An extension of Stone duality to fuzzy topologies and MV-algebras

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> *"In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics."*

Hermann Weyl (1885–1955)

Outline

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MV-algebras

Definition

An MV-algebra $\langle A, \oplus, ^*, 0 \rangle$ is an algebra of type (2,1,0) such that

- $\langle A, \oplus, 0 \rangle$ is a commutative monoid,
- $(x^*)^* = x$,
- $x \oplus 0^* = 0^*$,
- $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

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$\langle [0, 1], \oplus, ^*, 0\rangle$, with $x \oplus y := \min\{x + y, 1\}$ and $x^* := 1 - x$, is an MV-algebra, called standard. It generates the variety of MV-algebras both as a variety and as a quasi-variety.

Further operations and properties

Operations

- $x \leq y$ if and only if $x^* \oplus y = 1$,
- $1 = 0^*$,
- $x \odot y = (x^* \oplus y^*)^*$,
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Properties

- $\oplus$, $\odot$ and $\wedge$ distribute over any existing join.
- $\oplus$, $\odot$ and $\vee$ distribute over any existing meet.
- De Morgan laws hold both for weak and strong conjunction and disjunction:
  - $x \wedge y = (x^* \vee y^*)^*$ and $x \vee y = (x^* \wedge y^*)^*$,
  - $x \odot y = (x^* \oplus y^*)^*$ and $x \oplus y = (x^* \odot y^*)^*$.

MV and Boolean algebras

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- $a \oplus a = a$ iff $a \odot a = a$.
- a is Boolean iff a^* is.
- $\mathrm{B}(A) = \{a \in A \mid a \oplus a = a\}$ is a Boolean algebra, called the Boolean center of A. It is, in fact, the largest Boolean subalgebra of A.

Reducts of MV-algebras

[Di Nola–Gerla B., 2005]

For any MV-algebra A, $\langle A, \vee, \odot, 0, 1\rangle$ and $\langle A, \wedge, \oplus, 1, 0\rangle$ are (commutative, unital, additively idempotent) semirings, isomorphic under the negation.

So, if A is complete, $\langle A, \bigvee, \odot, 0, 1\rangle$ and $\langle A, \bigwedge, \oplus, 1, 0\rangle$ are isomorphic (commutative, unital) quantales.

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Moreover, also $\langle A, \vee, \oplus, 0\rangle$ and $\langle A, \wedge, \odot, 1\rangle$ are isomorphic semirings and, if A is complete, $\langle A, \bigvee, \oplus, 0\rangle$ and $\langle A, \bigwedge, \odot, 1\rangle$ are isomorphic quantales.

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It is worth noticing that, although $\mathcal{MV}^{ss}$ is NOT a variety (it is closed under $\mathbb{S}$ and $\mathbb{P}$, but not under $\mathbb{H}$), it contains $[0,1]$, $\mathcal{B}$oole, and free, projective, σ-complete and complete MV-algebras.

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- The map $\iota : a \in A \longmapsto \hat{a} \in [0,1]^{\operatorname{Max} A}$ is an MV-algebra embedding.

Hyperarchimedean algebras

Definition

Let A be an MV-algebra. An element $a \in A$ is archimedean if it satisfies the following equivalent conditions:

1. there exists a positive integer n such that $na \in \mathrm{B}(A)$;
2. there exists a positive integer n such that $a^* \vee na = 1$;
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Definition

An MV-algebra A is called hyperarchimedean if all of its elements are archimedean.

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Open sets

$\langle X, \Omega \rangle$ topological space

$\langle \{0,1\}^X, \bigvee, \bigwedge, ^*, \mathbf{0}, \mathbf{1} \rangle$ is a complete Boolean algebra.

$\langle X, \Omega \rangle$ MV-topological space

$\langle [0,1]^X, \bigvee, \bigwedge, \oplus, \odot, ^*, \mathbf{0}, \mathbf{1} \rangle$ is a complete MV-algebra.

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- $\langle \Omega, \bigvee, \oplus, \mathbf{0} \rangle$ is a subquantale of $\langle [0,1]^X, \bigvee, \oplus, \mathbf{0} \rangle$,
- $\langle \Omega, \wedge, \odot, \mathbf{1} \rangle$ is a subsemiring of $\langle [0,1]^X, \wedge, \odot, \mathbf{1} \rangle$.

Continuous maps

Preimage of a function

Let X, Y be sets and $f : X \longrightarrow Y$ a map. If we identify the subsets of X and Y with their membership functions, the preimage of f is

$$f^{\leftarrow} : \chi \in \{0,1\}^Y \longmapsto \chi \circ f \in \{0,1\}^X.$$

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MV-continuity

So, if $\langle X, \Omega_X \rangle$ and $\langle Y, \Omega_Y \rangle$ are MV-spaces, $f : X \longrightarrow Y$ is said to be MV-continuous if $f^{\leftarrow\!\!\sim}[\Omega_Y] \subseteq \Omega_X$.

