Scientific Legacy of Leo Esakia

Leo Esakia
1934-2010

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Leo was very fond of the movie, and recalled often how he helped his father in shooting different scenes of the movie.

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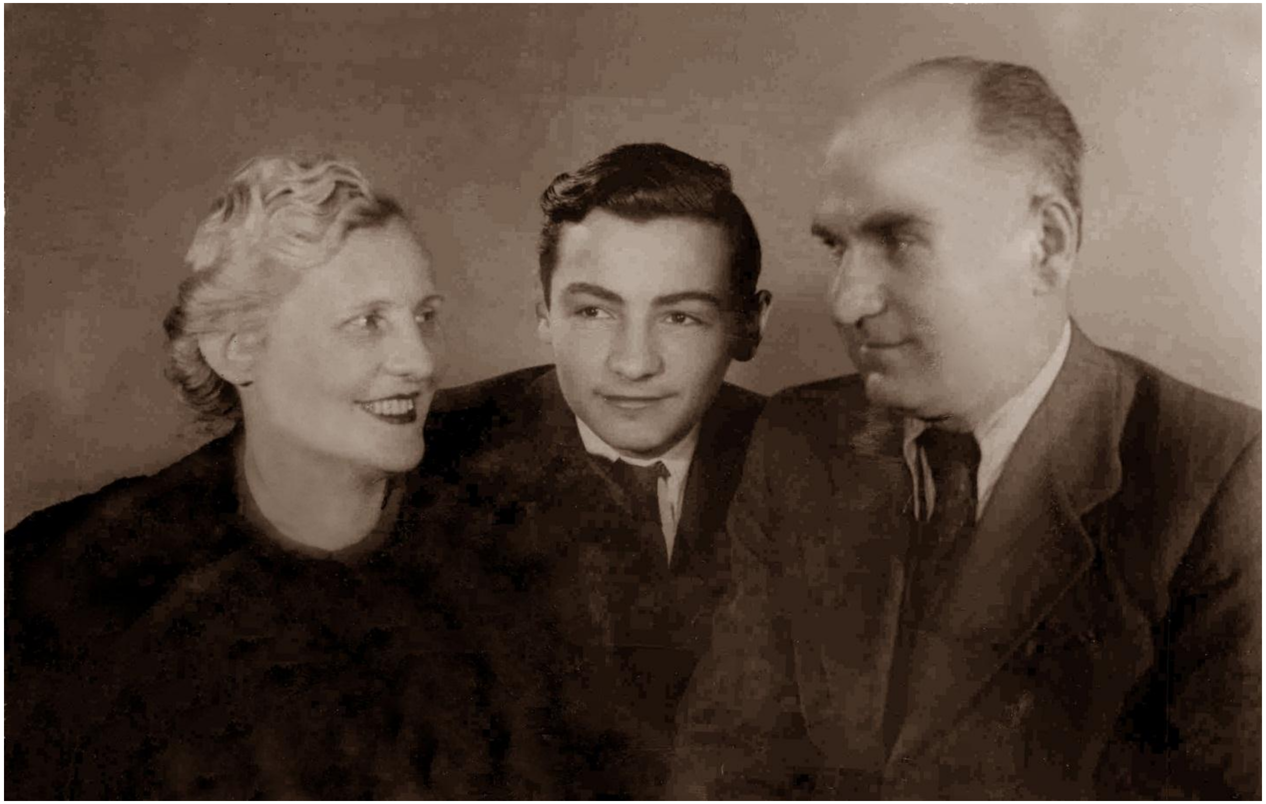
He was very much interested in the laws that govern the world around us. So he decided to study physics seriously.



16 year old Leo with his parents

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He worked at the institute for 5 years.

During this period of time his interests started to switch to mathematics, its foundations, and computer science, which at the time was a new and trendy branch of mathematics.

