An extension of Stone duality to fuzzy topologies and MV-algebras

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Topology, Algebra, and Categories in Logic

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“In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics.”

Hermann Weyl (1885–1955)
Outline

1. MV-algebras and their reducts
2. Semisimple and hyperarchimedean MV-algebras
3. MV-topologies
4. Stone MV-spaces and semisimple MV-algebras
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2. Semisimple and hyperarchimedean MV-algebras
3. MV-topologies
4. Stone MV-spaces and semisimple MV-algebras
Definition

An MV-algebra $\langle A, \oplus, \ast, 0 \rangle$ is an algebra of type (2,1,0) such that

- $\langle A, \oplus, 0 \rangle$ is a commutative monoid,
- $(x^*)^* = x$,
- $x \oplus 0^* = 0^*$,
- $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$. 

An MV-algebra $\langle [0, 1], \oplus, \ast, 0 \rangle$, with $x \oplus y := \min\{x + y, 1\}$ and $x^* := 1 - x$, is an MV-algebra, called standard. It generates the variety of MV-algebras both as a variety and as a quasi-variety.
MV-algebras

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### Further operations and properties

#### Operations

- $x \leq y$ if and only if $x^* \oplus y = 1$,
- $1 = 0^*$,
- $x \odot y = (x^* \oplus y^*)^*$,
- $\leq$ defines a structure of bounded lattice.
Further operations and properties

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Properties

- $\oplus$, $\odot$ and $\wedge$ distribute over any existing join.
- $\oplus$, $\odot$ and $\vee$ distribute over any existing meet.
- De Morgan laws hold both for weak and strong conjunction and disjunction:
  - $x \wedge y = (x^* \vee y^*)^*$ and $x \vee y = (x^* \wedge y^*)^*$,
  - $x \odot y = (x^* \oplus y^*)^*$ and $x \oplus y = (x^* \odot y^*)^*$. 
Boolean algebras form a subvariety of the variety of MV-algebras. They are the MV-algebras satisfying the equation $x \oplus x = x$. 

$\text{Boole} \subseteq \text{MV}$
**Boole ⊆ MV**

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**The Boolean center**

Let $A$ be an MV-algebra.

- $a \in A$ is called **idempotent** or **Boolean** if $a \oplus a = a$. 
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**The Boolean center**

Let $A$ be an MV-algebra.

- $a \in A$ is called **idempotent** or **Boolean** if $a \oplus a = a$.
- $a \oplus a = a$ iff $a \otimes a = a$.
- $a$ is Boolean iff $a^*$ is.
- $B(A) = \{a \in A \mid a \oplus a = a\}$ is a Boolean algebra, called the **Boolean center** of $A$. It is, in fact, the largest Boolean subalgebra of $A$. 

\[ B(A) = \{a \in A \mid a \oplus a = a\} \]
Reducts of MV-algebras

[Di Nola–Gerla B., 2005]

For any MV-algebra $A$, $\langle A, \lor, \circ, 0, 1 \rangle$ and $\langle A, \land, \oplus, 1, 0 \rangle$ are (commutative, unital, additively idempotent) semirings, isomorphic under the negation.

So, if $A$ is complete, $\langle A, \lor, \circ, 0, 1 \rangle$ and $\langle A, \land, \oplus, 1, 0 \rangle$ are isomorphic (commutative, unital) quantales.
For any MV-algebra $A$, $\langle A, \vee, \odot, 0, 1 \rangle$ and $\langle A, \wedge, \oplus, 1, 0 \rangle$ are (commutative, unital, additively idempotent) semirings, isomorphic under the negation.

