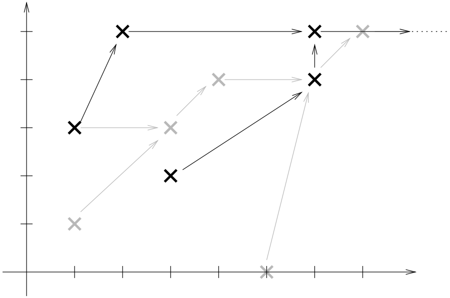


Now
here is another

—

with the same “limit”, right?

Comparison with Cauchy Completion



Instead of quotienting,
(as in Smyth-completion)
take the union
of all these equivalent
directed families

—

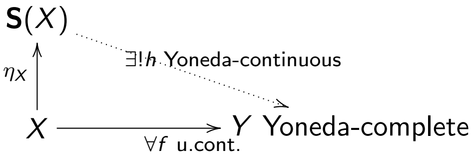
This is a
rounded ideal.

Universal Property

Theorem

S(X) *is the free Yoneda-complete space over* X*.*

I.e., letting $\eta_X(x) = \{(y, r) \mid (y, r) \prec (x, 0)\} \in \mathbf{S}(X)$ (unit):



Warning: morphisms:

q-metric spaces	uniformly continuous maps
Yoneda-compl. qms	u.c. + preserve d-limits ("Yoneda-continuity")
	(Yoneda-continuity=u.continuity in metric spaces)

Yoneda-Completion

- Let $[X \to \overline{\mathbb{R}}^+]_1 = \{$1-Lipschitz maps : $X \to \overline{\mathbb{R}}^+\}$, with sup quasi-metric $D(f, g) = \sup_{x \in X} d(f(x), g(x))$.
- Let $\eta^{\mathbf{Y}}_X(x) = d(\_, x) : X \to [X \to \overline{\mathbb{R}}^+]_1$

Definition (Yoneda completion [BvBR98])

$\mathbf{Y}(X) = D^{op}$-closure of $\mathrm{Im}(\eta^{\mathbf{Y}}_X)$ in $[X \to \overline{\mathbb{R}}^+]_1$

- Very natural from Lawvere's view of quasi-metric spaces as $\overline{\mathbb{R}}^{+op}$-enriched categories
  $+$ adequate version of Yoneda Lemma
  (…, i.e., $\eta^{\mathbf{Y}}_X$ is an isometric embedding)
- $\mathbf{Y}(X)$ also yields the free Yoneda-complete space over X

Formal Ball and Yoneda Completion

S and **Y** both build free Yoneda-complete space

Corollary

$\mathbf{S}(X) \cong \mathbf{Y}(X)$, *naturally in* X

Concretely:

- $$\begin{aligned} D \in \mathbf{S}(X) &\mapsto \lambda y \in X \cdot \limsup_{(x,r)\in D} d(y,x) \\ &= \lambda y \in X \cdot \inf{}^{\downarrow}_{(x,r)\in D}(d(y,x)+r) \end{aligned}$$
- Inverse much harder to characterize concretely (unique extension of $\eta^{\mathbf{Y}}_X : X \to \mathbf{Y}(X) \ldots$)

Smyth-Completeness Again (Pearl 6)

- $\mathbf{S} \cong \mathbf{Y}$ is a monad on quasi-metric spaces
- but not idempotent ($\mathbf{S}^2(X) \not\cong \mathbf{S}(X)$, except if X metric)

Theorem (JGL)

Let X be quasi-metric. The following are equivalent:

- $\eta_X : X \to \mathbf{S}(X)$ *is bijective*
- $\eta_X : X \to \mathbf{S}(X)$ *is an isometry*
- X *is Smyth-complete*

Example: $X = \overline{\mathbb{R}}^+$ Y-complete, not S-complete, so $\mathbf{S}(\overline{\mathbb{R}}^+) \supsetneq \overline{\mathbb{R}}^+$
Example: any dcpo X, with $d_\leq(x, y) = 0$ iff $x \leq y$, is Yoneda-complete, but $\mathbf{S}(X)$ is ideal completion of X ($\neq X$)

Outline

Conclusion

- Quasi-metrics needed for simulation
- Many theorems from the metric case adapt, e.g.

$$\text{“weak} = \text{open ball topol. of } d_{\mathrm{H}} \text{ on } \mathbf{V}_1(X)\text{”}$$

- And even generalize, e.g.

$$\text{“weak} = \text{open ball topol. of } d_{\mathrm{H}} \text{ on } \triangle\,\mathbf{P}(X)\text{”}$$

- Many recent advances.
- Demonstrated through

completeness for quasi-metric spaces now clarified
through
the unifying notion of formal balls

- Many other topics: fixpoint theorems [Rutten96], generalized Scott topology [BvBR98], Kantorovich-Rubinstein Theorem revisited [JGL08], models of Polish spaces [Martin03], etc.

And Remember. . .