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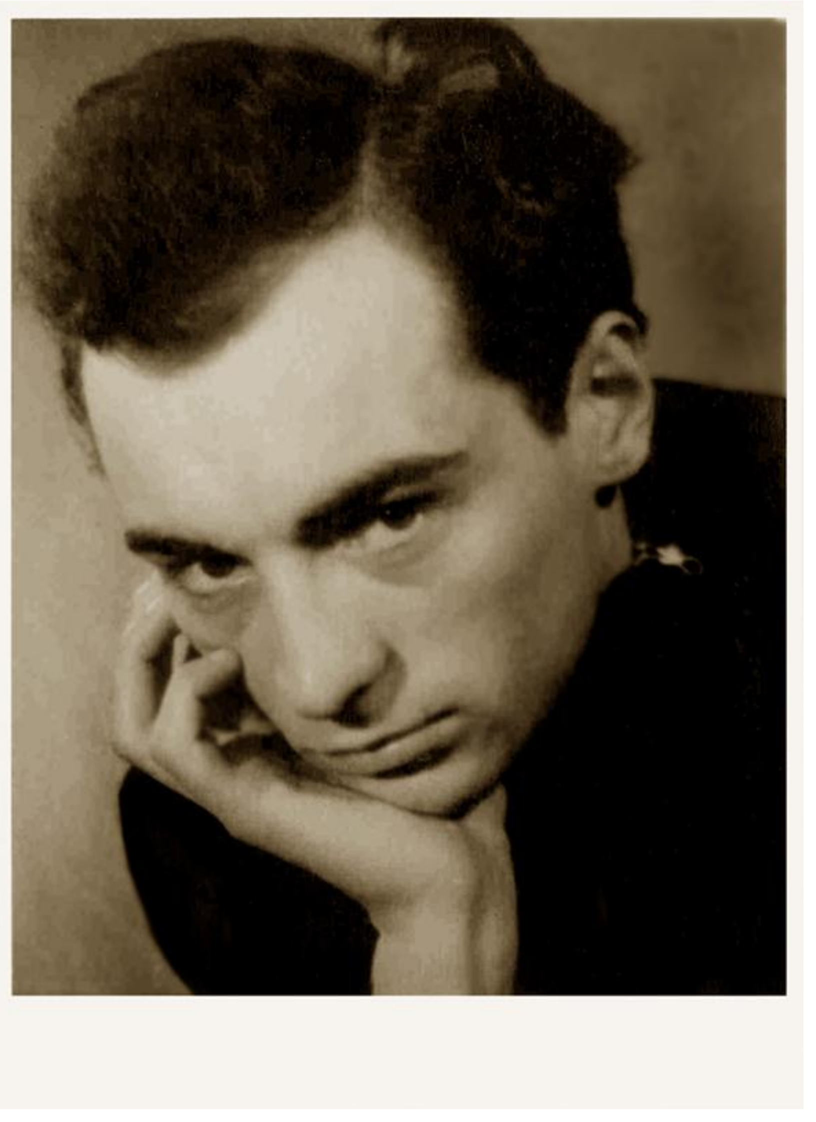
The 1960s is the time when Leo's main ideas started to form.



Leo in the 1960s

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Leo and Eteri in the 1960s

Leo and Eteri 40 years later

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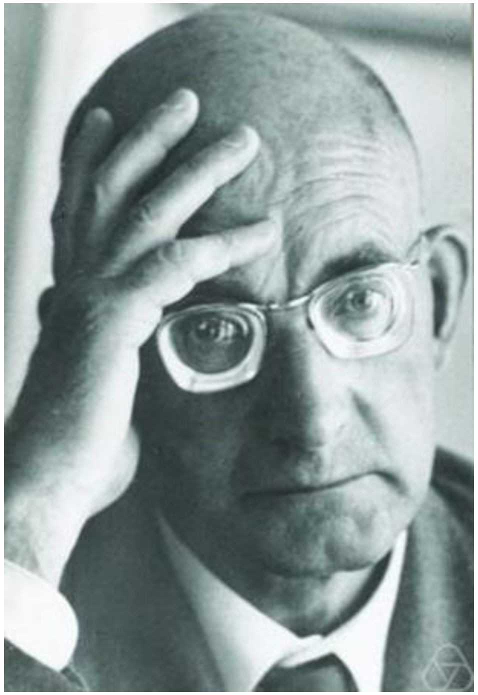

Pavel Alexandrov
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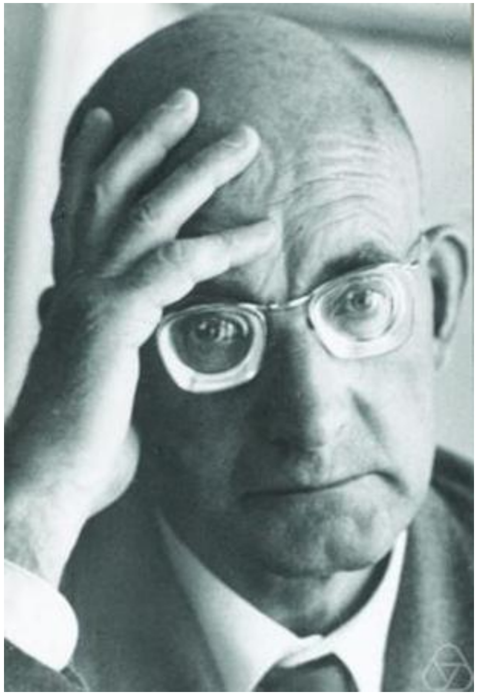

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Conversely, if H is a Heyting algebra, then $\boldsymbol{B}(H) = (B(H), \Box)$ is an S4-algebra, where $B(H)$ is the free Boolean extension of H and $\Box$ is defined as follows.