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Moreover, also $\langle A, \vee, \oplus, 0 \rangle$ and $\langle A, \wedge, \odot, 1 \rangle$ are isomorphic semirings and, if $A$ is complete, $\langle A, \vee, \oplus, 0 \rangle$ and $\langle A, \wedge, \odot, 1 \rangle$ are isomorphic quantales.
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Definition (from Universal Algebra)

An algebra \( A \) is called \textit{semisimple} if it is subdirect product of simple algebras.
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Proposition

An MV-algebra $A$ is semisimple if and only if
$\text{Rad } A := \bigcap \text{Max } A = \{0\}$. 

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$\mathcal{MV}^{\text{ss}}$

The class of semisimple **MV-algebras** form a full subcategory of $\mathcal{MV}$ that we shall denote by $\mathcal{MV}^{\text{ss}}$. 
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$MV^{ss}$

The class of semisimple $MV$-algebras form a full subcategory of $MV$ that we shall denote by $MV^{ss}$.

It is worth noticing that, although $MV^{ss}$ is NOT a variety (it is closed under $S$ and $P$, but not under $H$), it contains $[0, 1]$, Boole, and free, projective, $\sigma$-complete and complete $MV$-algebras.
Semisimple MV-algebras are algebras of fuzzy sets

Theorem [Belluce, 1986]

A is isomorphic to a subalgebra of $[0, 1]^{\text{Max} A}$, for any $A \in \mathcal{MV}^{\text{ss}}$. 
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**Sketch of the proof.**

- For any $M \in \text{Max} A$, $A/M$ is simple.
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- For any \( M \in \text{Max} A \), \( A/M \) is simple.
- [Chang, 1959]: Any simple MV-algebra is an archimedean chain, hence it is isomorphic to a (unique) subalgebra of \([0, 1]\).
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- Let \(\varphi_M : A \rightarrow A/M\) be the natural projection.
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- Let $\varphi_M : A \rightarrow A/M$ be the natural projection.
- $\forall a \in A$, let $\hat{a} : M \in \text{Max } A \mapsto \nu_M(\varphi_M(a)) \in [0, 1]$. 
Semisimple MV-algebras are algebras of fuzzy sets

Theorem [Belluce, 1986]

* A is isomorphic to a subalgebra of $[0, 1]^{\text{Max} A}$, for any $A \in \mathcal{M}V^{ss}$.

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- $\forall a \in A$, let $\hat{a} : M \in \text{Max} A \mapsto \iota_M(\varphi_M(a)) \in [0, 1]$.
- The map $\iota : a \in A \mapsto \hat{a} \in [0, 1]^{\text{Max} A}$ is an MV-algebra embedding.
Hyperarchimedean algebras

Definition

Let $A$ be an MV-algebra. An element $a \in A$ is archimedean if it satisfies the following equivalent conditions:

1. there exists a positive integer $n$ such that $na \in B(A)$;
2. there exists a positive integer $n$ such that $a^* \lor na = 1$;
3. there exists a positive integer $n$ such that $na = (n + 1)a$. 
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**Definition**

An MV-algebra $A$ is called **hyperarchimedean** if all of its elements are archimedean.
Open sets

\[ \langle X, \Omega \rangle \text{ topological space} \]

\[ \langle \{0, 1\}^X, \lor, \land, *, 0, 1 \rangle \text{ is a complete Boolean algebra.} \]

\[ \langle X, \Omega \rangle \text{ MV-topological space} \]

\[ \langle [0, 1]^X, \lor, \land, \oplus, \odot, *, 0, 1 \rangle \text{ is a complete MV-algebra.} \]
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- \[ \langle \Omega, \lor, 0 \rangle \text{ is a sup-sublattice of } \langle \{0, 1\}^X, \lor, 0 \rangle, \]

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\[ \langle \Omega, \land, \otimes, 1 \rangle \text{ is a subsemiring of } \langle [0, 1]^X, \land, \otimes, 1 \rangle. \]
Continuous maps

Preimage of a function

Let $X$, $Y$ be sets and $f : X \to Y$ a map. If we identify the subsets of $X$ and $Y$ with their membership functions, the preimage of $f$ is

$$f^\leftarrow : \chi \in \{0, 1\}^Y \mapsto \chi \circ f \in \{0, 1\}^X.$$
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Analogously, the fuzzy preimage of $f$ is defined by