Examples and bases

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- Let $d : X \longrightarrow [0, +\infty[$ be a metric on X and α a fuzzy point of X with support x. For any $r \in \mathbb{R}^+$, the open ball $B_r(\alpha)$ is
$$B_r(\alpha)(y) := \begin{cases} \alpha(x) & \text{if } d(x,y) < r \\ 0 & \text{if } d(x,y) \geq r \end{cases}.$$

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Definition

$\mathbf{T} = \langle X, \Omega\rangle \in {}^{\mathcal{MV}}\mathcal{T}\mathrm{op}$. $B \subseteq \Omega$ is called a base for $\mathbf{T}$ if, for all $o \in \Omega$, $o = \bigvee_{i \in I} b_i$, with $\{b_i\}_{i \in I} \subseteq B$.

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$\mathrm{Sh} : {}^{\mathcal{MV}}\mathcal{T}\mathrm{op} \longrightarrow \mathcal{T}\mathrm{op}$ is a functor. It is, in fact, the left-inverse of the inclusion $\mathcal{T}\mathrm{op} \subseteq {}^{\mathcal{MV}}\mathcal{T}\mathrm{op}$.

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The shadow of the MV-topology induced by a metric d is the topology induced by d.

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A more complex situation

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Compact spaces

An MV-space $\langle X, \Omega \rangle$ is said to be

- weakly compact if any open covering of X contains an additive covering, i.e., for any $\Omega' \subseteq \Omega$ such that $\bigvee \Omega' = \mathbf{1}$, there exists a finite subset $\{o_1, \ldots, o_n\}$ of Ω' such that $o_1 \oplus \cdots \oplus o_n = \mathbf{1}$;

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- compact if any open covering of X contains a finite covering.

Separation

T_2 axioms

An MV-space $\mathbf{T} = \langle X, \Omega \rangle$ is said to be weakly separated (or weakly Hausdorff) if for $x \neq y \in X$, there exist $o_x, o_y \in \Omega$ such that:

(i) $o_x(x) = o_y(y) = 1$,

(ii) $o_x(y) = o_y(x) = 0$,

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$\mathbf{T}$ is said to be separated if, for any $x \neq y \in X$, there exist $o_x, o_y \in \Omega$ satisfying (i) and

(iv) $o_x \wedge o_y = \mathbf{0}$.

T_2 definition do not need fuzzy points.

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Clopens and zero-dimensionality

Let $\mathbf{T} = \langle X, \Omega \rangle$ be an MV-space and $\Xi = \Omega^*$ be the family of closed fuzzy subsets. We denote by $\operatorname{Clop} \mathbf{T}$ the family $\Omega \cap \Xi$ of clopen fuzzy subsets of X. $\operatorname{Clop} \mathbf{T} \in \mathcal{MV}^{\mathrm{ss}}$, for any MV-space $\mathbf{T}$.

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Definition

A Stone MV-space is an MV-space which is weakly compact, weakly separated and zero-dimensional.

The MV-space $\langle \mathrm{Max}\, A, \Omega_A \rangle$

Remark

The category ${}^{\mathcal{MV}}\mathcal{S}$tone of Stone MV-spaces, with MV-continuous maps as morphisms, is a full subcategory of ${}^{\mathcal{MV}}\mathcal{T}$op.

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So, for any semisimple MV-algebra A, $\langle \operatorname{Max} A, \Omega_A \rangle$ denotes the MV-topological space on $\operatorname{Max} A$ having (an isomorphic copy of) A as a base.

A (proper) extension of Stone duality

Theorem

1. *The mappings*

$$\begin{array}{lccclcc} \Phi: & \mathbf{T} & \in & {}^{\mathcal{MV}}\mathcal{T}\mathrm{op} & \longmapsto & \operatorname{Clop}\mathbf{T} & \in & \mathcal{MV}^{\mathrm{ss}} \\ \Psi: & A & \in & \mathcal{MV}^{\mathrm{ss}} & \longmapsto & \langle \operatorname{Max} A, \Omega_A \rangle & \in & {}^{\mathcal{MV}}\mathcal{T}\mathrm{op} \end{array}$$

define two contravariant functors.

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   define two contravariant functors.