Each $x \in B(H)$ has the form

$$x = \bigwedge_{i=1}^{n} (\neg a_i \vee b_i)$$

where $a_i, b_i \in H$. Set

$$\Box x = \bigwedge_{i=1}^{n} (a_i \rightarrow b_i)$$

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As was shown by McKinsey and Tarski, it is this basic correspondence between Heyting algebras and S4-algebras that allows one to prove that Gödel's translation of **IPC** into **S4** is full and faithful.

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The *Kripke-Jonsson-Tarski representation* states that each S4-algebra is represented as a subalgebra of the S4-algebra ($P(X)$,$\Diamond_R$), where (X,R) is a *Kripke frame* with R reflexive and transitive and

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Leo realized that these two representations can be put together to obtain a full duality for S4-algebras, and as a result for Heyting algebras as well.

Let $\boldsymbol{B} = (B,\Diamond)$ be an S4-algebra, and let X be the Stone space of the Boolean algebra B. Then

$$\varphi: B \to Clopen(X)$$

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$$\varphi(\lozenge a) = \lozenge_R \varphi(a)$$

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First axiomatization: An Esakia space is a pair (*X*,*R*), where *X* is a Stone space and *R* is a reflexive and transitive relation on *X* such that:

1. *R*(*x*) is closed for all $x \in X$.
2. $U \in Clopen(X)$ *implies* $R^{-1}(U) \in Clopen(X)$.

For the second axiomatization, recall that the *Vietoris space V(X)* of *X* is the space of closed subsets of *X* with the *hit-or-miss* topology.

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Thus, the dual space of a Heyting algebra *H* is a partially ordered Esakia space, and each Heyting algebra is represented as the Heyting algebra of clopen upsets of a partially ordered Esakia space.

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This provides the object level of *Esakia duality* for Heyting algebras and S4-algebras.

For morphisms, the key was to show that if $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism of S4-algebras, and (*X*,*R*) and (*Y*,*Q*) are the dual spaces of ***A*** and ***B***, then $f\colon Y \to X$ given by

$$f(x) = h^{-1}(x)$$

is a continuous p-morphism; that is

1. xQy implies $f(x)Rf(y)$
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Esakia Lemma: If (*X*,*R*) is an Esakia space and $\{U_i\}$ is a downward directed family of clopen subsets of *X*, then

$$R^{-1} \bigcap U_i = \bigcap R^{-1} U_i$$

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All this (and many other interesting results) were contained in the following influential paper:

Leo Esakia, *Topological Kripke models*, Soviet Math. Dokl., 15 (1974), 147—151.

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Then V^* is a variety of Grzegorczyk algebras, and if $V \neq K$, then $V^* \neq K^*$.

This implies that the lattice of varieties of Heyting algebras embeds in the lattice of varieties of Grzegorczyk algebras.

But more is true: although not every Grzegorczyk algebra is of the form $\boldsymbol{B}(H)$ for some Heyting algebra H, each variety V of Grzegorczyk algebras is generated by the Grzegorczyk algebras in V of the form $\boldsymbol{B}(H)$.

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Also, the Tbilisi school of logic joined the ranks of such leading logic schools in the former Soviet union as Moscow and Novosibirsk (Russia) and Chisinau (Moldova).

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Leo and Slava proved that there are exactly *five* critical (pre-tabular) logics above **S4**. This result was important in drawing a line between the upper part of the lattice of normal extensions of **S4**, which is relatively easy to study, and it's lower part, which is highly complex.

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Leo and Revaz developed an elegant *coloring technique* for deciding whether or not a given Heyting or S4-algebra is finitely generated. This yielded an insight into the complex structure of finitely generated free Heyting and S4-algebras. This area is highly active now, and there will be several talks at the conference addressing similar issues.

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- Leo, Mamuka, and Dito developed and studied the concept of scattered topos.

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- One last topic that I'll mention concerned the concept of nucleus that has its origin in point-free topology, but can be thought of as an intuitionistic modality, and Leo was involved in developing the formal systems of logic to talk and reason about nuclei.

The Esakia seminar in the 1990s

Leo’s seminars were very special events. They were once a week, would start in the afternoon, and often would last for the rest of the day. All of Leo’s students have very fond memories of them. The common feeling was that these were incredible math journeys guided by a kind and gentle personality of Leo.

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We all got used that Leo would be always there, no matter what, but slowly his health started to deteriorate. It became painfully apparent in the late 2000s.

The last photograph of Leo
September, 2010

In spite of this, Leo managed to stay active until his death.

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This is how we remember Leo, surrounded by his students, who admired him. Leo being in his usual great mood, full of energy, and great math ideas.


რესტორანი
თანამგზავრი
Sputnik

Many of us are greatly indebted to him, including this great conference series

TACL

If it wasn’t for Leo, TACL wouldn’t have been what it is now.

Thank you, Leo!