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MV-continuity

So, if $\langle X, \Omega_X \rangle$ and $\langle Y, \Omega_Y \rangle$ are MV-spaces, $f : X \rightarrow Y$ is said to be MV-continuous if $f \leftarrow [\Omega_Y] \subseteq \Omega_X.$
Examples and bases

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- Any topology is an MV-topology.
- Let \( d : X \rightarrow [0, +\infty[ \) be a metric on \( X \) and \( \alpha \) a fuzzy point of \( X \) with support \( x \). For any \( r \in \mathbb{R}^+ \), the open ball \( B_r(\alpha) \) is
  \[
  B_r(\alpha)(y) := \begin{cases} 
  \alpha(x) & \text{if } d(x, y) < r \\
  0 & \text{if } d(x, y) \geq r 
  \end{cases}.
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The family of fuzzy subsets of $X$ that are joins of open balls is an MV-topology on $X$ that is said to be induced by $d$. 
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Definition

$T = \langle X, \Omega \rangle \in \mathcal{MVTop}$. $B \subseteq \Omega$ is called a base for $T$ if, for all $o \in \Omega$, $o = \bigvee_{i \in I} b_i$, with $\{b_i\}_{i \in I} \subseteq B$. 
The shadow topology

Definition

For any MV-space $T = \langle X, \Omega \rangle$, let $B(\Omega) := \Omega \cap \{0, 1\}^X$. 
The shadow topology

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For any MV-space $T = \langle X, \Omega \rangle$, let $B(\Omega) := \Omega \cap \{0, 1\}^X$. $Sh T = \langle X, B(\Omega) \rangle$ is a topology in the classical sense, called the shadow of $T$. 
The shadow topology

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For any MV-space $T = \langle X, \Omega \rangle$, let $B(\Omega) := \Omega \cap \{0, 1\}^X$. The shadow of $T$ is $Sh_T = \langle X, B(\Omega) \rangle$, which is a topology in the classical sense, called the shadow of $T$.

**Sh is a functor**

$Top$ is a full subcategory of $\mathcal{MN}\mathrm{Top}$.
The shadow topology

**Definition**

For any MV-space $T = \langle X, \Omega \rangle$, let $B(\Omega) := \Omega \cap \{0, 1\}^X$. Sh $T = \langle X, B(\Omega) \rangle$ is a topology in the classical sense, called the **shadow** of $T$.

**Sh is a functor**

$\mathcal{T}_{\text{op}}$ is a full subcategory of $\mathcal{MV}_{\text{Top}}$. The mapping Sh : $\mathcal{MV}_{\text{Top}} \rightarrow \mathcal{T}_{\text{op}}$ is a functor. It is, in fact, the left-inverse of the inclusion $\mathcal{T}_{\text{op}} \subseteq \mathcal{MV}_{\text{Top}}$. 
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**Definition**

For any MV-space $\mathbf{T} = \langle X, \Omega \rangle$, let $B(\Omega) := \Omega \cap \{0, 1\}^X$. $\text{Sh} \mathbf{T} = \langle X, B(\Omega) \rangle$ is a topology in the classical sense, called the **shadow** of $\mathbf{T}$.

**Sh is a functor**

$\text{Top}$ is a full subcategory of $\mathcal{MV}\text{Top}$. The mapping $\text{Sh} : \mathcal{MV}\text{Top} \rightarrow \text{Top}$ is a functor. It is, in fact, the left-inverse of the inclusion $\text{Top} \subseteq \mathcal{MV}\text{Top}$.

The shadow of the MV-topology induced by a metric $d$ is the topology induced by $d$. 
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A more complex situation

Due to the presence of two intersection and two union operations, compactness and each separation axiom can have at least two different MV-versions.
Compactness

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Due to the presence of two intersection and two union operations, compactness and each separation axiom can have at least two different MV-versions.