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A (proper) extension of Stone duality

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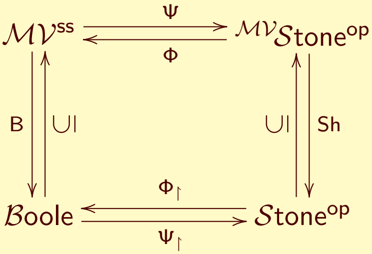
3. *The restriction of such a duality to Boolean algebras and Stone spaces coincide with the classical Stone duality.*

4. $\Phi\,\mathsf{Sh} = \mathsf{B}\,\Phi$ *and* $\Psi\,\mathsf{B} = \mathsf{Sh}\,\Psi$*.*

Graphically



Horizontal arrows: equivalences
Vertical arrows: inclusions of full subcategories and their left-inverses

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$$
\begin{array}{ccc}
\mathcal{MV}^{\mathrm{ss}} & \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} & {}^{\mathcal{MV}}\mathcal{S}\mathrm{tone}^{\mathrm{op}} \\
\mathrm{B}\downarrow\ \uparrow\cup\!| & & \cup\!|\uparrow\ \downarrow\mathrm{Sh} \\
\mathcal{B}\mathrm{oole} & \underset{\Psi_{\restriction}}{\overset{\Phi_{\restriction}}{\leftrightarrows}} & \mathcal{S}\mathrm{tone}^{\mathrm{op}}
\end{array}
$$

Horizontal arrows: equivalences
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Corollary

Separated Stone MV-spaces are dual to hyperarchimedean MV-algebras.

n-valued MV-algebras

The category $\mathcal{B}\text{oole}_n$

Objects of $\mathcal{B}\text{oole}_n$ are pairs $B_n = \langle B, (J_i)_{i=1}^{n-1} \rangle$ where B is a Boolean algebra and $(J_i)_{i=1}^{n-1}$ is a sequence of $n-1$ ideals of B such that

1. $J_i = J_{n-i}$ for all $i = 1, \ldots, n-1$, and
2. $J_h \cap J_{i-h} \subseteq J_i$, for all $i = 2, \ldots, n-1$ and $h = 1, \ldots, i-1$.

A morphism $f : \langle B, (J_i)_{i=1}^{n-1} \rangle \longrightarrow \langle B', (J_i')_{i=1}^{n-1} \rangle$ is a Boolean algebra homomorphism from B to B' s.t. $f[J_i] \subseteq J_i'$ for all i.

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Now, let $\mathcal{MV}_n$ denote the subvariety $\mathcal{V}(S_n)$ of $\mathcal{MV}$ generated by the $(n+1)$-element chain $S_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$.

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Theorem [Di Nola–Lettieri, 2000] (reformulated)

The categories $\mathcal{MV}_n$ and $\mathcal{B}$oole$_n$ are equivalent.

$\mathcal{MV}_n$ and Stone spaces

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The category $\mathcal{S}$tone$_n$

Objects of $\mathcal{S}$tone$_n$ are pairs $\tau_n = \langle\langle X, \Omega\rangle, (o_i)_{i=1}^{n-1}\rangle$ where $\langle X, \Omega\rangle$ is a Stone space and $(o_i)_{i=1}^{n-1}$ is a sequence of open subsets s.t.

1. $o_i = o_{n-i}$ for all $i = 1, \ldots, n-1$, and
2. $o_h \cap o_{i-h} \subseteq o_i$, for all $i = 2, 3, \ldots, n-1$ and $h = 1, \ldots, i-1$.

A morphism $f : \langle\langle X, \Omega\rangle, (o_i)_{i=1}^{n-1}\rangle \longrightarrow \langle\langle X', \Omega'\rangle, (o'_i)_{i=1}^{n-1}\rangle$ is a continuous map from X to X' such that $f^{\leftarrow}[o'_i] \subseteq o_i$ for all i.

$^{\mathcal{MV}}\mathcal{S}\mathrm{tone}_n$ and $\mathcal{S}\mathrm{tone}_n$

Theorem

The categories $\mathcal{B}\mathrm{oole}_n$ and $\mathcal{S}\mathrm{tone}_n$ are dually equivalent.

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Corollary

$\mathcal{MV}_n$ is dually equivalent to $\mathcal{S}\mathrm{tone}_n$.

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The categories $\mathcal{B}\mathrm{oole}_n$ and $\mathcal{S}\mathrm{tone}_n$ are dually equivalent.

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$\mathcal{MV}_n$ is dually equivalent to $\mathcal{S}\mathrm{tone}_n$.

From an MV-topological viewpoint, $\mathcal{MV}_n$ is dual to the category ${}^{\mathcal{MV}_n}\mathcal{S}\mathrm{tone}$ of Stone MV-spaces of fuzzy sets with S_n-valued membership functions.

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Corollary

$\mathcal{S}\mathrm{tone}_n$ and ${}^{\mathcal{MV}_n}\mathcal{S}\mathrm{tone}$ are equivalent.

"Point set topology is a disease from which the human race will soon recover."

Jules Henri Poincaré (1854–1912)

Thank you!

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