Compact spaces

An MV-space \( \langle X, \Omega \rangle \) is said to be

- weakly compact if any open covering of \( X \) contains an additive covering, i.e., for any \( \Omega' \subseteq \Omega \) such that \( \bigvee \Omega' = 1 \), there exists a finite subset \( \{o_1, \ldots, o_n\} \) of \( \Omega' \) such that \( o_1 \oplus \cdots \oplus o_n = 1 \);
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- **compact** if any open covering of $X$ contains a finite covering.
Separation

**T\textsubscript{2} axioms**

An MV-space \( T = \langle X, \Omega \rangle \) is said to be weakly separated (or weakly Hausdorff) if for \( x \neq y \in X \), there exist \( o_x, o_y \in \Omega \) such that:

(i) \( o_x(x) = o_y(y) = 1 \),

(ii) \( o_x(y) = o_y(x) = 0 \),

(iii) \( o_x \circ o_y = 0 \).
Separation

$T_2$ axioms

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(i) $o_x(x) = o_y(y) = 1$,
(ii) $o_x(y) = o_y(x) = 0$,
(iii) $o_x \odot o_y = 0$.

$T$ is said to be separated if, for any $x \neq y \in X$, there exist $o_x, o_y \in \Omega$ satisfying (i) and

(iv) $o_x \wedge o_y = 0$.

$T_2$ definition do not need fuzzy points.
Remark

Separation implies weak separation and they both collapse to classical \( T_2 \) in the case of crisp topologies. The same holds for compactness.
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**Clopens and zero-dimensionality**

Let $T = \langle X, \Omega \rangle$ be an MV-space and $\Xi = \Omega^*$ be the family of closed fuzzy subsets. We denote by Clop $T$ the family $\Omega \cap \Xi$ of clopen fuzzy subsets of $X$. Clop $T \in \mathcal{MV}^{ss}$, for any MV-space $T$. 
Stone MV-spaces

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Clopens and zero-dimensionality

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Definition

A Stone MV-space is an MV-space which is weakly compact, weakly separated and zero-dimensional.
The MV-space $\langle \text{Max } A, \Omega_A \rangle$

**Remark**

The category $\mathcal{M}_\text{Stone}$ of Stone MV-spaces, with MV-continuous maps as morphisms, is a full subcategory of $\mathcal{M}_\text{Top}$. 
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The Maximal MV-spectrum

Let $A$ be a semisimple MV-algebra. By Belluce representation theorem, there exists a canonical embedding $\iota : A \to [0, 1]^{\text{Max } A}$. 
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Remark

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The Maximal MV-spectrum

Let $A$ be a semisimple MV-algebra. By Belluce representation theorem, there exists a canonical embedding $\iota : A \rightarrow [0, 1]^{\Max A}$. Then $\iota[A]$ generates, as a base, an MV-topology on $\Max A$. The family of open sets of such a space is denoted by $\Omega_A$.

So, for any semisimple MV-algebra $A$, $\langle \Max A, \Omega_A \rangle$ denotes the MV-topological space on $\Max A$ having (an isomorphic copy of) $A$ as a base.
A (proper) extension of Stone duality

Theorem

The mappings

\[ \Phi : \mathcal{T} \in \mathcal{MV}_{\text{Top}} \rightarrow \text{Clop } \mathcal{T} \in \mathcal{MV}^{\text{ss}} \]
\[ \Psi : A \in \mathcal{MV}^{\text{ss}} \rightarrow \langle \text{Max } A, \Omega_A \rangle \in \mathcal{MV}_{\text{Top}} \]

define two contravariant functors.

They yield a duality between \( \mathcal{MV}^{\text{ss}} \) and \( \mathcal{MV}_{\text{Stone}} \), that is for every semisimple MV-algebra \( A \), \( \Psi_A \) is a Stone MV-space and \( A \) is isomorphic to the clopen algebra of such a space; conversely, every Stone MV-space \( \mathcal{T} = \langle X, \Omega \rangle \) is homeomorphic to \( \Psi \Phi \mathcal{T} \).

The restriction of such a duality to Boolean algebras and Stone spaces coincide with the classical Stone duality.

\[ \Phi \text{Sh} = B \Phi \] and \[ \Psi B = \text{Sh } \Psi \].

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A (proper) extension of Stone duality

**Theorem**

1. The mappings

\[ \Phi : \mathcal{MVT}_{\text{Top}} \ni T \mapsto \text{Clop} T \in \mathcal{MVS}_{\text{ss}} \]

\[ \Psi : \mathcal{MVS}_{\text{ss}} \ni A \mapsto \langle \text{Max } A, \Omega_A \rangle \in \mathcal{MVT}_{\text{Top}} \]

define two contravariant functors.

2. They yield a duality between \( \mathcal{MVS}_{\text{ss}} \) and \( \mathcal{MVS}_{\text{Stone}} \), that is
A (proper) extension of Stone duality

Theorem

1. The mappings

\[ \Phi : \mathcal{T} \in \mathcal{MV}_{\text{Top}} \mapsto \text{Clop } \mathcal{T} \in \mathcal{MV}^{ss} \]
\[ \Psi : A \in \mathcal{MV}^{ss} \mapsto \langle \text{Max } A, \Omega_A \rangle \in \mathcal{MV}_{\text{Top}} \]

define two contravariant functors.

2. They yield a duality between \( \mathcal{MV}^{ss} \) and \( \mathcal{MV}_{\text{Stone}} \), that is

- for every semisimple MV-algebra \( A \), \( \Psi A \) is a Stone MV-space
  and \( A \) is isomorphic to the clopen algebra of such a space;
A (proper) extension of Stone duality

Theorem

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   \[
   \Phi : \mathcal{T} \in \mathcal{MV_{Top}} \mapsto \text{Clop} \mathcal{T} \in \mathcal{MV_{ss}}
   \]
   \[
   \Psi : A \in \mathcal{MV_{ss}} \mapsto \langle \text{Max } A, \Omega_A \rangle \in \mathcal{MV_{Top}}
   \]

   define two contravariant functors.

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   - for every semisimple MV-algebra \( A \), \( \Psi A \) is a Stone MV-space and \( A \) is isomorphic to the clopen algebra of such a space;
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A (proper) extension of Stone duality

Theorem

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\[ \Phi : \mathcal{T} \in \mathcal{MV}_{\text{Top}} \mapsto \text{Clop} \mathcal{T} \in \mathcal{MV}_{\text{ss}} \]
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   - conversely, every Stone MV-space \( \mathcal{T} = \langle X, \Omega \rangle \) is homeomorphic to \( \Psi \Phi \mathcal{T} \).

3. The restriction of such a duality to Boolean algebras and Stone spaces coincide with the classical Stone duality.

4. \( \Phi \text{Sh} = B \Phi \) and \( \Psi B = \text{Sh} \Psi \).
Graphically

Horizontal arrows: equivalences
Vertical arrows: inclusions of full subcategories and their left-inverses

Corollary
Separated Stone MV-spaces are dual to hyperarchimedean MV-algebras.
Graphically

Horizonal arrows: equivalences
Vertical arrows: inclusions of full subcategories and their left-inverses

Corollary

*Separated Stone MV-spaces are dual to hyperarchimedean MV-algebras.*
The category $\text{Boole}_n$

Objects of $\text{Boole}_n$ are pairs $B_n = \langle B, (J_i)_{i=1}^{n-1} \rangle$ where $B$ is a Boolean algebra and $(J_i)_{i=1}^{n-1}$ is a sequence of $n - 1$ ideals of $B$ such that

1. $J_i = J_{n-i}$ for all $i = 1, \ldots, n - 1$, and
2. $J_h \cap J_{i-h} \subseteq J_i$, for all $i = 2, \ldots, n - 1$ and $h = 1, \ldots, i - 1$.

A morphism $f : \langle B, (J_i)_{i=1}^{n-1} \rangle \longrightarrow \langle B', (J'_i)_{i=1}^{n-1} \rangle$ is a Boolean algebra homomorphism from $B$ to $B'$ s.t. $f[J_i] \subseteq J'_i$ for all $i$. 
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Now, let $\mathcal{MV}_n$ denote the subvariety $\mathcal{V}(S_n)$ of $\mathcal{MV}$ generated by the $(n+1)$-element chain $S_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$.
$n$-valued MV-algebras

The category $\text{Boole}_n$

Objects of $\text{Boole}_n$ are pairs $B_n = \langle B, (J_i)_{i=1}^{n-1} \rangle$ where $B$ is a Boolean algebra and $(J_i)_{i=1}^{n-1}$ is a sequence of $n-1$ ideals of $B$ such that

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Theorem [Di Nola–Lettieri, 2000] (reformulated)
The categories $\mathcal{MV}_n$ and $\text{Boole}_n$ are equivalent.
$\text{MV}_n$ and Stone spaces

A purely topological duality for $n$-valued MV-algebras is achieved through the introduction of the category of Stone spaces with distinguished open sets.
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The category $\text{Stone}_n$

Objects of $\text{Stone}_n$ are pairs $\tau_n = \langle \langle X, \Omega \rangle, (o_i)_{i=1}^{n-1} \rangle$ where $\langle X, \Omega \rangle$ is a Stone space and $(o_i)_{i=1}^{n-1}$ is a sequence of open subsets s.t.

1. $o_i = o_{n-i}$ for all $i = 1, \ldots, n-1$, and
2. $o_h \cap o_{i-h} \subseteq o_i$, for all $i = 2, 3, \ldots, n-1$ and $h = 1, \ldots, i-1$.

A morphism $f : \langle \langle X, \Omega \rangle, (o_i)_{i=1}^{n-1} \rangle \rightarrow \langle \langle X', \Omega' \rangle, (o'_i)_{i=1}^{n-1} \rangle$ is a continuous map from $X$ to $X'$ such that $f^{\leftarrow}[o'_i] \subseteq o_i$ for all $i$. 
Theorem

The categories $\mathcal{B}oule_n$ and $\text{Stone}_n$ are dually equivalent.
**Theorem**

The categories $\mathcal{Boole}_n$ and $\mathcal{Stone}_n$ are dually equivalent.

**Corollary**

$\mathcal{M}\mathcal{V}_n$ is dually equivalent to $\mathcal{Stone}_n$. 

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*MV*$_n$ and *Stone*$_n$
**Theorem**

The categories $\text{Boole}_n$ and $\text{Stone}_n$ are dually equivalent.

**Corollary**

$\mathcal{MV}_n$ is dually equivalent to $\text{Stone}_n$.

From an MV-topological viewpoint, $\mathcal{MV}_n$ is dual to the category $\mathcal{MV}_n^{\text{Stone}}$ of Stone MV-spaces of fuzzy sets with $S_n$-valued membership functions.
The categories $\mathcal{Boole}_n$ and $\text{Stone}_n$ are dually equivalent.

Corollary

$\mathcal{MV}_n$ is dually equivalent to $\text{Stone}_n$.

From an MV-topological viewpoint, $\mathcal{MV}_n$ is dual to the category $\mathcal{MV}_n\text{Stone}$ of Stone MV-spaces of fuzzy sets with $S_n$-valued membership functions.

Corollary

$\text{Stone}_n$ and $\mathcal{MV}_n\text{Stone}$ are equivalent.
“Point set topology is a disease from which the human race will soon recover.”

Jules Henri Poincaré (1854–1912)
Thank you!
References


